Bol. Soc. Paran. Mat.
(3s.) **v. 20** 1/2 (2002): 51–58.
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Long time existence of a class of contact discontinuities for second order hyperbolic balance laws

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1. Introduction

The study of long time existence of classical smooth solutions for second order quasilinear wave equations has received much attention (see e.g. [1] and the references given there). Results were also obtained for continuous semilinear waves with gradient jumps on a characteristic hypersurface in [2] and for C^1 quasilinear waves with second order derivatives jumps on a characteristic hypersurface in [3], [4]. In this paper we show how the methods and results of [2] can be extended to a class of continuous weak solutions, with gradient jumps on a characteristic hypersurface, for some second order quasilinear balance laws.

2. Statement of the results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set lying locally on one side of its boundary $\partial\Omega$, where $\partial\Omega$ is a C^∞ manifold of dimension $(N-1)$. We shall consider the balance law

$$\Box z = \sum_{0\leq i\leq N} \partial_i(F^i(z')) + f(z') \tag{2.1}$$

if $t > 0$, $x \in \mathbb{R}^N$, where $x = (x_1, \dots, x_N)$ is the space variable, t (sometimes called x_0) is the time variable, $\partial_i = \dfrac{\partial}{\partial x_i}$ if $0 \leq i \leq N$, $z' = {}^{\mathrm{tr}}(\partial_t z, \partial_1 z, \dots, \partial_N z)$, where ${}^{\mathrm{tr}}$ means transpose, $\Box = \partial_t^2 - \sum_{1\leq j\leq N} \partial_j^2$. F^i, f are C^∞ in an open neighborhood in $\mathbb{R}^{N+1}$ of the closed ball $\{p \in \mathbb{R}^{N+1}, |p| \leq R\}$ where $R > 0$. Put $f^{ij}(p) = \dfrac{1}{2}(\partial_i F^j + \partial_j F^i)(p)$. We shall assume that

$$f^{ij}(0) = 0 \text{ if } 0 \leq i, j \leq N, (\partial^\alpha f)(0) = 0 \text{ if } |\alpha| \leq 1, \tag{2.2}$$

and also that the following “null condition” holds :

$$\begin{array}{l}\text{if } p = {}^{\mathrm{tr}}(p_0, \dots, p_N) \text{ satisfies } |p| \le R \text{ and} \\ p_0^2 - \sum_{1\le j\le N} p_j^2 = 0, \text{ then } \sum_{0\le i,j\le N} f^{ij}(p)p_ip_j = 0.\end{array} \tag{2.3}$$

For example, (2.3) is satisfied if $\sum_{0\le i\le N} \partial_i(F^i(z'))$ is $\partial_l(\partial_k z\partial_j z) - \partial_j(\partial_l z\partial_k z)$.

We shall consider weak solutions to (2.1) which satisfy the initial condition

$$\partial_t^j z = z_j \tag{2.4}$$

if $t = 0$, $j = 0, 1$, where $z_0 \in C(\mathbb{R}^N)$, $z_0 = z_1 = 0$ outside Ω, $z_j \in C^\infty(\bar\Omega)$ if $j = 0, 1$, $\sum_{1\le j\le N} |\partial_j z_0| + |z_1| < R_1$, where $R_1 < R$ is small enough.

The initial data will have to satisfy appropriate compatibility conditions. To describe those conditions, let $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be such that $\psi < 0$ in Ω, $\psi > 0$ in $\mathbb{R}^N \setminus \bar\Omega$, $d\psi \ne 0$ at each point of $\partial\Omega$, and let φ be the (at least local near $\{0\} \times \partial\Omega$) solution to $\varphi_t + |\varphi_x| = 0$, $\varphi|_{t=0} = \psi$. Then $\Sigma = \varphi^{-1}(0)$ is the outgoing characteristic hypersurface of $\Box$ through $\{0\} \times \partial\Omega$. Put $X = \partial_t - \sum_{1\le j\le N} \frac{\partial_j\varphi}{\partial_t\varphi}\partial_j$. X is tangent to Σ. We shall restrict ourselves to solutions to (2.1), (2.4) which satisfy the conditions

$$\lim_{\Omega\ni x\to a} X^m z(0, x) = 0 \tag{2.5}$$

for all $m \in \mathbb{N}$ and all $a \in \partial\Omega$. Of course (2.5) can be expressed in terms of z_0, z_1 only. Put $\Sigma(t) = \{(s, x) \in \Sigma,\ s = t\}$, $D(t) = \bigcup_{0<s<t} (\{s\} \times \Omega(s))$, where $\{s\} \times \Omega(s)$ is the bounded connected component of $(\{s\} \times \mathbb{R}^N_x) \setminus \Sigma(s)$, $E(t) = \{(s, x) \in \mathbb{R} \times \mathbb{R}^N,\ 0 < s < t\}$, $S(t) = \bigcup_{0<s<t} \Sigma(s)$. We have the following local existence result.

