Bol. Soc. Paran. Mat.
(3s.) **v. 20** 1/2 (2002): 21–28.

Some Problems Concerning Definitions in Mathematics and Physics

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Keywords: Axiomatic method, definitions, foundations of physics, formal language.

Contents

ABSTRACT: Time is a concept that can be defined by means of other primitive concepts in some of the most important classical physical theories. In this paper we discuss about some problems associated to this result.

1. Introduction

There are several kinds of definition in mathematics. For a brief reference see, for example, [10]. In the present paper we are concerned with definitions in the sense of Leśniewski, as presented by Patrick Suppes in [12], with some minor modifications. In this paper we present the notions of "definition" and "definability" in a very intuitive form, but sufficiently rigorous for our purposes.

In an axiomatic system S [11] a primitive term or concept c is definable by means of the remaining primitive ones if there is an appropriate formula, provable in the system, that fixes the meaning of c in function of the other primitive terms of S. When c is not definable in S, it is said to be independent of the the other primitive terms.

There is a method, introduced by A. Padoa [9], which can be employed to show the independence of concepts. In fact, Padoa's method gives a necessary and sufficient condition for independence [1,12,14].

In order to present Padoa's method, some preliminary remarks are necessary. Loosely speaking, if we are working in set theory, as our basic theory, an axiomatic system S characterizes a species of mathematical structures in the sense of Bourbaki [2]. Actually there is a close relationship between Bourbaki's species of structures and Suppes predicates [13]; for details see [4]. On the other hand, if our underlying logic is higher-order logic (type theory), S determines a usual higher-order structure [3]. In the first case, our language is the first order language of set theory, and, in the second, it is the language of (some) type theory. Tarski showed that Padoa's method is valid in the second case [14], and Beth that it is applicable in the first [1].

From the point of view of applications of the axiomatic method, for example in the foundations of physics, it is easier to assume that our mathematical systems and structures are contructed in set theory [4].

A simplified and sufficiently rigorous formulation of the method, adapted to our exposition, is described in the next paragraphs.

Let S be an axiomatic system whose primitive concepts are c_1, c_2, ..., c_n. One of these concepts, say c_i, is independent from the remaining if and only if there are two models of S in which c_1, ..., c_{i-1}, c_{i+1}, ..., c_n have the same interpretation, but the interpretations of c_i in such models are different.

Of course a model of S is a set-theoretical structure in which all axioms of S are true, according to the interpretation of its primitive terms [8].

It is important to recall that, according to the theory of definition of Leśniewski [12], a definition should satisfy the *criterion of eliminability*. That means that a defined symbol should always be eliminable from any formula of the theory. Some authors prefer to say that any definable concept is superfluous or dispensable, as synonimous of eliminable.

2. Topology Without Topological Space

In the present section we use Padoa's principle in topology, as an example to ilustrate our ideas.

The standard definition of a topological space is as follows:

Definition 1 *A* topological space *is an ordered pair $\langle X, T\rangle$, such that the following axioms are satisfied:*

T1 *X is a set.*

T2 *T is a set of subsets of X.*

T3 *$\emptyset \in T$ and $X \in T$, where $\emptyset$ denotes the empty set.*

T4 *Any arbitrary union of elements of T belongs to T.*

T5 *If O_i and O_j belong to T, then $O_i \cap O_j$ also belongs to T.*

Some remarks should be done:

1. T is not necessarily the power set of X.
2. The elements of T are called the *open sets* of X and T is a set called the *topology* of X.
3. There are other equivalent definitions for topological space.
4. Many authors use to call X as the *topological space* itself, if there is no risk of confusion.

5. By a trivial topological space we mean a topological space $\langle X, T\rangle$ such that its topology T is $\{\emptyset, X\}$.

Now we can state the following theorem:

Theorem 1 *In a topological space $\langle X, T\rangle$, the topology is an independent concept (so, undefinable), except in the trivial case where $X = \emptyset$.*

Proof: According to Padoa's method, we have to exhibt two models M_1 and M_2 for a topological space where X has the same interpretation, but the topology T allows different interpretations. Let M_1 be the interpretation where X is the standard metric space of real numbers and $T = \{\emptyset, \Re\}$. Now, let M_2 be the interpretation where X is the same metric space, but T is the standard topology of $\Re$. It is easy to see that M_1 and M_2 are both models of a topological space.

Theorem 2 *In any topological space $\langle X, T\rangle$, X is a definable concept.*

Proof: Suppose that X is independent. In that case it would be possible to show two models M_1 and M_2 for the given topological space where the topology T corresponds to the same interpretation, but X allows at least two different interpretations. Nevertheless, this is impossible. For, if we change the interpretation of X, that would entail a change in the interpretation of T, since one of the elements of T is alway X, according to axiom **T3**. Hence, X is a dependent concept, i.e., a definable concept.

