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Relative controllability of the nonlinear fractional dynamical systems with multiple delays in control*

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ABSTRACT: Relative controllability of linear and nonlinear fractional systems with time-variable delays in control variables for finite-dimensional spaces is considered. Sufficient and necessary circumstances for the controllability of a linear fractional system are offered. Employing Schauder's fixed point theorem, sufficient circumstances for the controllability of nonlinear fractional systems are presented.

Key Words: Fractional differential equations, relative controllability, control delay, Mittag-Leffler matrix function, Schauder's fixed-point theorem.

Contents

1. Introduction

Although factional calculus seeing as a generalisation of integer calculus has been studied by the mathematicians for the last three centuries, it has been drawn to many applied areas of science. It was discovered that interdisciplinary applications can be more sensitively formulated via the fractional derivatives. Fractional integrals and derivatives offer more accurate formulation of systems. The reason of this may be the numerical value of the fraction parameter provides a closer characterisation of uncertainties in systems. Several researchers have examined applications of fractional calculus in nonlinear oscillation of earthquakes [14], signal processing [15], interfaces between substrate and nanoparticles [16], bioengineering [17], circuit theory [18], behaviour of visco-elastic materials [20,21], statistic mechanics [22]. It is shown in [23] that fractional differential equations are eminent tools in order to formulate several physical problems.

Prabhakar fractional calculus have shown up in the literature in the last ten years. The fractional integral operator firstly defining in [3] gives rise to the Prabhakar fractional calculus. The integral operator is elegantly studied and interrogated in [2] and extended to the notion of fractional derivatives in [4]. It is used to pure and applied mathematics [1,5] and several applications [6,7]. The Prabhakar fractional derivatives contain within distinct sorts of fractional operators like Riemann-Liouville, the Lorenzo-Hartly, Gorenflo-Minerdi, the Miller-Ros, Caputo fractional operators, etc.

It is easily realized that an interest in the field of control theory has been increased. To put it simply, controllability means that it enables to steer a system to any final state from any initial state by employing

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(admissible) control functions. Modelling of dynamical equations mostly involve delays in control variables or in state variables. Controllability outcomes for linear fractional systems have been investigated by many researchers [13,24,26,27,28]. Balachandran and Dauer [29] debated the controllability of nonlinear dynamical systems by means of fixed point approach. Several monographs have expressed controllability of integer order nonlinear system with several kinds of delays in control variables, e.g., point constant delays [19,25,30], point time-variables delays, distributed delays. The researchers in [31,32] investigated controllability of nonlinear dynamical equations with time-varying multiple delays in control variables. Klamka in [33,34] examined controllability of nonlinear dynamical system with distinct kinds of delays in control variables. Balachandran et al., in [12] considered linear Caputo fractional system multiple delays in control and shown it controllable. It should be stressed that the theory of controllability for nonlinear fractional systems is not yet satisfactory and it is even now under development. Inspired by this reality and the cited-above works, we consider the below nonlinear Prabhakar fractional differential dynamical systems of Caputo-type with time-varying multiple delays in control variables:

$$\begin{cases} {}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho(\varsigma) = Z\rho(\varsigma) + \sum_{i=0}^{M} H_i u(r_i(\varsigma)) + \daleth(\varsigma,\rho(\varsigma),u(\varsigma)), \quad \varsigma \in (0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{1.1}$$

where ${}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}$ symbolizes the Prabhakar fractional derivative of Caputo-type of orders $0<\alpha\le 1$, $\rho\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $Z\in\mathbb{R}^{n\times n}$, and $H_i\in\mathbb{R}^{n\times m}$ for each $i=0,1,2,\ldots,M$.

We will build our theoretical results on the below assumptions.

$\mathbf{A_1}$: Let $r_j:[0,T]\to\mathbb{R}$, $j=0,1,\ldots,M$ be so twice continuously differentiable and strictly increasing that $r_j(\varsigma)\le\varsigma$, $\varsigma\in[0,T]$, $j=0,1,\ldots,M$.

$\mathbf{A_2}$: Let $r_0(\varsigma)=\varsigma$ and the below inequalities be hold for $\varsigma=T$

$$r_M(T)\le r_{M-1}(T)\le\ldots\le r_{m+1}(T)=0<r_m(T)=\ldots=r_0(T)=T.$$

$\mathbf{A_3}$: Let the time-lead function $h_j:[r_j(0),r_j(T)]\to[0,T]$, $j=0,1,\ldots,M$ be defined by $h_j(r_j(\varsigma))=\varsigma$, $\varsigma\in[0,T]$, $j=0,1,\ldots,M$.