Theorem 2.1 *Assume that (2.3), (2.5) hold. If $T > 0$ is small, (2.1), (2.4) has a unique weak solution $z \in C^\infty(\overline{D(T)}) \cap C(\overline{E(T)})$ which vanishes outside $D(T)$. Moreover $S(T)$ is characteristic for $z|_{D(T)}$ (and for $z|_{E(T)\setminus\overline{D(T)}}$ which is 0).*

The solution z described in Thm 2.1 is a contact discontinuity. To obtain long time existence results, we assume that

$$\begin{array}{l}\Omega \text{ is convex and the total curvature of} \\ \partial\Omega \text{ in the normal direction is nonvanishing(so } N \ge 2)\end{array} \tag{2.6}$$

Then Σ is global in $t > 0$ (cf. [2]). We also introduce the following smallness assumptions. We assume that z_0, z_1 depend on a small parameter $\epsilon > 0$, and that

$$\begin{array}{l}\text{for some } \varepsilon_0 > 0 \text{ and all } \alpha \in \mathbb{N}^N, \text{ one can find } C_\alpha > 0 \text{ such that} \\ |\partial^\alpha z_j| \le C_\alpha\varepsilon \text{ in } \Omega \text{ if } j = 0, 1 \text{ and } 0 < \varepsilon \le \varepsilon_0.\end{array} \tag{2.7}$$

Denote by T_ε the supremum of all $T > 0$ such that Thm 2.1 holds. Then we have the following long time existence result.

Theorem 2.2 *Assume that (2.3), (2.5), (2.6), (2.7) hold. One can find* $\epsilon_0, C > 0$ *such that the following holds : if* $\varepsilon \leq \varepsilon_0$*, then* $T_\varepsilon \geq C/\varepsilon^2$ *if* $N = 2$*,* $T_\varepsilon \geq e^{C/\varepsilon}$ *if* $N = 3$*,* $T_\varepsilon = +\infty$ *if* $N \geq 4$*.*

Rem. If all F^i are identically constant, Thm 2.2 is contained in [2]. In [3], [4], long time existence results were proved for C^1 piecewise C^2 waves.

3. Proof of Theorem 2.1

It is enough to find a $C^\infty(\overline{D(T)})$ solution z to (2.1) in $D(T)$, with $|z'| \leq R$, such that

$$z = 0 \text{ on } S(T). \tag{3.1}$$

Indeed if we put $\tilde{z} = z$ in $\overline{D(T)}$, $\tilde{z} = 0$ in $E(T) \setminus \overline{D(T)}$, and if z satisfies (2.1) in $D(T)$ and (3.1) holds, let us check that $\tilde{z}$ is a weak solution to (2.1), that is, that the Rankine-Hugoniot condition

$$\sum_{0 \leq i \leq N} \left[\eta^{ii}\partial_i \tilde{z} - F^i(\tilde{z}')\right] \partial_i \varphi = 0 \tag{3.2}$$