According to the last theorem, we introduce the following definition:

Definition 2 *In a topological space $\langle X, T\rangle$, we can define X as it follows:*

$$X = \bigcup_{O_i \in T} O_i,$$

i.e., the union of all elements of the topology T.

Now we can define topological space by means of its topology only:

Definition 3 *A* topological space *is a set T, such that the following axioms are satisfied:*

NT1 *T is a set whose elements are sets.*

NT2 $\emptyset \in T$.

NT3 *Any arbitrary union of elements of T belongs to T.*

NT4 *If O_i and O_j belong to T, then $O_i \cap O_j \in T$.*

One straighforward consequence from **NT3** is the following fact:

$$\bigcup_{O_i \in T} O_i \in T,$$

which corresponds to one part of axiom **T3** and guarantees the definability of X.

In other words, topology is indispensable, but the topological space X is a derived concept.

3. Classical Physical Theories Without Time

In two recent papers [5] [6] it has been proven that time is eliminable (since it is definable) in many important physical theories like newtonian particle mechanics (the non-relativistic case), hamiltonian mechanics, classical gauge theories, general relativity, Maxwell's electromagnetism, Dirac's electron, and even classical thermodynamics, which is a theory that copes with physical phenomena that are irreversible with respect to time.

In the present paper we intend to motivate some questions which may be interesting for those who are interested on some kind of link between logic and differential equations.

In order to put our ideas in a simple form, consider a very simple physical theory, namely, the classical particle mechanics, in the newtonian formalism, as introduced by McKinsey, Sugar, and Suppes in 1953 [7]. We call this McKinsey-Sugar-Suppes system of classical particle mechanics and abbreviate this terminology as MSS system.

MSS system has six primitive notions: P, T, m, $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$. P and T are sets, m is a real-valued unary function defined on P, $\mathbf{s}$ and $\mathbf{g}$ are vector-valued functions defined on the Cartesian product $P \times T$, and $\mathbf{f}$ is a vector-valued function defined on the Cartesian product $P \times P \times T$. Intuitivelly, P corresponds to the set of particles and T is to be physically interpreted as a set of real numbers measuring elapsed times (in terms of some unit of time, and measured from some origin of time). $m(p)$ is to be interpreted as the numerical value of the mass of $p \in P$. $\mathbf{s}_p(t)$, where $t \in T$, is a 3-dimensional vector which is to be physically interpreted as the position of particle p at instant t. $\mathbf{f}(p, q, t)$, where p, $q \in P$, corresponds to the internal force that particle q exerts over p, at instant t. And finally, the function $\mathbf{g}(p, t)$ is to be understood as the external force acting on particle p at instant t.

Next we present the axiomatic formulation for MSS system:

Definition 4 $\mathcal{P} = \langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$ *is an MSS system if and only if the following axioms are satisfied:*

P1 P *is a non-empty, finite set.*

P2 T *is an interval of real numbers.*

P3 *If* $p \in P$ *and* $t \in T$, *then* $\mathbf{s}_p(t)$ *is a* 3*-dimensional vector* ($\mathbf{s}_p(t) \in \Re^3$) *such that* $\frac{d^2\mathbf{s}_p(t)}{dt^2}$ *exists.*

P4 *If $p \in P$, then $m(p)$ is a positive real number.*

P5 *If $p, q \in P$ and $t \in T$, then $\mathbf{f}(p, q, t) = -\mathbf{f}(q, p, t)$.*

P6 *If $p, q \in P$ and $t \in T$, then $[\mathbf{s}_p(t), \mathbf{f}(p, q, t)] = -[\mathbf{s}_q(t), \mathbf{f}(q, p, t)]$.*

P7 *If $p, q \in P$ and $t \in T$, then $m(p)\frac{d^2\mathbf{s}_p(t)}{dt^2} = \sum_{q \in P} \mathbf{f}(p, q, t) + \mathbf{g}(p, t)$.*

The brackets [,] in axiom **P6** denote external product.

Axiom **P5** corresponds to a weak version of Newton's Third Law: to every force there is always a counterforce. Axioms **P6** and **P5**, correspond to the strong version of Newton's Third Law. Axiom **P6** establishes that the direction of force and counterforce is the direction of the line defined by the coordinates of particles p and q.

Axiom **P7** corresponds to Newton's Second Law.

