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Complementary Degree Equitable Sets in Graphs

L. Muthusubramanian, R. Sundareswaran, V. Swaminathan

ABSTRACT: Let $G = (V, E)$ be a simple graph. The concept of equitability was first introduced by W.Meyer. [3] . A proper colouring is called equitable if the cardinalities of any two colour classes differ by one. Inspired by this concept, E.Sampathkumar formulated degree equitability. Two vertices in a simple graph are said to be degree equitable if the difference of the degrees of the two vertices differ by one. This concept of degree equitability was used to develop degree equitable sets, degree equitable domination and its variations and equitable colouring in [1,2,7,8,9]. A subset S is said to be complementary equitable if for any $u, v \in V - S, |deg(u) - deg(v)| \leq 1$. In this paper, a study of complementary degree equitable sets is initiated.

Key Words: Equitable dominating functions, set of zeros, k-equitable regular graphs

Contents

1. Introduction

Let G be a simple, finite and unordered graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $u \in V(G)$, the set of neighbours of u in G is denoted by $N(u)$ and $|N(u)|$ is called the degree of $u \in G$ and is denoted by $deg(u)$. A subset S of $V(G)$ is a dominating set of $V(G)$ if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $uv \in E(G)$ [5,6]. An equitable neighbor of u is a vertex which is adjacent with u and whose degree difference from that of u is at most one. The equitable neighbourhood of u, denoted by $N^e(u)$ is defined as $N^e(u) = \{v \in V(G) : v \in N(u) \text{ and } |deg(u) - deg(v)| \leq 1\}$. u is called an equitable isolate if $N^e(u) = \emptyset$. The equitable degree of u denoted by $deg^e_G(u)$ is $|N^e(u)|$. The set of all equitable isolates of G is denoted by $I_e(G)$. The maximum and minimum equitable degree of vertices of G is denoted respectively by $\Delta^e(G)$ and $\delta^e(G)$. That is, $\Delta^e(G) = \max_{u \in V(G)} \{deg^e_G(u)\}$ and $\delta^e(G) = \min_{u \in V(G)} \{deg^e_G(u)\}$. A subset S of $V(G)$ is said to be degree equitable if any two vertices of S are degree equitable. Such sets were defined and studied in [2]. For general notations and terminologies, those given in F.Harary's book [4] are followed.

2. Complementary Degree Equitable Set

Definition 2.1 A subset S of $V(G)$ is called complementary degree equitable set(cde-set) if $V - S$ is degree equitable. The minimum (maximum) cardinality of a minimal cde-set is called the lower (upper) complementary degree equitable number of G and is denoted by $cd^e(G)(CD^e(G))$.

Remark 2.1

- For any $u, v \in V - S$, $|deg_G(u) - deg_G(v)| \leq 1$.
- For any graph G, $V(G)$ is cde-set.
- The property of complementary degree equitability is super hereditary. That is, if S is a cde-set, then any super set of S is also a cde-set.
- A cde-set S is minimal if and only if it is 1-minimal.

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Proposition 2.1 $cd^e(G)$ and $CD^e(G)$ for standard graphs

1. $cd^e(K_n) = 0$, $CD^e(K_n) = 0$.
2. $cd^e(\overline{K_n}) = 0$, $CD^e(\overline{K_n}) = 0$.
3. $cd^e(K_{1,n}) = 1$, $CD^e(K_{1,n}) = \begin{cases} n+1 & \text{if } n = 2 \text{ or } 3 \\ n & \text{if } n \geq 4 \end{cases}$
4. $cd^e(P_n) = 0$, $CD^e(P_n) = 0$.
5. $cd^e(C_n) = 0$, $CD^e(C_n) = 0$.
6. *If* $n \leq 5$, $cd^e(W_n) = 0$, $CD^e(W_n) = 0$.
   If $n \geq 6$, $cd^e(W_n) = 1$, $CD^e(W_n) = n - 1$.
7. $cd^e(P) = 0$, $CD^e(P) = 0$ *where* P *is the Petersen graph.*
8. $cd^e(K_{m,n}) = \begin{cases} 0 & \text{if } |m-n| \leq 1 \\ \min\{m,n\} & \text{if } |m-n| \geq 2 \end{cases}$
   $CD^e(K_{m,n}) = \begin{cases} 0 & \text{if } |m-n| \leq 1 \\ \max\{m,n\} & \text{if } |m-n| \geq 2 \end{cases}$
9. $cd^e(D_{r,s}) = \begin{cases} 2 & \text{if } |r-s| \leq 1 \\ 2 & \text{if } |r-s| \geq 2 \end{cases}$
   $CD^e(D_{r,s}) = \begin{cases} r+s & \text{if } |r-s| \leq 1 \\ r+s+1 & \text{if } |r-s| \geq 2 \end{cases}$ *The double star* $D_{r,s}$ *is defined as the graph obtained by joining the centres of two stars* K_r *and* K_s.
10. *Let* G *be a graph with degree set* $\{d, d+1\}$. *Then* $cd^e(G) = 0$.