$\mathbf{A_4}$: Let the function u_ς, $\varsigma\in[0,T]$ be defined by $u_\varsigma(s)=u(\varsigma+s)$, $s\in[-r,0)$ with functions $u:[-r,T]\to\mathbb{R}^m$ for given $r>0$.

2. Preliminaries

In this section, fundamental concepts on which we will build our theoretical findings are offered.

$\mathbb{N}$ and $\mathbb{C}$ stand for the natural numbers and the complex numbers, respectfully. $\mathbb{R}^m$ is a Euclidean space for $m\in\mathbb{N}$. For $a<b$, $AC^m(a,b)$ consists of such a real-valued function ρ that $\rho^{(n-1)}$ exists on (a,b) in addition to being absolutely continuous.

Definition 2.1 *[1,2,3] For $\alpha,\eta,w,\delta\in\mathbb{C}$, $Re(\eta)>0$ and $Re(\alpha)>0$, the Prabhakar fractional integral is defined as noted below*

$$\left({}_0\mathfrak{I}^{\nu,\delta}_{\eta,\alpha}\rho\right)(\varsigma) = \int_0^\varsigma (\varsigma-s)^{\alpha-1} e^{\delta}_{\eta,\alpha}\left(\nu(\varsigma-s)^\eta\right)\rho(s)\,ds,$$

where the reputed three-parameter Mittag-Leffler function is given by

$$e^{\delta}_{\eta,\alpha}(\varsigma) = \sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(k\eta+\alpha)}\frac{\varsigma^k}{k!},\ \ Re(\eta)>0,$$

here, $\Gamma(.)$ is the well-known gamma function and $(\delta)_i$ is the Pochhammer notation, that is, $(\delta)_k=\frac{\Gamma(\delta+k)}{\Gamma(\delta)}$ or

$$(\delta)_0=1,\quad (\delta)_k=\delta(\delta-1)\ldots(\delta-k+1),\quad k=0,1,2,\ldots.$$

Definition 2.2 *[4] The Prabhakar factional derivatives of Riemann-Liouville-type of order $0<\alpha<1$ is defined by*

$$\left({}^{pr}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho\right)(\varsigma)=\frac{d}{d\varsigma}\left(\mathcal{I}^{\nu,-\delta}_{\eta,1-\alpha}\rho\right)(\varsigma)=\frac{d}{d\varsigma}\int_0^{\varsigma}(\varsigma-s)^{-\alpha}e^{-\delta}_{\eta,1-\alpha}\left(\nu(\varsigma-s)^{\eta}\right)\rho(s)\,ds,$$

and the Prabhakar factional derivatives of Caputo-type of order $0<\alpha<1$ is defined by

$$\left({}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho\right)(\varsigma)=\mathcal{I}^{\nu,-\delta}_{\eta,1-\alpha}\left(\frac{d}{d\varsigma}\rho\right)(\varsigma)=\int_0^{\varsigma}(\varsigma-s)^{-\alpha}e^{-\delta}_{\eta,1-\alpha}\left(\nu(\varsigma-s)^{\eta}\right)\frac{d}{ds}\rho(s)\,ds,$$

where $\rho\in AC^1(0,T)$, $\eta,\alpha,\nu,\delta\in\mathbb{C}$, $Re(\alpha)\geq 0$, and $Re(\eta)>0$.

Remark 2.1 *The Prabhakar derivative of Caputo type ${}^{PC}\mathfrak{D}^{\nu,\delta}_{\eta,\alpha}$ for $\delta=0$ corresponds to Caputo fractional derivative of order α.*

Lemma 2.1 *[4] There is a tie between Riemann-Liouville and Caputo types derivatives in the Prabhakar sense for $0<\alpha<1$,*

$$\left({}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho\right)(\varsigma)={}^{pr}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\left(\rho(\varsigma)-\rho(0)\right).$$

Lemma 2.2 *[9] For $\nu,\eta,\alpha>0$, the laplace transform of the three-parameter Mittag-Leffler function $e^{\delta}_{\eta,\alpha}(\nu\varsigma^{\eta})$ is*

$$\mathfrak{L}\left\{\varsigma^{\alpha-1}e^{\delta}_{\eta,\alpha}(\nu\varsigma^{\eta})\right\}(s)=s^{-\alpha}\left(1-\nu s^{-\eta}\right)^{-\delta},\ Re(s)>\|\nu\|^{\frac{1}{\alpha}},$$

where $\mathfrak{L}$ stands for the well-known laplace transform.