holds on $S(T)$, where $[g](t,x) = \lim_{\overline{\Omega(t)} \not\ni y \to x} g(t,y) - \lim_{\Omega(t) \ni y \to x} g(t,y)$ if $(t,x) \in S(T)$ and $\eta^{00} = 1$, $\eta^{ii} = -1$ if $i > 0$. Since Σ is a characteristic hypersurface for $\Box$, (3.2) follows easily from (2.3). To solve (2.1) in $D(T)$ with (2.4) in $\{0\} \times \Omega$ and (3.1), we are going to rewrite (2.1) as a first order system. We assume that $|z'| \leq R_1$ where R_1 is so small that $1 - f^{00}(z') > 0$. Put $g^{i,j} = f^{ij}$ if i and $j \neq 0$ or if $(i,j) = (0,0)$, $g^{0,j} = 0$ if $j > 0$, $g^{j,0} = 2f^{0j}$ if $j > 0$. Write $u_j = \partial_j z$ if $0 \leq j \leq N$, $u = {}^{\mathrm{tr}}(u_0, u_1, \ldots, u_N)$. Define $(N+1) \times (N+1)$ matrices $E(u)$, $A_i(u)$, $1 \leq i \leq N$, and a $(N+1) \times 1$ matrix $G(u)$ in the following way : $E^{i,j} = \delta^{ij}$ if $(i,j) \neq (0,0)$, $E^{0,0} = (1 - g^{0,0})^{-1}$; $A_i = B_i + \tilde{B}_i$, where $B_i^{j,k} = -1$ if $(j,k) = (i+1,1)$ or $(1,i+1)$ and $B_i^{j,k} = 0$ otherwise, $\tilde{B}_i^{j,k} = 0$ if $j \neq 1$ and $\tilde{B}_i^{1,k} = -g^{i,k-1}$; $G_1 = (1 - g^{0,0})^{-1} f$, $G_i = 0$ if $i \geq 2$. From (2.1) it follows that

$$\partial_t u + \sum_{1 \leq j \leq N} (EA_j)(u) \partial_j u = G(u) \text{ if } (t,x) \in D(T). \tag{3.3}$$

Define $\bar{u}_0 = z_1$, $\bar{u}_j = \partial_j z_0$ if $1 \leq j \leq N$, $\bar{u} = {}^{\mathrm{tr}}(\bar{u}_0, \ldots, \bar{u}_N)$. Then

$$u = \bar{u} \text{ if } t = 0,\ x \in \Omega. \tag{3.4}$$

Henceforth we shall put $\Lambda_{ij} = \dfrac{\partial_i \varphi}{\partial_t \varphi} \partial_j - \dfrac{\partial_j \varphi}{\partial_t \varphi} \partial_i$ if $0 \leq i,\ j \leq N$. (3.1) implies that

$$u_j = \frac{\partial_j \varphi}{\partial_t \varphi} u_0,\ 1 \leq j \leq N, \text{ if } (t,x) \in S(T). \tag{3.5}$$

Moreover, if z satisfies (2.5), $u = z'$ should satisfy

$$\lim_{\Omega \ni x \to a} X^k (u_j - \frac{\partial_j \varphi}{\partial_t \varphi} u_0)(0,x) = 0, \quad 1 \leq j \leq N, \tag{3.6}$$

for all $m \in \mathbb{N}$. Indeed, since $\Lambda_{0k} = -\dfrac{\partial_k \varphi}{\partial_t \varphi} X - \sum_{1 \le j \le N} \dfrac{\partial_j \varphi}{\partial_t \varphi} \Lambda_{kj}$, (3.6) follows from the commutation properties of X^m with Λ_{ij} $(1 \le i, j \le N)$ (cf. Prop A.4 of [2]). To prove Thm 2.1, it is enough to show that (3.3)-(3.5) has a unique $C^\infty(\overline{D(T)})$ solution if $T > 0$ is small enough (if (3.6) holds). Let us show why. If u is a smooth solution to (3.3)-(3.5), it follows from (3.3) that $\partial_t u_j = \partial_j u_0$ if $1 \le j \le N$. Hence $\partial_k \partial_t u_j = \partial_k \partial_j u_0$ if $1 \le k \le N$, so using (3.3) again, we find that $\partial_t(\partial_k u_j - \partial_j u_k) = 0$ in $D(T)$. Now $\partial_k u_j = \partial_j u_k$ if $t = 0$, $x \in \Omega$. Let us show that

$$\partial_k u_j = \partial_j u_k \text{ if } (t, x) \in S(T). \tag{3.7}$$

(3.7) is equivalent to $\Lambda_{0j} u_k + \dfrac{\partial_j \varphi}{\partial_t \varphi} \partial_k u_0 = \Lambda_{0k} u_j + \dfrac{\partial_k \varphi}{\partial_t \varphi} \partial_j u_0$. But this last relation is easily seen to hold if we make use of the relations $\partial_t u_l = \partial_l u_0$, $1 \le l \le N$, which follow from (3.3), and of (3.5). Taking (3.7) into account, we finally conclude that $\partial_k u_j = \partial_j u_k$ in $D(T)$, $0 \le j \le k \le N$. Let $z \in C^\infty(\overline{D(T)})$ be such that $z' = u$ and such that z vanishes at some point of $\partial\Omega$. It is easily seen that z satisfies (2.1) in $D(T)$, (2.4) on $\{0\} \times \Omega$ and (3.1).

