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Certain Fractional Integral with Generalized Mittag-Leffler-Type Function of Arbitrary Order

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ABSTRACT: In this paper, we define a fractional integral operator involving the generalized Mittag-Leffler function in the kernel. We establish the boundedness and composition properties of this new operator. The Laplace and Mellin transforms of this operator are obtained. Applying the Laplace transform, we solve certain fractional differential equations. Additionally, some special cases of the established results are presented.

Key Words: Mittag-Leffler function, Integral operator, Laplace transform, Mellin transform.

Contents

1. Introduction

The theory of fractional calculus has recently attracted considerable attention due to its wide-ranging applications in various scientific fields. Fractional integral operators involving special functions have gained significance in physics and engineering. The motivation for this research is recent studies on various types of integral operators (see [1,2,3,4]).

Definition 1.1 *[14] The Mittag-Leffler function $E_\alpha(z)$ is defined by*

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha,z\in\mathbb{C},\Re(\alpha)>0. \tag{1.1}$$

Definition 1.2 *[23] The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0. \tag{1.2}$$

Definition 1.3 *[16] The three-parameter Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ is defined by*

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0, \tag{1.3}$$

where $(\gamma)_n$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function Γ by (see, e.g., [21])

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}:=\{1,2,...\}).\end{cases}$$

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Definition 1.4 *[17] The generalized Mittag-Leffler function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ is defined by*

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)(\delta)_n} z^n, \tag{1.4}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$.

Definition 1.5 *[15] The Mittag-Leffler-type function of arbitrary order $E^{j,k}_{\alpha,\beta}(z)$ is defined by*

$$E^{j,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(\beta + \alpha(nj+k))}, \tag{1.5}$$

where $\alpha, \beta, z \in \mathbb{C}$, $\Re(\alpha) > 0$,$\Re(\beta) > 0$, $j \geq 1$, $k \geq 0$.

Definition 1.6 *[5] The generalized arbitrary order Mittag-Leffler-type function $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ is defined by*

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\beta + \alpha(nj+k))} z^{nj+k}, \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$.

In the same paper, we expressed $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ as the Mellin-Barnes integral in the following form:

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z) = \frac{z^k \Gamma(\delta)}{2\pi i\ \Gamma(\gamma)} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma - s)}{\Gamma(\delta - s)\Gamma(\beta + \alpha k - \alpha j s)} (-z^j)^{-s} ds, \tag{1.7}$$

where the contour of integration L joins $-i\infty$ to $+i\infty$, and splitting all the poles at $s = -n$, $(n = 0, 1, 2, \ldots)$ to the left and the poles at $s = n + 1$ and at $s = \gamma + n$, $(n = 0, 1, 2, \ldots)$ to the right.

Definition 1.7 *[22] The Fox-Wright function is defined as*

$${}_p\Psi_q \left[\begin{array}{c} (d_1, D_1), ..., (d_p, D_p) \\ (e_1, E_1), ..., (e_q, E_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(d_i + D_i n)}{\prod_{j=1}^{q} \Gamma(e_j + E_j n)} \frac{z^n}{n!}, \tag{1.8}$$

where $d_i, D_i, e_j, E_j, z \in \mathbb{C}$, $\Re(d_i) > 0, \Re(D_i) > 0$, $i = 1, ..., p$, $\Re(e_i) > 0, \Re(E_i) > 0$, $j = 1, ..., q$ and $1 + \Re\left(\sum_{j=1}^{q} E_j - \sum_{i=1}^{p} D_i\right) \geq 0$.

Definition 1.8 *[12] The H-function is defined as*

$$H^{M,N}_{P,Q}\left[z \middle| \begin{array}{c} (A_1, \alpha_1), ..., (A_P, \alpha_P) \\ (B_1, \beta_1), ..., (B_Q, \beta_Q) \end{array} \right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{M} \Gamma(B_j + \beta_j s) \prod_{i=1}^{N} \Gamma(1 - A_i - \alpha_i s)}{\prod_{i=N+1}^{P} \Gamma(A_i + \alpha_i s) \prod_{j=M+1}^{Q} \Gamma(1 - B_j - \beta_j s)} z^{-s} ds, \tag{1.9}$$

where M, N, P, Q are integers such that $0 \leq M \leq Q$, $0 \leq N \leq P$, and the parameters $A_i, B_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}^+ (i = 1, ..., p; j = 1, ..., q)$ with the contour L suitably chosen, and an empty product, if it occurs, is taken to be unity.