Lemma 2.3 *[9] The laplace transform of Prabhakar Caputo-type fractional derivative of order $0<\alpha<1$ is given by*

$$\mathfrak{L}\left\{{}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho(\varsigma)\right\}(s)=s^{\alpha}\left(1-\nu s^{-\eta}\right)^{\delta}\mathfrak{L}\left\{\rho(\varsigma)\right\}(s)-s^{\alpha-1}\left(1-\nu s^{-\eta}\right)^{\delta}\rho(0).$$

Lemma 2.4 *[8] The laplace integral transform of the convolution of ρ and $\urcorner$ on $[0,\infty)$ is given by*

$$\mathfrak{L}\left\{(\rho*\urcorner)(\varsigma)\right\}(s)=\mathfrak{L}\left\{\rho(\varsigma)\right\}(s)\,\mathfrak{L}\left\{\urcorner(\varsigma)\right\}(s),\quad s\in\mathbb{C},$$

provided that the laplace transforms of the functions ρ and $\urcorner$ are available.

Definition 2.3 *[10] A (control) function $u(t)\in\mathbb{R}^m$ is admissible if it is bounded and measurable on every finite time interval.*

Lemma 2.5 *[11, Proposition 1] If the function $\urcorner$ is locally bounded in $\mathbb{R}^n\times\mathbb{R}^m$ and satisfies*

$$\lim_{|(v,u)|\to\infty}\frac{|\urcorner(t,v,u)|}{|(v,u)|}=0,$$

uniformly in $[0,T]$ then, for each pair of constants a and b, there exists a constat r such that if $\|(v,u)\|\leq r$, then

$$a\,|\urcorner(t,v,u)|+b\leq r \text{ for all } t\in[0,T].$$

3. A solution to a linear system

In this section, we look for an explicit solution to the just-below given linear Prabhakar Caputo-type fractional differential system.

$$\begin{cases}{}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho(\varsigma)=Z\rho(\varsigma)+\urcorner(\varsigma) & \varsigma\in(0,T],\\ \rho(0)=\rho_0, & \end{cases}\tag{3.1}$$

where ${}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}$ symbolizes the Prabhakar fractional derivative of Caputo-type of orders $0<\alpha\leq 1$, $\rho\in\mathbb{R}^n$, $Z\in\mathbb{R}^{n\times n}$, and $\urcorner:[0,T]\to\mathbb{R}^n$ is continuous.

In order to obtain the desired solution of the system, we will use the reputed laplace transform, although it is an old technique, since it is adapted to the Prabhakar fractional derivatives of Caputo type. Obtaining this solution with such an old technique does not diminish anything from the novelty of the solution and the paper.

We start with applying the laplace transform to the both sides of the system (3.1) to get the solution

$$s^{\alpha}(1-\nu s^{-\eta})^{\delta}R(s) - s^{\alpha-1}(1-\nu s^{-\eta})^{\delta}\rho_0 = ZR(s) + D(s),$$

where $R(s) = \mathfrak{L}\{\rho(\varsigma)\}(s)$ and $D(s) = \mathfrak{L}\{\urcorner(\varsigma)\}(s)$. If it is rearranged and $R(s)$ is left alone, one can get

$$\begin{aligned} R(s) &= s^{-\alpha}(1-\nu s^{-\eta})^{-\delta}(I - s^{-\alpha}(1-\nu s^{-\eta})^{-\delta}Z)^{-1}D(s) \\ &\quad + s^{-1}(I - s^{-\alpha}(1-\nu s^{-\eta})^{-\delta}Z)^{-1}\rho_0, \end{aligned}$$

where I is the identity matrix. If $\|s^{-\alpha}\left(1-\nu s^{-\eta}\right)^{-\delta} Z\| < 1$, by the well-known Neumann series one can obtain

$$R(s) = \sum_{k=0}^{\infty} s^{-k\alpha-\alpha}(1-\nu s^{-\eta})^{-k\delta-\delta}Z^k D(s) + \sum_{k=0}^{\infty} s^{-k\alpha-1}(1-\nu s^{-\eta})^{-k\delta}Z^k\rho_0.$$

Applying the inverse laplace transform and using the definition of the reputed convolution of two functions, one can get the desired solution as follows

$$\begin{aligned} \rho(\varsigma) &= \sum_{k=0}^{\infty} \int_0^{\varsigma} (\varsigma - s)^{(k+1)\alpha-1} e^{(k+1)\delta}_{\eta,(k+1)\alpha}\left(\nu\left(\varsigma - s\right)^{\eta}\right) Z^k \urcorner(s)ds \\ &\quad + \sum_{k=0}^{\infty} \varsigma^{k\alpha} e^{k\delta}_{\eta,k\alpha+1}\left(\nu\varsigma^{\eta}\right) Z^k\rho_0. \end{aligned}$$

4. Controllability of a linear system with multiple delays in control

In this section, we will offer the necessary and sufficient circumstances to show the just-below given system relatively controllable.