To solve (3.3)-(3.5), we are going to make use of the results of [5]. To do this we shall check that the system in (3.3) is symmetrizable hyperbolic, that $S(T)$ is characteristic of constant multiplicity 1 and that the boundary conditions in (3.5) are maximal dissipative. Take $a \in \partial\Omega$. Then $\partial_j \varphi(0, a) \ne 0$ for some j and it is no restriction to assume that $j = N$. Define the change of variables $y_j = x_j$, $0 \le j < N$, $y_N = -\varphi(t, x)$. Writing $y = (y_1, \ldots, y_N)$, $v(t, y) = u(t, x)$, $\mu_j(t, y) = \partial_j \varphi(t, x)$ if $0 \le j \le N$, $b = (a_1, \ldots, a_{N-1}, 0)$, we obtain from (3.3) that

$$\partial_t v + \sum_{1 \le j \le N-1} B_j(v) \partial_{y_j} v + B(t, y, v) \partial_{y_N} v = G(v) \tag{3.8}$$

if $y_N > 0$ and (t, y) is close to $(0, b)$, where $B_j = EA_j$ and $B(t, y, v) = -\mu_0(t, y) - \sum_{1 \le j \le N} \mu_j(t, y) B_j(v)$. (3.5) implies that

$$v_j = \frac{\mu_j}{\mu_0} v_0, \quad 1 \le j \le N \tag{3.9}$$

if $y_N = 0$ and (t, y) is close to $(0, b)$. The system in (3.8) is symmetrizable hyperbolic. Indeed let $S(v)$, $|v|$ small, be the $(N+1) \times (N+1)$ matrix defined by $S^{1,1}(v) = 1$, $S^{1,i}(v) = S^{i,1}(v) = 0$ if $2 \le i \le N+1$, $S^{i,j}(v) = -A^{1,i}_{j-1}(v)(1 - g^{0,0}(v))^{-1}$ if $2 \le i$, $j \le N+1$. Then $S(v)$ is symmetric positive definite and each $(SB_j)(v)$ is symmetric. Now rank $B(t, y, 0) = N$ if (t, y) is close to $(0, b)$, and a computation using (2.3) shows that $B(t, y, v)\mu = 0$ if v satisfies (3.9) and $\mu = {}^{\mathrm{tr}}(\mu_0, \mu_1, \ldots, \mu_N)$. Hence dim Ker $B(t, y, v) = 1$ near $(0, b, 0)$ if $y_N = 0$ and v satisfies (3.9). Furthermore it is easy to check that the boundary conditions in (3.9) are maximal dissipative near $(0, b, 0)$. Recall that this means that $S(v)B(t, y, v)$ is ≤ 0 on $\mathcal{E}_{t,y}$ if $y_N = 0$, if (t, y, v) is close to $(0, b, 0)$ and if v satisfies (3.9), where $\mathcal{E}_{t,y} = \{w \in \mathbb{R}^{N+1}, w_j = \dfrac{\mu_j}{\mu_0}(t, y) w_0,$

$1 \leq j \leq N\}$, and that $\mathcal{E}_{t,y}$ is maximal with this property. Thm 2.1 now readily follows from the results of [5].

4. Proof of Theorem 2.2

When each F^i is identically constant, Thm 2.2 has been proved in [2]. We are going to use the same method as in [2] in order to prove estimates which will enable us to obtain Thm 2.2 by a continuation method. If $h(t,x)$ is a function of t,x and $U \subset \mathbb{R}^N_x$, we shall put $|h(t)|_U = \sup_{x\in U} |h(t,x)|$, $||h(t)||_U = \left(\int_U |h(t,x)|^2\,dx\right)^{1/2}$. We have the following energy estimate (where X, Λ_{ij} are as before).