Definition 1.9 *For real or complex valued functions, the Lebesgue measurable space is defined by*

$$L(a, b) = \left(f : ||f||_1 = \int_a^b |f(x)| dx < \infty \right). \tag{1.10}$$

Definition 1.10 *[18] The Riemann-Liouville fractional integral operator I_{a+}^{ν} is defined as follows*

$$\left(I_{a+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)dt, \quad \left(\Re(\nu) > 0, x > a\right). \tag{1.11}$$

Then, for $\nu, \alpha, \beta, \gamma, \delta, \omega \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$, we can show that [5]

$$\left(I_{a+}^{\nu}\left[(t-a)^{\beta-1} E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x) = (x-a)^{\beta+\nu-1} E_{\alpha,\beta+\nu,\gamma,\delta}^{j,k}\left(\omega(x-a)^{\alpha}\right). \tag{1.12}$$

Definition 1.11 *[18] The Riemann-Liouville fractional derivative operator D_{a+}^{ν} is defined as follows*

$$\left(D_{a+}^{\nu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{a+}^{m-\nu} f\right)(x), \quad \left(\Re(\nu) > 0, m = [\Re(\nu)] + 1\right), \tag{1.13}$$

where $\Re(\nu)$ denotes the real part of the complex number $\nu \in \mathbb{C}$ and $[\Re(\nu)]$ represents the integral part of $\Re(\nu)$.

Hilfer [11] defined the following fractional differential operator:

$$\left(D_{a+}^{\nu,\eta} f\right)(x) = \left(I_{a+}^{\eta(1-\nu)} \frac{d}{dx}\left(I_{a+}^{(1-\eta)(1-\nu)} f\right)\right)(x), \tag{1.14}$$

where $0 < \nu < 1$ denotes the order and $0 \leq \eta \leq 1$ denotes the type of integration with respect to x. The distinction between different types of fractional derivatives is evident in the following formula that utilizes the Laplace transformation [11]:

$$\mathcal{L}\left[D_{0+}^{\nu,\eta} f(x)\right](s) = s^{\nu} \mathcal{L}\left[f(x)\right](s) - s^{\eta(1-\nu)}\left(I_{0+}^{(1-\eta)(1-\nu)} f\right)(0+), \quad (0 < \nu < 1), \tag{1.15}$$

where the initial value term $\left(I_{0+}^{(1-\eta)(1-\nu)} f\right)(0+)$ includes the Riemann-Liouville fractional integral of order $(1-\eta)(1-\nu)$, evaluated in the limit as $t \to 0+$. Here, as usual

$$\mathcal{L}\left[f(x)\right](s) = \int_0^{\infty} e^{-sx} f(x)dx, \tag{1.16}$$

provided that the defining integral in (1.16) exists.

Fubini's theorem (Dirichlet formula) [18]

$$\int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx. \tag{1.17}$$

The paper is organized as follows. In Section 2, we introduce a new fractional integral operator along with its basic properties. In Section 3, we derive the Laplace and Mellin transforms of the fractional integral operator. In Section 4, we solve some fractional differential equations utilizing the Laplace transform. Finally, in Section 5, we present the conclusions of this research.

2. An integral operator involving the function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ and its properties

In this section, we define the following fractional integral operator with the generalized arbitrary order Mittag-Leffler-type function as its kernel:

$$\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w} f\right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(x-t)^{\alpha}\right) f(t)dt, \quad x > a, \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta, w \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$.
If $j = 1$ and $k = 0$ in (2.1), then we hav the following fractional integral operator:

$$\left(\mathbf{E}_{\alpha,\beta;a+}^{\gamma,\delta;w} f\right)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta}\left(w(x-t)^{\alpha}\right) f(t)dt, \quad x > a. \tag{2.2}$$

If we set $\delta = 1$ in (2.2), we obtain the following fractional integral operator defined by Prabhakar [16]:

$$\mathbf{E}\left(\alpha, \beta; \gamma; w\right) f(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(w(x-t)^{\alpha}\right) f(t)dt. \tag{2.3}$$

When $w = 0$, the integral operator in (2.3) reduces to the Riemann-Liouville fractional integral operator given in (1.11).

Theorem 2.1 *Suppose* $\alpha, \beta, \gamma, \delta, \lambda, w \in \mathbb{C}$*;* $\Re(\alpha) > 0$*,* $\Re(\beta) > 0$*,* $\Re(\gamma) > 0$*,* $\Re(\delta) > 0$*,* $\Re(\lambda) > 0$ *and* $j \geq 1$*,* $k \geq 0$*, then the following result holds true:*

$$\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w} (t-a)^{\lambda-1}\right)(x) = (x-a)^{\beta+\lambda-1}\Gamma(\lambda)E_{\alpha,\beta+\lambda,\gamma,\delta}^{j,k}\left(w\left(x-a\right)^{\alpha}\right). \tag{2.4}$$