The linear fractional dynamical system with multiple delays in control is given by

$$\begin{cases} {}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}\rho\left(\varsigma\right) = Z\rho\left(\varsigma\right) + \sum_{i=0}^{M} H_i u\left(r_i(\varsigma)\right), \quad \varsigma \in (0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{4.1}$$

where ${}^{pc}\mathcal{T}^{\nu,\delta}_{\eta,\alpha}$ symbolizes the Prabhakar fractional derivative of Caputo-type of orders $0 < \alpha \leq 1$, $\rho \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n\times n}$, and $H_i \in \mathbb{R}^{n\times m}$ for each $i = 0, 1, 2, \ldots, M$.

Definition 4.1 *The system is relatively controllable if, for an arbitrary initial control $u_0(\varsigma)$, $\varsigma \in [-r, 0]$, and the final state $\rho_T \in \mathbb{R}^n$ with time T, then there exists an admissible control function $u(\varsigma)$, $\varsigma \in [0, T]$ such that the corresponding solution $\rho(\varsigma)$, $\varsigma \in [0, T]$ to the system satisfies $\rho(T) = \rho_T$.*

Based on the section 3, an explicit solution to the system 4.1 can be given as follows

$$\begin{aligned} \rho(\varsigma) &= \sum_{i=0}^{M}\sum_{k=0}^{\infty} \int_0^{\varsigma} (\varsigma - s)^{(k+1)\alpha-1} e^{(k+1)\delta}_{\eta,(k+1)\alpha}\left(\nu\left(\varsigma - s\right)^{\eta}\right) Z^k H_i u\left(r_i(s)\right)ds \\ &\quad + \sum_{k=0}^{\infty} \varsigma^{k\alpha} e^{k\delta}_{\eta,k\alpha+1}\left(\nu\varsigma^{\eta}\right) Z^k\rho_0. \end{aligned}$$

For simplicity, we set

$$\Omega^Z_1(\varsigma) = \sum_{k=0}^{\infty} \varsigma^{k\alpha} e^{k\delta}_{\eta,k\alpha+1}\left(\nu\varsigma^{\eta}\right) Z^k, \ \Omega^Z_2(\varsigma) = \sum_{k=0}^{\infty} \varsigma^{(k+1)\alpha-1} e^{(k+1)\delta}_{\eta,(k+1)\alpha}\left(\nu\varsigma^{\eta}\right) Z^k.$$

One can rewrite a representation of the solution as follows

$$\rho(\varsigma) = \Omega_1^Z(\varsigma)\rho_0 + \sum_{i=0}^{M} \int_0^{\varsigma} \Omega_2^Z(\varsigma - s) H_i u\left(r_i(s)\right) ds.$$

Now, we remove the delay parameters from the control function in the solution by applying the transformation $x = r_i(s)$. Then, the solution is written by

$$\rho(\varsigma) = \Omega_1^Z(\varsigma)\rho_0 + \sum_{i=0}^{M} \int_{r_i(0)}^{r_i(\varsigma)} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u\left(s\right) ds.$$

Based on $\mathbf{A_2}$, we may separate the summation symbol as follows

$$\begin{aligned}\rho(\varsigma) = \Omega_1^Z(\varsigma)\rho_0 &+ \sum_{i=m+1}^{M} \int_{r_i(0)}^{r_i(\varsigma)} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u_0\left(s\right) ds \\ &+ \sum_{i=0}^{m} \int_{r_i(0)}^{0} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u_0\left(s\right) ds \\ &+ \sum_{i=0}^{m} \int_{0}^{\varsigma} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u\left(s\right) ds.\end{aligned}$$

Let us introduce the following formulas

$$\begin{aligned}J(\varsigma) := \Omega_1^Z(\varsigma)\rho_0 &+ \sum_{i=m+1}^{M} \int_{r_i(0)}^{r_i(\varsigma)} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u_0\left(s\right) ds \\ &+ \sum_{i=0}^{m} \int_{r_i(0)}^{0} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s) u_0\left(s\right) ds,\end{aligned}$$

and

$$\Omega_3^{H,Z}(\varsigma, s) := \sum_{i=0}^{m} \Omega_2^Z(\varsigma - r_i(s)) H_i r_i^{'}(s). \tag{4.2}$$

We can describe the Gram matrix as follows

$$W[0,T] := \int_0^T \Omega_3^{H,Z}(T,s)\Omega_3^{H^*,Z^*}(T,s) ds,$$

where $(\Omega_3^{H,Z}(\varsigma, s))^* = \Omega_3^{H^*,Z^*}(\varsigma, s)$, the symbol $.^*$ stands for the transpose of a matrix.