Proposition 4.1 *One can find $\delta, C > 0$ such that the following holds. If $T > 0$, $\bar f^{ij} \in C^\infty(\overline{D(T)})$, $0 \leq i,j \leq N$, with $\bar f^{ij} = \bar f^{ji}$, $\sum_{0\leq i,j\leq N} |\bar f^{ij}| \leq \delta$ in $\overline{D(T)}$, $\sum_{0\leq i,j\leq N} \bar f^{ij}\partial_i\varphi\partial_j\varphi = 0$ on $S(T)$, $L = \Box - \sum_{0\leq i,j\leq N} \bar f^{ij}\partial^2_{ij}$ and $w \in C^\infty(\overline{D(T)})$, then*

$$
\begin{aligned}
||w'(T)||^2_{\Omega(T)} \leq C\Bigg(&||w'(0)||^2_\Omega + \iint_{D(T)} \Big(|Lw|\,|\partial_t w| + \sum_{|\alpha|=1} |\partial^\alpha \bar f^{ij}(t)|_{\Omega(t)}|w'|^2\Big)\,dt\,dx \\
&+ \int_{S(T)}\Big((Xw)^2 + \sum_{1\leq i<j\leq N}(\Lambda_{ij}w)^2 \\
&+ \Big(\sum_{0\leq i,j\leq N}|\bar f^{ij}|\Big)\Big(\sum_{1\leq q\leq N}|\Lambda_{0q}w|\Big)\Big(|\partial_t w| + \sum_{1\leq q\leq N}|\Lambda_{0q}w|\Big)\Big)d\sigma\Bigg),
\end{aligned}
\tag{4.1}
$$

where $d\sigma$ is the canonical hypersurface measure on $S(T)$.

Proof of Prop 4.1. One writes $Lw\cdot\partial_t w$ as the sum of a divergence and a quadratic form in w', and integrates over $D(T)$. This is done as in the proof of Prop 5.1 of [2] (see also Prop 3.4 of [4]). We may omit the details.

Denote by $\Gamma_1,\ldots,\Gamma_n$ the vector fields $x_i\partial_j - x_j\partial_i$ $(1 \leq i < j \leq N)$, $t\partial_j + x_j\partial_t$ $(1 \leq j \leq N)$, $t\partial_t + \sum_{1\leq j\leq N} x_j\partial_j$, introduced in [6]. If $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, put $\Gamma^\alpha = \Gamma_1^{\alpha_1}\ldots\Gamma_n^{\alpha_n}$. If $f(t,x)$ is a function of (t,x), write $||f(t)||_{k,U} = \sum_{|\alpha|\leq k}\left(\int_U |(\Gamma^\alpha f)(t,x)|^2\,dx\right)^{1/2}$, $|f(t)|_{k,U} = \sum_{|\alpha|\leq k}\sup_{x\in U}|\Gamma^\alpha f(t,x)|$.

If $z \in C^\infty(\overline{D(T)})$ satisfies $\Box z = \sum_{0\leq i,j\leq N} f^{ij}(z')\partial^2_{ij}z + f(z')$ in $D(T)$, it follows that

$$(\Box - \sum_{0\leq i,j\leq N} f^{ij}(z')\partial^2_{ij})\Gamma^\alpha z = f_\alpha \qquad \text{in } D(T),$$

where

$$f_\alpha = [\Box, \Gamma^\alpha]z + \left(\Gamma^\alpha\left(\sum_{0\le i,j\le N} f^{ij}(z')\partial^2_{ij}z\right) - \sum_{0\le i,j\le N} f^{ij}(z')\Gamma^\alpha\partial^2_{ij}z\right)$$
$$+ \sum_{0\le i,j\le N} f^{ij}(z')[\Gamma^\alpha, \partial^2_{ij}]z + \Gamma^\alpha f(z').$$

Applying Prop 4.1 with $\bar{f}^{ij} = f^{ij}(z')$ and $w = \Gamma^\alpha z$, $|\alpha| \le k$, we obtain if $z \in C^\infty(\overline{D(T)})$ satisfies $\Box z = \sum_{0\le i,j\le N} f^{ij}(z')\partial^2_{ij}z + f(z')$, if $|z'| \le R$, and if $t \le T$:

$$\begin{aligned}&||z'(t)||^2_{k,\Omega(t)} \le C\Big(||z'(0)||^2_{k,\Omega}\\ &+ \int_0^t (||f_\alpha(s)||_{\Omega(s)}||z'(s)||_{k,\Omega(s)} + |z''(s)|_{\Omega(s)}||z'(s)||^2_{k,\Omega(s)})ds + J_1 + J_2\Big),\end{aligned} \tag{4.2}$$