Definition 5 *Let $\mathcal{P} = \langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$ be a MSS system, let P' be a non-empty subset of P, let $\mathbf{s}'$, $\mathbf{g}'$, and m' be, respectively, the restrictions of functions $\mathbf{s}$, $\mathbf{g}$, and m with their first arguments restricted to P', and let $\mathbf{f}'$ be the restriction of $\mathbf{f}$ with its first two arguments restricted to P'. Then $\mathcal{P}' = \langle P', T, \mathbf{s}', m', \mathbf{f}', \mathbf{g}' \rangle$ is a subsystem of $\mathcal{P}$ if $\forall p, q \in P'$ and $\forall t \in T$,*

$$m'(p)\frac{d^2\mathbf{s}'_p(t)}{dt^2} = \sum_{q \in P'} \mathbf{f}'(p, q, t) + \mathbf{g}'(p, t). \tag{1}$$

Theorem 3 *Every subsystem of an MSS system is again an MSS system.*

Definition 6 *Two MSS systems*

$$\mathcal{P} = \langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$$

and

$$\mathcal{P}' = \langle P', T', \mathbf{s}', m', \mathbf{f}', \mathbf{g}' \rangle$$

are equivalent if and only if $P = P'$, $T = T'$, $\mathbf{s} = \mathbf{s}'$, and $m = m'$.

Definition 7 *A MSS system is isolated if and only if for every $p \in P$ and $t \in T$, $\mathbf{g}(p, t) = \langle 0, 0, 0 \rangle$.*

Theorem 4 *If*

$$\mathcal{P} = \langle P, T, \mathbf{s}, m, \mathbf{f}, \mathbf{g} \rangle$$

and

$$\mathcal{P}' = \langle P', T', \mathbf{s}', m', \mathbf{f}', \mathbf{g}' \rangle$$

are two equivalent systems of particle mechanics, then for every $p \in P$ and $t \in T$

$$\sum_{q \in P} \mathbf{f}(p, q, t) + \mathbf{g}(p, t) = \sum_{q \in P'} \mathbf{f}'(p, q, t) + \mathbf{g}'(p, t).$$

The embedding theorem is the following:

Theorem 5 *Every MSS system is equivalent to a subsystem of an isolated system of particle mechanics.*

The next theorem can easily be proved by Padoa's method:

Theorem 6 *Mass and internal force are each independent of the remaining primitive notions of MSS system.*

We let the proof as an exercise to the reader.

The next theorem is rather important for our discussion on the dependence of time with respect to the remaining primitive concepts.

Theorem 7 *Time is definable from the remaining primitive concepts of MSS system.*

Proof: According to Padoa's principle, the primitive concept T in MSS system is independent from the remaining primitive concepts (mass, position, internal force, and external force) iff there are two models of MSS system such that T has two interpretations and the remaining primitive symbols have the same interpretation. But these two interpretations are not possible, since position $\mathbf{s}$, internal force $\mathbf{f}$, and external force $\mathbf{g}$ are functions whose domains depend on T. If we change the interpretation of T, then we will change the interpretation of other primitive concepts, namely, $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$. So, time is not independent and hence it can be defined.□

The reader will note that our proof does not show how to define time. But that is not necessary if we want to show that time is dispensable in MSS system, since a definition does satisfy the criterion of eliminability.

4. Questions

Some questions that we want to raise are:

1. What is the reason for the existence of so many dispensable concepts in ordinary mathematical and physical theories? We believe that the main "vilain" is the standard set-theoretical foundation that grounds mathematics and theoretical physics. We know that it is possible to give a formal picture for those theories without definable concepts. But the price for that is high, at least from the pedagogical point of view. So, is it possible to present an axiomatic formulation for mathematical and physical theories with less primitive concepts, but simpler than the usual framework? Do we really need this set-theoretical language in mathematics and physics?

2. What does the mathematician mean by "autonomous system" in the theory of differential equations? When a mathematician says that parameter time is definable by means of other variables, what is the meaning of that? Within this context, does the word "definable" have the same meaning given in logic?

If time is definable in several physical theories, can we say that these physical theories do not allow the existence of non-autonomous systems? If the answer to this question is positive, then the usual axiomatic framework given to classical physical theories have very serious limitations, since autonomous systems are demanded in many applications. Nevertheless, if the answer is negative, then we should establish in a very cristal clear form the meaning of terms like "time dependence" and "time independence".

No matter what answer we get to these questions, there is a lot of work to do.

5. Acknowledgments

I want to express my admiration for all my friends and colleagues, members of the editorial board of the *Boletim da Sociedade Paranaense de Matemática*, for their effort to keep this journal alive after 50 years of existence, despite the difficulties faced in the state of Paraná, Brazil.

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