Theorem 4.1 *Suppose that $0.5 < \alpha < 1$. The linear fractional system described in (4.1) is relatively controllable if and only if the Gram matrix $W[0,T]$ is nonsingular.*

Proof: Due to the non-singularity of the Gram matrix $W := W[0,T]$, it is of the inverse W^{-1}. Let ρ_ς be the craved final state at time T. Then we allege that the system (4.1) is relatively controllable in terms of the following control function

$$u(\varsigma) = \Omega_3^{H^*,Z^*}(T,\varsigma) W^{-1}[\rho_\varsigma - J(T)].$$

It is easy to confirm that as follows

$$\begin{aligned}\rho(T) &= J(T) + \int_0^T \Omega_3^{H,Z}(T,s) u(s) ds \\ &= J(T) + \int_0^T \Omega_3^{H,Z}(T,s)\Omega_3^{H^*,Z^*}(T,s) W^{-1}[\rho_\varsigma - J(T)] ds \\ &= J(T) + WW^{-1}[\rho_\varsigma - J(T)] \\ &= \rho_\varsigma.\end{aligned}$$

To prove sufficiency, we use the method of the reductio ad absurdum. Suppose that the system (4.1) is relatively controllable, but, W is singular. Due to the singularity of the Gram matrix W, there is at least one nonzero vector $\upsilon \in \mathbb{R}^n$ such that $W\upsilon = 0$. Then, one can easily obtain the following equality

$$\upsilon^* W \upsilon = 0 = \int_0^T \upsilon^* \Omega_3^{H,Z}(T,s)\Omega_3^{H^*,Z^*}(T,s)\upsilon ds,$$

which provides that $\upsilon^*\Omega_3^{H,Z}(T,s) = 0$ for $0 \leq s \leq T$. Let 0 and υ be the final states for the time T. Based on the relative controllability of the system (4.1), there exist two distinct control functions u_1 and u_2 such that

$$\rho(T) = J(T) + \int_0^T \Omega_3^{H,Z}(T,s)u_1(s)ds = 0, \;\; \rho(T) = J(T) + \int_0^T \Omega_3^{H,Z}(T,s)u_2(s)ds = \upsilon.$$

One can acquire the following equation

$$\upsilon = \int_0^T \Omega_3^{H,Z}(T,s)(u_2(s) - u_1(s))ds.$$

The just-above obtained information $\upsilon^*\Omega_3^{H,Z}(T,s) = 0$ for $0 \leq s \leq T$ provides $\|\upsilon\|^2 = \upsilon^*\upsilon = 0$. So, $\upsilon = 0$ contradicts with υ being a nonzero. So, the Gram matrix W is nonsingular. The proof is completed. $\square$

Before investigating the relative controllability of the whole system (1.1), we need to make estimations for the functions $\Omega_1^Z(\varsigma)$, $\Omega_2^Z(\varsigma)$ and $\Omega_3^{H,Z}(\varsigma,s)$.

We know from [3] that the three-parameter Mittag-Leffler function is bounded in a closed interval, that is

$$\|e_{\eta,\alpha}^{\delta}(\nu\varsigma^{\eta})\| \leq \mathcal{H}, \quad \varsigma \in [a,b],$$

where $\mathcal{H}, a, b \in \mathbb{R}$. Based on this information, for the fixed $k \in \mathbb{N}$, we have

$$\|e_{\eta,k\alpha+1}^{k\delta}(\nu\varsigma^{\eta})\| \leq \mathcal{H}_1, \quad \|e_{\eta,(k+1)\alpha}^{(k+1)\delta}(\nu\varsigma^{\eta})\| \leq \mathcal{H}_2, \quad \varsigma \in [0,T],$$

where $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{R}$. Then, possible estimations for the functions $\Omega_1^Z(\varsigma)$, $\Omega_2^Z(\varsigma)$ and $\Omega_3^{H,Z}(\varsigma,s)$ can be given as follows

$$\|\Omega_1^Z(\varsigma)\| = \|\sum_{k=0}^{\infty} \varsigma^{k\alpha}\mathcal{H}_1 Z^k\| \leq \mathcal{H}_1 \sum_{k=0}^{\infty}(T^{\alpha}\|Z\|)^k = \mathcal{H}_1 e^{(T^{\alpha}\|Z\|)},$$