where $z''(s,x) = \{\partial^\alpha z(s,x), |\alpha| = 2\}$,

$$J_1 = \sum_{|\alpha|\le k}\int_{S(T)}\left((X\Gamma^\alpha z)^2 + \sum_{1\le i<j\le N}(\Lambda_{ij}\Gamma^\alpha z)^2\right)d\sigma,$$
$$J_2 = \sum_{|\alpha|\le k}\int_{S(T)} |z'|\,|\Lambda\Gamma^\alpha z|\,(|\partial_t\Gamma^\alpha z| + |\Lambda\Gamma^\alpha z|)\,d\sigma,$$

with the notation $|\Lambda f| = \sum_{1\le q\le N}|\Lambda_{0q}f|$. Using the calculus properties of the derivatives Γ^α (cf. [6]), we find that

$$||f_\alpha(s)||_{\Omega(s)} \le C_{r,k}|z'(s)|_{[\frac{k+1}{2}],\Omega(s)}||z'(s)||_{k,\Omega(s)} \text{ if } |\alpha| \le k \text{ and } |z'(s)|_{[\frac{k}{2}],\Omega(s)} \le r; \tag{4.3}$$

$[\lambda]$ means $\sup\{\nu \in \mathbb{Z}, \nu \le \lambda\}$ and r is small. To estimate $J_1 + J_2$ in (4.2), we may use the following result.

Proposition 4.2 *One can find $\varepsilon_0, C_\alpha > 0$ ($\alpha \in \mathbb{N}^n$) such that the following holds: if $T > 0$ and $z \in C^\infty(\overline{D(T)})$ is a solution to (2.1) in $D(T)$, to (2.4) on $\{0\}\times\Omega$, and to (3.1), and if (2.6) and (2.7) with $\varepsilon\int_0^T(1+s)^{\frac{1-N}{2}}\,ds \le \varepsilon_0$ hold, then*

$$|X\Gamma^\alpha z| + \sum_{0\le i<j\le N}|\Lambda_{ij}\Gamma^\alpha z| \le C_\alpha\varepsilon(1+t)^{-\frac{N+1}{2}}\log^{|\alpha|+1}(2+t) \text{ on } S(T),$$

$$|(\Gamma^\alpha z)'| \le C_\alpha\varepsilon(1+t)^{-\frac{N-1}{2}}\log^{|\alpha|}(2+t) \text{ on } S(T).$$

Admitting Prop 4.2 for a moment, and using it to estimate $J_1 + J_2$, we obtain from (4.2), (4.3), if ε is small :

$$||z'(t)||^2_{k,\Omega(t)} \le C_k\varepsilon^2 + C_{k,r}\int_0^t |z'(s)|_{[\frac{k+1}{2}],\Omega(s)}||z'(s)||^2_{k,\Omega(s)}\,ds \tag{4.4}$$

if $0 \le t \le T$, $|z'(s)|_{[\frac{k}{2}],\Omega(s)} \le r$ for $0 \le s \le t$, and $k \ge 1$. Now, as proved in Prop B.1 of [2], we have the following variation on an inequality of [7]: if $k_0 = [\frac{N}{2}] + 1$, one can find $C > 0$ such that for all $U \in C^\infty(\overline{D(T)})$ and all $(t,x) \in D(T)$:

$$|U(t,x)| \le C(1+t)^{-\frac{N-1}{2}} ||U(t)||_{k_0,\Omega(t)}. \tag{4.5}$$

Put $\psi(t) = \sup_{0\le s\le t} ||z'(s)||_{k,\Omega(s)}$. Making use of (4.5) to bound $|z'(s)|_{[\frac{k+1}{2}],\Omega(s)}$ and applying the Gronwall inequality to (4.4), we deduce that

$$\psi(t) \le C_k \varepsilon e^{C_{k,r}\psi(t)\left(\int_0^t (1+s)^{-\frac{N-1}{2}} ds\right)}$$

if $k \ge \left[\dfrac{k+1}{2}\right] + k_0$ and $\sup_{0\le s\le t} |z'(s)|_{[\frac{k}{2}],\Omega(s)} \le r$. So finally we obtain that

$$||z'(t)||_{k,\Omega(t)} \le C_k \varepsilon \text{ if } 0 \le t \le T, \tag{4.6}$$