Proof: From (1.6) and (2.1), we find

$$\begin{aligned}
&\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w} (t-a)^{\lambda-1}\right)(x) \\
&= \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(x-t)^{\alpha}\right)(t-a)^{\lambda-1} dt \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n \Gamma\left(\beta+\alpha(nj+k)\right)} \left(\int_a^x (x-t)^{\beta+\alpha(nj+k)-1}(t-a)^{\lambda-1} dt\right) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n} \frac{1}{\Gamma\left(\beta+\alpha(nj+k)\right)} \int_a^x (x-t)^{\beta+\alpha(nj+k)-1}(t-a)^{\lambda-1} dt \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n} I_{a+}^{\beta+\alpha(nj+k)}\left[(t-a)^{\lambda-1}\right](x) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n} \frac{\Gamma(\lambda)}{\Gamma\left(\beta+\lambda+\alpha(nj+k)\right)} (x-a)^{\beta+\lambda+\alpha(nj+k)-1} \\
&= (x-a)^{\beta+\lambda-1}\Gamma(\lambda)E_{\alpha,\beta+\lambda,\gamma,\delta}^{j,k}\left(w\left(x-a\right)^{\alpha}\right),
\end{aligned}$$

which completes the required proof. □

Let $a = 0$ and $x = 1$, then we obtain the following result.

Corollary 2.1 *If* $\alpha, \beta, \gamma, \delta, \lambda, w \in \mathbb{C}$*;* $\Re(\alpha) > 0$*,* $\Re(\beta) > 0$*,* $\Re(\gamma) > 0$*,* $\Re(\delta) > 0$*,* $\Re(\lambda) > 0$ *and* $j \geq 1$*,* $k \geq 0$*, then*

$$\frac{1}{\Gamma(\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\beta-1} E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(1-t)^{\alpha}\right) dt = E_{\alpha,\beta+\lambda,\gamma,\delta}^{j,k}(w). \tag{2.5}$$

Set $\delta = j = 1$ and $k = 0$ in (2.5), then we obtain the well-known result [16].

Theorem 2.2 *Suppose* $\alpha, \beta, \gamma, \delta, \lambda, \sigma, w \in \mathbb{C}$*;* $\Re(\alpha) > 0$*,* $\Re(\beta) > 0$*,* $\Re(\gamma) > 0$*,* $\Re(\delta) > 0$*,* $\Re(\lambda) > 0$*,* $\Re(\sigma) > 0$ *and* $j \geq 1$*,* $k \geq 0$*, then the following result holds true:*

$$\begin{aligned}
\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w} t^{\frac{(\lambda+\gamma)}{\sigma}-1}\right)(x) &= \frac{w^k x^{\beta+\alpha k+\frac{(\lambda+\gamma)}{\sigma}}\Gamma(\delta)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma(\gamma)} \\
&\times {}_2\Psi_2\left[\begin{array}{c}(1,1),(\gamma,1) \\ (\delta,1),(\beta+\alpha k+(\lambda+\gamma)/\sigma,\alpha j)\end{array}\middle| (wx^{\alpha})^j\right].
\end{aligned} \tag{2.6}$$

Proof: Using the fractional integral operator (2.1), we obtain

$$\begin{aligned}&\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}t^{\frac{(\lambda+\gamma)}{\sigma}-1}\right)(x)\\&=\int_0^x (x-t)^{\beta-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(w(x-t)^{\alpha}\right)t^{\frac{(\lambda+\gamma)}{\sigma}-1}dt\\&=\int_0^x (x-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}\left(w(x-t)^{\alpha}\right)^{nj+k}t^{\frac{(\lambda+\gamma)}{\sigma}-1}dt\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}x^{\beta+\alpha(nj+k)-1}\int_0^x\left(1-\frac{t}{x}\right)^{\beta+\alpha(nj+k)-1}t^{\frac{(\lambda+\gamma)}{\sigma}-1}dt.\end{aligned}$$

Setting $t/x=u$, we get

$$\begin{aligned}&\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}t^{\frac{(\lambda+\gamma)}{\sigma}-1}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}x^{\beta+\alpha(nj+k)-1}\int_0^1(1-u)^{\beta+\alpha(nj+k)-1}(xu)^{\frac{(\lambda+\gamma)}{\sigma}-1}x\,du\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}x^{\beta+\alpha(nj+k)+\frac{(\lambda+\gamma)}{\sigma}}\int_0^1(1-u)^{\beta+\alpha(nj+k)-1}u^{\frac{(\lambda+\gamma)}{\sigma}-1}du\\&=\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\Gamma(\gamma+n)w^{nj+k}}{\Gamma(\gamma)\Gamma(\delta+n)\Gamma\left(\beta+\alpha(nj+k)\right)}x^{\beta+\alpha(nj+k)+\frac{(\lambda+\gamma)}{\sigma}}\frac{\Gamma\left(\beta+\alpha(nj+k)\right)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma\left(\beta+\alpha(nj+k)+(\lambda+\gamma)/\sigma\right)}\\&=\frac{w^k x^{\beta+\alpha k+\frac{(\lambda+\gamma)}{\sigma}}\Gamma(\delta)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\delta+n)\Gamma\left(\beta+\alpha(nj+k)+(\lambda+\gamma)/\sigma\right)}w^{nj}x^{\alpha nj}\\&=\frac{w^k x^{\beta+\alpha k+\frac{(\lambda+\gamma)}{\sigma}}\Gamma(\delta)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(\gamma+n)}{\Gamma(\delta+n)\Gamma\left(\beta+\alpha k+(\lambda+\gamma)/\sigma+\alpha nj\right)}\frac{(w^j x^{\alpha j})^n}{n!}\\&=\frac{w^k x^{\beta+\alpha k+\frac{(\lambda+\gamma)}{\sigma}}\Gamma(\delta)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma(\gamma)}{}_2\Psi_2\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),(\beta+\alpha k+(\lambda+\gamma)/\sigma,\alpha j)\end{array}\middle|(wx^{\alpha})^j\right].\end{aligned}$$