$$\|\Omega_2^Z(\varsigma)\| = \|\sum_{k=0}^{\infty} \varsigma^{(k+1)\alpha-1}\mathcal{H}_2 Z^k\| \leq \mathcal{H}_2 T^{\alpha-1} \sum_{k=0}^{\infty}(T^{\alpha}\|Z\|)^k = \mathcal{H}_2 T^{\alpha-1} e^{(T^{\alpha}\|Z\|)},$$

and

$$\|\Omega_3^{H,Z}(\varsigma,s)\| = \sum_{i=0}^{m} \mathcal{H}_2 T^{\alpha-1} e^{(T^{\alpha}\|Z\|)}\|H_i\||r_i^{'}(s)| \leq \mathcal{H}_2\mathcal{H}_3 T^{\alpha-1} e^{(T^{\alpha}\|Z\|)},$$

where $\mathcal{H}_3 := \sum_{i=0}^{m}\|H_i\| \max_{s\in[0,T]}|r_i^{'}(s)|$. As a result, one can easily find such upper bounds to show that the matrix functions $\Omega_1^Z(\varsigma)$, $\Omega_2^Z(\varsigma)$ and $\Omega_3^{H,Z}(\varsigma,s)$ are bounded.

5. Controllability of a nonlinear system with multiple delays in control

In this section, we will offer the sufficient circumstances to show the system (1.1) relatively controllable.

With $C([0,T],\mathbb{R}^n)$ being the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}^n$, it is also well-known that $E = C([0,T],\mathbb{R}^n) \times C([0,T],\mathbb{R}^m)$ is the Banach space with the norm $\|(\rho,u)\| = \|\rho\| + \|u\|$, where $\|\rho\| = \sup\{|\rho(\varsigma)| : \varsigma \in [0,T]\}$.

In the similar manner in the previous section, one can easily acquire the following equation as a solution to the system (1.1) for $\varsigma = T$,

$$\begin{aligned}\rho(T) &= \Omega_1^Z(T)\rho_0 + \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds \\ &+ \sum_{i=m+1}^{M}\int_{r_i(0)}^{r_i(T)} \Omega_2^Z(T-r_i(s))H_i r_i^{'}(s)u_0(s)\,ds \\ &+ \sum_{i=0}^{m}\int_{r_i(0)}^{0} \Omega_2^Z(T-r_i(s))H_i r_i^{'}(s)u_0(s)\,ds \\ &+ \sum_{i=0}^{m}\int_{0}^{T} \Omega_2^Z(T-r_i(s))H_i r_i^{'}(s)u(s)\,ds.\end{aligned}$$

Let the pair of $\rho \in \mathbb{R}^n, u \in \mathbb{R}^m$ be a solution pair to the following nonlinear integral equations:

$$\rho(\varsigma) = J(\varsigma) + \int_0^\varsigma \Omega_3^{H,Z}(\varsigma,s)u(s)ds + \int_0^\varsigma \Omega_2^Z(\varsigma-s)\daleth(s,\rho(s),u(s))ds, \tag{5.1}$$

$$u(\varsigma) = \Omega_3^{H^*,Z^*}(T,\varsigma)W^{-1}[\hat{\rho} - \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds], \tag{5.2}$$

where $\hat{\rho} = \rho_T - J(T)$. We claim that $\rho(\varsigma)$ is a solution to the system (1.1) corresponding to the control function u on $[0,T]$. It is quite easy to verify that as follows

$$\begin{aligned}\rho(T) &= J(T) + \int_0^T \Omega_3^{H,Z}(T,s)u(s)ds + \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds \\ &= J(T) + WW^{-1}[\rho_T - J(T) - \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds] \\ &+ \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds \\ &= \rho_T.\end{aligned}$$

Our curiosity then is to identify sufficient circumstances which guarantee the existence of a solution pair to the nonlinear integral equations (5.1) and (5.2). The following theorem satisfies the curiosity.