if $k \ge \left[\dfrac{k+1}{2}\right] + k_0, \varepsilon \displaystyle\int_0^T (1+s)^{-\frac{N-1}{2}} ds \le \bar{\varepsilon}_{k,r}$ with $\bar{\varepsilon}_{k,r}$ small, and $\sup_{0\le t\le T} |z'(t)|_{[\frac{k}{2}],\Omega(t)}$ $\le r$. Actually this last inequality is automatically satisfied if $k \ge \left[\dfrac{k+1}{2}\right] + k_0$, $\varepsilon \displaystyle\int_0^T (1+s)^{-\frac{N-1}{2}} ds \le \bar{\varepsilon}_{k,r}$, and $\varepsilon \le \bar{\bar{\varepsilon}}_{k,r}$ (with $\bar{\bar{\varepsilon}}_{k,r}$ small), as a simple argument using (4.5) and (4.6) shows. Since (4.6) holds, it follows from the results of [5] (and from well known results for the classical Cauchy problem) that we may continue z up to $t = T + \eta$, for some $\eta > 0$ (as a solution to (2.1) in $D(T+\eta)$ satisfying (3.1) on $S(T+\eta)$), provided that ε and $\varepsilon \displaystyle\int_0^T (1+s)^{-\frac{N-1}{2}} ds$ are small. A standard reasoning then gives the lower bounds for T_ε stated in Theorem 2.2, and Theorem 2.2 is proved. So it remains to prove Prop 4.2. To prove Prop 4.2 we may proceed as in the proof of Prop 4.2 of [2] (in which $f^{ij} \equiv 0$ for all i,j). Denote by $M_1, \ldots, M_l$ the vector fields Λ_{ij}, $1 \le i < j \le N$. One first proves by induction that one can find $\varepsilon_0 > 0$, and $C_{\beta k \alpha} > 0$ for any $\beta \in \mathbb{N}^l$, $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{N+1}$, with ε_0 and $C_{\beta k\alpha}$ independent of T, such that

$$|M^\beta X^k \partial^\alpha z'| \le C_{\beta k \alpha} \varepsilon (1+t)^{-\frac{N-1}{2} - |\beta| - k} \log^{|\beta|}(2+t) \tag{4.7}$$

on $S(T)$, if $0 < \varepsilon \le \varepsilon_0$, $\beta \in \mathbb{N}^l$, $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{N+1}$. Then estimates involving Γ^α can be deduced (see [2]). In [2], the jump of $\partial_t z$ across $S(T)$ satisfies a differential equation along the integral curves of X; in the present situation, it satisfies a first order quasilinear partial differential equation on $S(T)$. Indeed, put again $u_j = \partial_j z$, $0 \le j \le N$. In $D(T)$ we have

$$\partial_t u_0 - \sum_{1\le j\le N} \partial_j u_j = \sum_{0\le j,k\le N} f^{jk}(u) \partial_j u_k + f(u), \tag{4.8}$$

$$\partial_t u_k - \partial_k u_0 = 0, 1 \leq k \leq N. \tag{4.9}$$

Adding $\sum_{1\leq k\leq N} \frac{\partial_k\varphi}{\partial_t\varphi}(\partial_t u_k - \partial_k u_0)$ (which is 0 by (4.9)) to (4.8), and using that $\partial_j u_k = \Lambda_{0j}u_k + \frac{\partial_j\varphi}{\partial_t\varphi}\Lambda_{0k}u_0 + \frac{\partial_j\varphi\partial_k\varphi}{(\partial_t\varphi)^2}\partial_t u_0$ and that $u_k = \frac{\partial_k\varphi}{\partial_t\varphi}u_0$ if $1 \leq k \leq N$, we finally obtain the equation

$$(Z+H)u_0 = \frac{1}{2}\sum_{0\leq j,k\leq N} f^{jk}(u_0, u_0\frac{\partial_x\varphi}{\partial_t\varphi})(\Lambda_{0j}\frac{\partial_k\varphi}{\partial_t\varphi})u_0 + \frac{1}{2}f(u_0, u_0\frac{\partial_x\varphi}{\partial_t\varphi}), \tag{4.10}$$

where $Z = X - \sum_{\substack{0<k,m\leq N\\0\leq j\leq N}} f^{jk}(u_0, u_0\frac{\partial_x\varphi}{\partial_t\varphi})\frac{\partial_j\varphi\partial_m\varphi}{(\partial_t\varphi)^2}\Lambda_{mk}$ and $H = \frac{\Box\varphi}{2\partial_t\varphi}$. Estimates of H are given in [2]. From (4.10) it is not hard to deduce that (4.7) is true if $\beta = k = \alpha = 0$. The general case of (4.7) follows by obvious adaptations of the reasonings of [2]. This completes the proof of Prop 4.2. Hence the proof of Theorem 2.2 is also complete.

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