This is the desired result. □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 2.2 *If* $\alpha,\beta,\gamma,\delta,\lambda,\sigma,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\lambda)>0$ *and* $\Re(\sigma)>0$, *then*

$$\left(\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;0+}t^{\frac{(\lambda+\gamma)}{\sigma}-1}\right)(x)=\frac{x^{\beta+\frac{(\lambda+\gamma)}{\sigma}}\Gamma(\delta)\Gamma\left((\lambda+\gamma)/\sigma\right)}{\Gamma(\gamma)}{}_2\Psi_2\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),(\beta+(\lambda+\gamma)/\sigma,\alpha)\end{array}\middle|wx^{\alpha}\right].\qquad(2.7)$$

Theorem 2.3 *Suppose* $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ *and* $j\geq 1$, $k\geq 0$, *then the operator* $\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}$ *is bounded on* $L(a,b)$ *and*

$$\left\|\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right\|_1\leq A\left\|f\right\|_1,\qquad(2.8)$$

where

$$A=(b-a)^{\Re(\beta)}\sum_{n=0}^{\infty}\frac{|(\gamma)_n|}{[\Re(\beta)+\Re(\alpha)(nj+k)]\left|\Gamma(\beta+\alpha(nj+k))\right|}\frac{\left|w(b-a)^{\Re(\alpha)}\right|^{nj+k}}{|(\delta)_n|}.\qquad(2.9)$$

Proof: From equations (1.10) and (2.1), we have

$$\left|\left|\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right|\right|_1 = \int_a^b \left|\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right)(x)\right| dx$$
$$= \int_a^b \left|\int_a^x (x-t)^{\beta-1} E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(w(x-t)^{\alpha}\right) f(t)dt\right| dx.$$

By interchanging the order of integration and applying Dirichlet's formula (1.17), we obtain

$$\left|\left|\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right|\right|_1 \leq \sum_{n=0}^{\infty} \frac{|(\gamma)_n|\,|w|^{nj+k}}{|(\delta)_n|\,|\Gamma(\beta+\alpha(nj+k))|} \int_a^b \left[\int_t^b (x-t)^{\Re(\beta)+\Re(\alpha)(nj+k)-1}\,dx\right] |f(t)|\,dt.$$

By putting $u=(x-t)$, we obtain

$$\left|\left|\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right|\right|_1 \leq \sum_{n=0}^{\infty} \frac{|(\gamma)_n|\,|w|^{nj+k}}{|(\delta)_n|\,|\Gamma(\beta+\alpha(nj+k))|} \int_a^b \left[\int_0^{b-t} u^{\Re(\beta)+\Re(\alpha)(nj+k)-1} du\right] |f(t)|\,dt$$
$$\leq \sum_{n=0}^{\infty} \frac{|(\gamma)_n|\,|w|^{nj+k}}{|(\delta)_n|\,|\Gamma(\beta+\alpha(nj+k))|} \int_a^b \left[\frac{u^{\Re(\beta)+\Re(\alpha)(nj+k)}}{\Re(\beta)+\Re(\alpha)(nj+k)}\right]_0^{b-a} |f(t)|\,dt$$
$$= \left\{(b-a)^{\Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_n|}{[\Re(\beta)+\Re(\alpha)(nj+k)]\,|\Gamma(\beta+\alpha(nj+k))|} \frac{\left|w(b-a)^{\Re(\alpha)}\right|^{nj+k}}{|(\delta)_n|}\right\}$$
$$\times \int_a^b |f(t)|\,dt$$
$$= A\,||f||_1\,,$$

this completes the proof. □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 2.3 *If $\alpha,\beta,\gamma,\delta,w \in \mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\Re(\delta)>0$, then the operator $\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;a+}$ is bounded on $L(a,b)$ and*

$$\left|\left|\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;a+}f\right|\right|_1 \leq A\,||f||_1\,, \tag{2.10}$$

where

$$\mathcal{A} = (b-a)^{\Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_n|}{[\Re(\beta)+\Re(\alpha)n]\,|\Gamma(\beta+\alpha n)|} \frac{\left|w(b-a)^{\Re(\alpha)}\right|^{n}}{|(\delta)_n|}. \tag{2.11}$$