Theorem 5.1 *Assume that $1 > \alpha > 0.5$ and the continuous function $\daleth$ fulfills $\lim_{|(\rho,u)|\to\infty} \frac{|\daleth(\varsigma,\rho,u)|}{|(\rho,u)|}$ uniformly in $\varsigma \in [0,T]$. Then, the nonlinear fractional dynamical system (1.1) is relatively controllable provided that the linear fractional dynamical system (4.1) is relatively controllable.*

Proof: We will start the proof with defining an operator $\gimel : E \to E$ by $\gimel(\rho,u) = (z,v)$, where

$$z(\varsigma) = J(\varsigma) + \int_0^\varsigma \Omega_3^{H,Z}(\varsigma,s)v(s)ds + \int_0^\varsigma \Omega_2^Z(\varsigma-s)\daleth(s,\rho(s),u(s))ds,$$

$$v(\varsigma) = \Omega_3^{H^*,Z^*}(T,\varsigma)W^{-1}[\hat{\rho} - \int_0^T \Omega_2^Z(T-s)\daleth(s,\rho(s),u(s))ds].$$

Introduce the following formulations:

$$\begin{aligned}&\lambda = \max\{T\|\Omega_3^{H,Z}(T,0)\|, 1\}, \\ &b_1 = 4\lambda\|\Omega_3^{H,Z}(T,0)\|\|W^{-1}\|\|\hat{\rho}\|, \qquad b_2 = 4\|J(T)\|, \\ &a_1 = 4\lambda T\|\Omega_3^{H,Z}(T,0)\|\|W^{-1}\|\|\Omega_2^Z(T)\|, \quad a_2 = 4T\|\Omega_2^Z(T)\|, \\ &a = \max\{a_1,a_2\}, b = \{b_1,b_2\}.\end{aligned}$$

Based on Lemma 2.5 and the statements of this theorem, there exists such a constant $\varepsilon > 0$ such that if $\|(\rho, u)\| \leq \varepsilon$, then $a|\daleth(\varsigma, \rho, u)| + b \leq \varepsilon$ for all $\varsigma \in [0, T]$. We will demonstrate $\sqsupset(B_\varepsilon) \subseteq B_\varepsilon$, where $B_\varepsilon = \{(\rho, u) \in E : \|(\rho, u)\| \leq \varepsilon\}$.

$$
\begin{aligned}
\|v(\varsigma)\| &\leq \|\Omega_3^{H,Z}(T,0)\|\|W^{-1}\|\|\hat{\rho}\| + T\|\Omega_3^{H,Z}(T,0)\|\|W^{-1}\|\|\Omega_2^{Z}(T)\| \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| \\
&\leq b_1(4\lambda)^{-1} + a_1(4\lambda)^{-1} \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| \\
&\leq (4\lambda)^{-1}(a \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| + b) \\
&\leq (4\lambda)^{-1}\varepsilon \leq \frac{\varepsilon}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
\|z(\varsigma)\| &\leq \|J(T)\| + T\|\Omega_3^{H,Z}(T,0)\|\|v\| + T\|\Omega_2^{Z}(T)\| \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| \\
&\leq \frac{b_2}{4} + \lambda\|v\| + \frac{a_2}{4} \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| \\
&\leq \frac{1}{4}(a \sup_{\varsigma\in[0,T]} |\daleth(\varsigma, \rho, u)| + b) + \lambda\frac{\varepsilon}{4\lambda} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}
$$

Hence, $\|(z, v)\| = \|z\| + \|v\| \leq \frac{3\varepsilon}{4}$, which gives that the desired result,$\sqsupset(B_\varepsilon) \subseteq B_\varepsilon$. In fact, $\sqsupset$ maps the convex closure of $\sqsupset(B_\varepsilon)$ into itself. Because of the boundedness of $\daleth$ on B_ε, $\sqsupset(B_\varepsilon)$ is equicontinuous. According to the Schauder fixed point theorem, $\sqsupset$ is of a fixed point in B_ε.

This fixed point (z, v) of $\sqsupset$ is a solution pair to the nonlinear integral equations (5.1) and (5.2). Since the initial control function u_0 and the final state ρ_T are arbitrary, the nonlinear system (1.1) is relatively controllable. □

Remark 5.1 *All discussed results of the paper have matched up with those of the work [12] when it is taken* $\delta = 0$.

6. Numerical verifications

In this section, we numerically verify our theoretical findings.