Theorem 2.4 *Suppose $\alpha,\beta,\gamma,\delta,\nu,w \in \mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$ and $j\geq 1$, $k\geq 0$, $f(x)\in L(a,b)$, then the following result holds true:*

$$\left(I^{\nu}_{a+}\left[\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right]\right)(x) = \left(\mathbf{E}^{j,k;w}_{\alpha,\beta+\nu,\gamma,\delta;a+}f\right)(x). \tag{2.12}$$

Proof: Using equations (1.11) and (2.1), we have

$$\left(I^{\nu}_{a+}\left[\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right]\right)(x) = \frac{1}{\Gamma(\nu)}\int_a^x \int_a^u (x-u)^{\nu-1}(u-t)^{\beta-1} E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(w(u-t)^{\alpha}\right) f(t)dtdu.$$

Applying the Dirichlet formula (1.17), we obtain

$$\left(I^{\nu}_{a+}\left[\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;a+}f\right]\right)(x) = \int_a^x \left[\frac{1}{\Gamma(\nu)}\int_t^x (x-u)^{\nu-1}(u-t)^{\beta-1} E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(w(u-t)^{\alpha}\right) du\right] f(t)dt.$$

Setting $u-t=\eta$, we get

$$\left(I_{a+}^{\nu}\left[\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w}f\right]\right)(x)=\int_a^x\left[\frac{1}{\Gamma(\nu)}\int_0^{x-t}(x-t-\eta)^{\nu-1}(\eta)^{\beta-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(\eta)^{\alpha}\right)d\eta\right]f(t)dt.$$

Using equation (1.11) again, we obtain

$$\left(I_{a+}^{\nu}\left[\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w}f\right]\right)(x)=\int_a^x\left(I_{a+}^{\nu}\left[\eta^{\beta-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w\eta^{\alpha}\right)\right]\right)(x-t)\,f(t)dt.$$

Applying equation (1.12), we have

$$\left(I_{a+}^{\nu}\left[\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w}f\right]\right)(x)=\int_a^x(x-t)^{\beta+\nu-1}E_{\alpha,\beta+\nu,\gamma,\delta}^{j,k}\left(w(x-t)^{\alpha}\right)f(t)dt.$$

From (2.1), we obtain

$$\left(I_{a+}^{\nu}\left[\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w}f\right]\right)(x)=\left(\mathbf{E}_{\alpha,\beta+\nu,\gamma,\delta;a+}^{j,k;w}f\right)(x).$$

This is the proof of (2.12). □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 2.4 *If $\alpha,\beta,\gamma,\delta,\nu,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$ and $f(x)\in L(a,b)$, then*

$$\left(I_{a+}^{\nu}\left[\mathbf{E}_{\alpha,\beta;a+}^{\gamma,\delta;w}f\right]\right)(x)=\left(\mathbf{E}_{\alpha,\beta+\nu;a+}^{\gamma,\delta;w}f\right)(x). \tag{2.13}$$

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3. Integral transform of the operator $\mathbf{E}_{\alpha,\beta,\gamma,\delta;a+}^{j,k;w}$

Theorem 3.1 *(Mellin transform) Suppose $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j\geq 1$, $k\geq 0$, then the following result holds true:*

$$\begin{aligned}\mathcal{M}\left[\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w}f\right)(x);s\right]=&\frac{w^k\Gamma(\delta)}{\Gamma(1-s)\Gamma(\gamma)}H_{2,3}^{1,2}\left[-(wt^{\alpha})^j\,\middle|\,\begin{matrix}(0,1),(1-\gamma,1)\\(0,1),(1-\delta,1),(1-s-\beta-\alpha k,\alpha j)\end{matrix}\right]\\&\times\mathcal{M}\left[t^{\beta+\alpha k}f(t);s\right].\end{aligned}\tag{3.1}$$

Proof: Using the definition of the Mellin transform, we have

$$\mathcal{M}\left[\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w}f\right)(x);s\right]=\int_0^{\infty}x^{s-1}\int_0^x(x-t)^{\beta-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(x-t)^{\alpha}\right)f(t)dtdx.$$

By interchanging the order of integrations, we obtain

$$\mathcal{M}\left[\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w}f\right)(x);s\right]=\int_0^{\infty}f(t)\int_t^{\infty}x^{s-1}(x-t)^{\beta-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(w(x-t)^{\alpha}\right)dxdt.$$

By putting $(x-t)=u$, we get

$$\mathcal{M}\left[\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w}f\right)(x);s\right]=\int_0^{\infty}f(t)\int_t^{\infty}(u+t)^{s-1}u^{\beta-1}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(wu^{\alpha}\right)dudt.$$

By applying (1.7), we have

$$\begin{aligned}\mathcal{M}\left[\left(\mathbf{E}_{\alpha,\beta,\gamma,\delta;0+}^{j,k;w}f\right)(x);s\right]=&\int_0^{\infty}\frac{w^k\Gamma(\delta)}{2\pi i\ \Gamma(\gamma)}f(t)\int_{-i\infty}^{+i\infty}\frac{\Gamma(r)\Gamma(1-r)\Gamma(\gamma-r)}{\Gamma(\delta-r)\Gamma(\beta+\alpha k-\alpha jr)}(-w^j)^{-r}\\&\times\int_0^{\infty}(u+t)^{s-1}u^{\beta+\alpha k-\alpha jr-1}dudrdt.\end{aligned}$$

Now, by using the following formula [9]:

$$\int_0^\infty x^{\nu-1}(x+\lambda)^{-\mu}\,dx = \frac{\Gamma(\nu)\Gamma(\mu-\nu)}{\Gamma(\mu)}\lambda^{\nu-\mu},$$

we obtain

$$\begin{aligned}\mathcal{M}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}f\right)(x);s\right] &= \frac{w^k\Gamma(\delta)}{2\pi i\,\Gamma(1-s)\Gamma(\gamma)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(r)\Gamma(1-r)\Gamma(\gamma-r)\Gamma(1-s-\beta-\alpha k+\alpha jr)}{\Gamma(\delta-r)}\\ &\times(-w^jt^{\alpha j})^{-r}dr\int_0^\infty t^{s+\beta+\alpha k-1}f(t)dt\\ &= \frac{w^k\Gamma(\delta)}{\Gamma(1-s)\Gamma(\gamma)}H^{1,2}_{2,3}\left[-(wt^\alpha)^j\left|\begin{array}{c}(0,1),(1-\gamma,1)\\(0,1),(1-\delta,1),(1-s-\beta-\alpha k,\alpha j)\end{array}\right.\right]\\ &\times\mathcal{M}\left[t^{\beta+\alpha k}f(t);s\right],\end{aligned}$$

this is the required proof of (3.1). □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 3.1 *If $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $\Re(\delta)>0$, then*

$$\begin{aligned}\mathcal{M}\left[\left(\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;0+}f\right)(x);s\right] &= \frac{\Gamma(\delta)}{\Gamma(1-s)\Gamma(\gamma)}H^{1,2}_{2,3}\left[-wt^\alpha\left|\begin{array}{c}(0,1),(1-\gamma,1)\\(0,1),(1-\delta,1),(1-s-\beta,\alpha)\end{array}\right.\right]\\ &\times\mathcal{M}\left[t^\beta f(t);s\right].\end{aligned}\tag{3.2}$$

Set $\delta=1$ in (3.2), then we obtain the well-known result [20].

Theorem 3.2 *(Laplace transform) Suppose $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(p)>0$ and $j\geq 1$, $k\geq 0$, then the following result holds true:*

$$\mathcal{L}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}f\right)(x);p\right] = \frac{w^kp^{-(\beta+\alpha k)}\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_1\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),\end{array}\left|\left(\frac{w}{p^\alpha}\right)^j\right.\right]F(p).\tag{3.3}$$

Proof: By using (1.16) and (2.1), we have

$$\begin{aligned}\mathcal{L}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}f\right)(x);p\right] &= \int_0^\infty e^{-px}\left[\int_0^x(x-t)^{\beta-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(w(x-t)^\alpha\right)f(t)dt\right]dx\\ &= \int_0^\infty e^{-px}\left[\int_0^x(x-t)^{\beta-1}\sum_{n=0}^\infty\frac{(\gamma)_n\left(w(x-t)^\alpha\right)^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}f(t)dt\right]dx.\end{aligned}$$

By changing the order of integration and applying Dirichlet formula, we get

$$\begin{aligned}\mathcal{L}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}f\right)(x);p\right] &= \int_0^\infty e^{-px}\int_t^\infty(x-t)^{\beta-1}\sum_{n=0}^\infty\frac{(\gamma)_n\left(w(x-t)^\alpha\right)^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}dxf(t)dt\\ &= \sum_{n=0}^\infty\frac{(\gamma)_nw^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^\infty\int_t^\infty e^{-px}(x-t)^{\beta+\alpha(nj+k)-1}dxf(t)dt.\end{aligned}$$