Example 6.1 *One can consider the following fractional dynamical systems wit multiple delays in control*

$$
\begin{cases} {}^{pc}\mathcal{T}^{1,1}_{1,0.7}\rho(\varsigma) = Z\rho(\varsigma) + H_0 u(\varsigma) + H_1 u(\varsigma - 2) & \varsigma \in (0, 3], \\ \rho(0) = \rho_0, \end{cases} \tag{6.1}
$$

here, $Z = \begin{pmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{pmatrix}$, $H_0 = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{pmatrix}$, $H_1 = \begin{pmatrix} 0.8 & 0.7 \\ 0.9 & 1 \end{pmatrix}$, $\rho_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. *It is easy to compute* $\Omega_3^{H,Z}(\varsigma, s)$ *from (4.2):*

$$
\begin{aligned}
\Omega_3^{H,Z}(\varsigma, s) := &\sum_{k=0}^{\infty} (t-s)^{0.7k-0.3} e^{k+1}_{1,0.7k+0.7}(t-s) Z^k H_0 \\
&+ \sum_{k=0}^{\infty} (t-(s+2))^{0.7k-0.3} e^{k+1}_{1,0.7k+0.7}(t-(s+2)) Z^k H_1.
\end{aligned}
$$

The corresponding Gramian matrix is as follows:

$$
\begin{aligned}
W[0,3] :&= \int_0^3 \Omega_3^{H,Z}(3,s)\Omega_3^{H^*,Z^*}(3,s) ds \\
&= \begin{pmatrix} 0.205 & 0.339 \\ 0.339 & 0.562 \end{pmatrix},
\end{aligned}
$$

whose determinant is nonzero. So, the corresponding Gramian matrix is nonsingular. Then, Theorem 4.1 guarantees that the system (6.1) is relatively controllable.

Example 6.2 *One can investigate the following fractional dynamical systems with multiple delays in control*

$$\begin{aligned} &{}^{pc}\mathcal{T}^{0.5,0.2}_{0.1,0.9}\rho(\varsigma) = Z\rho(\varsigma) + H_0u(\varsigma) + H_1u(\varsigma-1) + H_2u(\varsigma-1.5) + \urcorner(\varsigma,\rho,u), \varsigma \in (0,2], \\ &\rho(0) = \rho_0, \end{aligned} \tag{6.2}$$

here, $Z = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, $H_0 = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$, $H_1 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, $H_2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\rho_0(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\urcorner(t) = \begin{pmatrix} \frac{1}{1+\rho_1+u_1} \\ \frac{\rho_2}{1+\rho_2^2+u_2^2} \end{pmatrix}$ *where* $\rho(\varsigma) = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$, $u(\varsigma) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. *It is easy to compute* $\Omega_3^{H,Z}(\varsigma, s)$ *from (4.2):*

$$\begin{aligned} \Omega_3^{H,Z}(\varsigma,s) := &\sum_{k=0}^{\infty}(t-s)^{0.9k-0.1}e^{0.2k+0.2}_{0.1,0.9k+0.9}(0.5(t-s)^{0.2})Z^kH_0 \\ &+\sum_{k=0}^{\infty}(t-(s+1))^{0.9k-0.1}e^{0.2k+0.2}_{0.1,0.9k+0.9}(0.5(t-(s+1))^{0.2})Z^kH_1 \\ &+\sum_{k=0}^{\infty}(t-(s+1.5))^{0.9k-0.1}e^{0.2k+0.2}_{0.1,0.9k+0.9}(0.5(t-(s+1.5))^{0.2})Z^kH_2. \end{aligned}$$

The corresponding Gramian matrix is as follows:

$$\begin{aligned} W[0,2] := &\int_0^2 \Omega_3^{H,Z}(2,s)\Omega_3^{H^*,Z^*}(2,s)ds \\ = &\begin{pmatrix} 1394 & 3085 \\ 3085 & 6829 \end{pmatrix}, \end{aligned}$$

whose determinant is 2401. So, the corresponding Gramian matrix is nonsingular. Then Theorem 4.1 guarantees that the system (6.2) without the nonlinear function $\urcorner$ *is relatively controllable. Since the nonlinear function* $\urcorner$ *also satisfies all statements in Theorem 5.1, the nonlinear system (6.2) is relatively controllable.*

7. Conclusion

In brief, by employing Gramian matrix, necessary and sufficient circumstances for controllability of linear fractional system with time-varying multiple delays in control variables are determined and by applying Schauder fixed point theorem, sufficient circumstances for controllability of nonlinear fractional system with time-varying multiple delays in control variables are identified. In a sense, the obtained findings of the paper can be seen as a generalization to several time-variable delays (or constant delays) in control, of the results published so far.

Since the results of the paper are comprehensive, we expect, it will take many citations. There is a rich range of problems on this subject originating from the Prabhakar calculus. For these reasons, we are sure that this study will inspire many researchers in a very short time. As a future work, all acquired findings can be extended to distinct kinds of systems in the Prabhakar's sense such as semilinear system with distributed delays in control [24], integro differential control system [27] [28]. Furthermore, the examined system is expandable by adding a delay parameter in state variable, which such a fractional system with a delay in state and multiple delays in control has not been considered so far as far as we know.

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