Putting $(x-t)=u$, we have

$$\begin{aligned}\mathcal{L}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}f\right)(x);p\right] &= \sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^{\infty} f(t)dt\int_0^{\infty} e^{-p(u+t)}u^{\beta+\alpha(nj+k)-1}du\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^{\infty} e^{-pt}f(t)dt\int_0^{\infty} e^{-pu}u^{\beta+\alpha(nj+k)-1}du\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_n w^{nj+k}}{(\delta)_n p^{\beta+\alpha(nj+k)}}\int_0^{\infty} e^{-pt}f(t)dt\\ &= \frac{w^k p^{-(\beta+\alpha k)}\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\delta+n)}\left(\frac{w^j}{p^{\alpha j}}\right)^n\int_0^{\infty} e^{-pt}f(t)dt\\ &= \frac{w^k p^{-(\beta+\alpha k)}\Gamma(\delta)}{\Gamma(\gamma)}{}_2\Psi_1\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),\end{array}\middle|\left(\frac{w}{p^{\alpha}}\right)^j\right]F(p),\end{aligned}$$

where $F(p)$ is the Laplace transform of $f(t)$. This completes the proof of Theorem 3.2. □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 3.2 *If $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $\Re(p)>0$, then*

$$\mathcal{L}\left[\left(\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;0+}f\right)(x);p\right]=\frac{p^{-\beta}\Gamma(\delta)}{\Gamma(\gamma)}{}_2\Psi_1\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),\end{array}\middle|\frac{w}{p^{\alpha}}\right]F(p). \tag{3.4}$$

Set $\delta=1$ in (3.4), then we obtain the well-known result [20].

4. Fractional differential equations

In this section, we presents certain solutions of some differential equations based upon the Hilfer derivative.

Theorem 4.1 *Suppose $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $0<\nu<1$, $0\le\eta\le1$, $j\ge1$, $k\ge0$, then*

$$\left(D^{\nu,\eta}_{0+}y\right)(x)=\sigma\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}\right)(x)+f(x), \tag{4.1}$$

with the initial condition

$$\left(I^{(1-\eta)(1-\nu)}_{0+}y\right)(0+)=c,$$

has the solution given by

$$y(x)=c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma\left(\nu-\eta+\nu\eta\right)}+\sigma x^{\beta+\nu}E^{j,k}_{\alpha,\beta+\nu+1,\gamma,\delta}\left(wx^{\alpha}\right)+\frac{1}{\Gamma\left(\nu\right)}\int_0^x(x-t)^{\nu-1}f(t)dt, \tag{4.2}$$

where c is an arbitrary constant.

Proof: Applying the Laplace transform to both sides of (4.1) and using (1.15) and (3.3) along with the Laplace convolution theorem yields

$$\begin{aligned}p^{\nu}Y(p)-cp^{\eta(1-\nu)} &= \sigma\mathcal{L}\left[\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}\right);p\right]\mathcal{L}\left(1;p\right)+F(p)\\ &= \frac{w^k\sigma p^{-(\beta+\alpha k+1)}\Gamma(\delta)}{\Gamma(\gamma)}{}_2\Psi_1\left[\begin{array}{c}(1,1),(\gamma,1)\\(\delta,1),\end{array}\middle|\left(\frac{w}{p^{\alpha}}\right)^j\right]+F(p)\\ &= \frac{w^k\sigma p^{-(\beta+\alpha k+1)}\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\delta+n)}\left(\frac{w^j}{p^{\alpha j}}\right)^n+F(P),\end{aligned}$$

which easily gives

$$Y(p) = cp^{\eta(1-\nu)-\nu} + \frac{w^k \sigma p^{-(\beta+\nu+\alpha k+1)}\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\delta+n)} \left(\frac{w^j}{p^{\alpha j}}\right)^n + p^{-\nu}F(P). \tag{4.3}$$

Taking the inverse Laplace transform of both sides of (4.3) yields

$$\begin{aligned} y(x) &= c\mathcal{L}^{-1}\left(p^{\eta(1-\nu)-\nu};x\right) + \sigma \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n} \mathcal{L}^{-1}\left(p^{-(\beta+\nu+\alpha(nj+k)+1)};x\right) + \mathcal{L}^{-1}\left(p^{-\nu}F(P);x\right) \\ &= c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta(1-\nu))} + \sigma x^{\beta+\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_n (wx^{\alpha})^{nj+k}}{(\delta)_n \Gamma(\beta+\nu+1+\alpha(nj+k))} + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt \\ &= c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta+\nu\eta)} + \sigma x^{\beta+\nu} E^{j,k}_{\alpha,\beta+\nu+1,\gamma,\delta}(wx^{\alpha}) + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt. \end{aligned}$$

Theorem 4.1 is proved. □

Let $j=1$ and $k=0$, then we obtain the following result.

Corollary 4.1 *If $\alpha,\beta,\gamma,\delta,w \in \mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $0<\nu<1$, $0\le\eta\le1$, then*

$$\left(D^{\nu,\eta}_{0+}y\right)(x) = \sigma\left(\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;0+}\right)(x) + f(x), \tag{4.4}$$

with the initial condition

$$\left(I^{(1-\eta)(1-\nu)}_{0+}y\right)(0+) = c,$$

has the solution given by

$$y(x) = c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta+\nu\eta)} + \sigma x^{\beta+\nu} E^{\gamma,\delta}_{\alpha,\beta+\nu+1}(wx^{\alpha}) + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \tag{4.5}$$

where c is an arbitrary constant.

Theorem 4.2 *Suppose $\alpha,\beta,\gamma,\delta,w \in \mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $0<\nu<1$, $0\le\eta\le1$, $j\ge1$, $k\ge0$, then*

$$\left(D^{\nu,\eta}_{0+}y\right)(x) = \sigma\left(\mathbf{E}^{j,k;w}_{\alpha,\beta,\gamma,\delta;0+}\right)(x) + x^{\beta} E^{j,k}_{\alpha,\beta+1,\gamma,\delta}(wx^{\alpha}), \tag{4.6}$$

with the initial condition

$$\left(I^{(1-\eta)(1-\nu)}_{0+}y\right)(0+) = c,$$

has the solution given by

$$y(x) = c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta+\nu\eta)} + (\sigma+1)\, x^{\beta+\nu} E^{j,k}_{\alpha,\beta+\nu+1,\gamma,\delta}(wx^{\alpha}), \tag{4.7}$$

where c is an arbitrary constant.

Proof: Substituting $f(t) = t^{\beta} E^{j,k}_{\alpha,\beta+1,\gamma,\delta}(wt^{\alpha})$ in Theorem 4.1, we obtain

$$y(x) = c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta+\nu\eta)} + \sigma x^{\beta+\nu} E^{j,k}_{\alpha,\beta+\nu+1,\gamma,\delta}(wx^{\alpha}) + \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\beta} E^{j,k}_{\alpha,\beta+1,\gamma,\delta}(wt^{\alpha})\, dt. \tag{4.8}$$

Here,

$$\int_0^x (x-t)^{\nu-1} t^{\beta} E^{j,k}_{\alpha,\beta+1,\gamma,\delta}(wt^{\alpha})\,dt = \int_0^x (x-t)^{\nu-1} t^{\beta} \sum_{n=0}^{\infty} \frac{(\gamma)_n (wt^{\alpha})^{nj+k}}{(\delta)_n \Gamma(\beta+\alpha(nj+k)+1)} dt$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k}}{(\delta)_n \Gamma(\beta+\alpha(nj+k)+1)} \int_0^x (x-t)^{\nu-1} t^{\beta+\alpha(nj+k)} dt.$$

Let $t = xu$, then we have

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^{nj+k} x^{\beta+\nu+\alpha(nj+k)}}{(\delta)_n \Gamma(\beta+\alpha(nj+k)+1)} \int_0^1 (1-u)^{\nu-1} u^{\beta+\alpha(nj+k)} du$$
$$= x^{\beta+\nu}\Gamma(\nu) \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\beta+\nu+\alpha(nj+k)+1)} (wx^{\alpha})^{nj+k}$$
$$= x^{\beta+\nu}\Gamma(\nu) E^{j,k}_{\alpha,\beta+\nu+1,\gamma,\delta}(wx^{\alpha}).$$

Applying this result in (4.8) yields (4.7), completing the proof of Theorem 4.2. □

Let $j = 1$ and $k = 0$, then we obtain the following result.

Corollary 4.2 *If* $\alpha, \beta, \gamma, \delta, w \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ *and* $0 < \nu < 1$, $0 \le \eta$, *then*

$$\left(D^{\nu,\eta}_{0+} y\right)(x) = \sigma \left(\mathbf{E}^{\gamma,\delta;w}_{\alpha,\beta;0+}\right)(x) + x^{\beta} E^{\gamma,\delta}_{\alpha,\beta+1}(wx^{\alpha}), \tag{4.9}$$

with the initial condition

$$\left(I^{(1-\eta)(1-\nu)}_{0+} y\right)(0+) = c,$$

has the solution given by

$$y(x) = c\frac{x^{\nu-\eta(1-\nu)-1}}{\Gamma(\nu-\eta+\nu\eta)} + (\sigma+1)\, x^{\beta+\nu} E^{\gamma,\delta}_{\alpha,\beta+\nu+1}(wx^{\alpha}), \tag{4.10}$$

where c *is an arbitrary constant.*

5. Conclusion

In this paper, we define and study a new fractional integral operator, which contain the generalized arbitrary order Mittag-Leffler-type function as its kernel. This operator can be updated for set-valued mappings. By adopting a similar technique, some more operators can be introduced.

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