Remark 2.2 Let S be a *cde*-set of G. Then S is minimal if and only if for any $u \in S$, there exists a vertex $v \in V - S$ such that $|deg_G(u) - deg_G(v)| \geq 2$.

Definition 2.2 [2] Let $G = (V, E)$ be a simple graph. A subset S of $V(G)$ is called a degree equitable set if the degrees of any two vertices in S differ by at most one. The maximum cardinality of a degree equitable set in G is called the degree equitable number of G and is denoted by $D_e(G)$. The minimum cardinality of a maximal degree equitable set in G is called the lower degree equitable number of G and is denoted by $d_e(G)$.

Remark 2.3 It may be felt that $cd^e(G) = n - D^e(G)$. But it may not be so. In the case of $K_{m,n}$ with $|m-n| \geq 2$, $d_e(K_{m,n}) = \min\{m,n\}$ and $cd^e(K_{m,n}) = \min\{m,n\}$. Also, $D_e(K_{m,n}) = \max\{m,n\}$ if $|m-n| \geq 2$ and $CD^e(K_{m,n}) = \max\{m,n\}$. In the case of double star $D_{r,s}$ with $|r-s| \geq 2$, $d_e(D_{r,s}) = 1, D_e(D_{r,s}) = r+s, cd^e(D_{r,s}) = 2, CD^e(D_{r,s}) = r+s+1$. When $n = 4$ or 5, $d_e(W_n) = n$ and $cd^e(W_n) = 0$. Also, $CD^e(W_n) = 0, D_e(W_n) = n$.

Remark 2.4 There is no relationship between $cd^e(G)$ and $d_e(G)$ and between $CD^e(G)$ and $D_e(G)$.

Remark 2.5 Let G be a connected graph and let $H = G \circ K_1$. Let $S_1(H) = \{v : v \in V(H), deg(v) = 1$ or $2\}$. Let $D = H - S_1(H)$. Then $V(H) - D = S_1(H)$ which is degree equitable. If D_1 is a proper subset of $H - S_1(H)$, then its complement is not degree equitable since $S_1(H)$ contains degree one vertices and the complement of D_1 contains degree three or more vertices. Therefore, $cd^e(H) = |H - S_1(H)| = 2n - |S_1(H)|$.

Remark 2.6 Let S be a complementary equitable in G. Then S is complementary degree equitable in $\overline{G}$. The converse is also true. Therefore, $cd^e(G) = cd^e(\overline{G})$ and $CD^e(G) = CD^e(\overline{G})$.

Remark 2.7 Let $\delta \leq i \leq \Delta - 1$. Let $S_i = \{v \in V(G) : deg(v) = i \text{ or } i+1\}$. Let A be a non empty minimal complementary degree equitable set of G. Then $A = V(G) - S_i$ for some i, $\delta \leq i \leq \Delta - 1$. Hence $cd^e(G) = \min\{n - |S_i|, S_i \neq \phi\}$ and $CD^e(G) = \max\{n - |S_i|, S_i \neq \phi\}$. Hence $cd^e(G)$ and $CD^e(G)$ can be computed in a linear time.

Remark 2.8 Let n and k be positive integers with $k \leq n$. Then there exists a graph G of order n with $cd^e(G) = k$.

Example 2.1 *Let n and k be positive integers with $k \leq n$. Let $G = K_{n-k}$. Attach one pendant vertex at each of k vertices of K_n. Let H be the resulting graph. Let $n - k \geq 4$. Then G is regular with degree of each vertex three. The degree of each pendant vertex is one. Therefore, $cd^e(H) = k$ and order of H is equal to n.*

Theorem 2.1 *Suppose $cd^e(G) = n - 2$. Then $D_e(G) = 2$.*

Proof. Let S be a cd^e-set of G. Then $|V - S| = 2$ and $V - S$ is degree equitable. Therefore, $D_e(G) \geq 2$. Suppose $D_e(G) > 2$. Let T be a D_e-set of G. Then $V - T$ is complementary degree equitable and $|V - T| < n - 2$. That is, $cd^e(G) < n - 2$, a contradiction. Therefore, $D_e(G) = 2$. Then $G = K_2$ or $\overline{K_2}$ (By theorem 2.8 [2]).

Theorem 2.2 *Let G be a graph. Then $cd^e(G) = n - D_e(G)$.*

Proof. Suppose $cd^e(G) = k$. Let D be a cd^e-set of G. Then $V - D$ is degree equitable. Therefore, $D_e(G) \geq |V - D| = n - k$. Suppose $D_e(G) > n - k$. Let T be a D_e-set of G. Consider $V - T$. $V - T$ is complementary degree equitable and $|V - T| < k$. But $cd^e(G) = k$, a contradiction. Therefore, $D_e(G) = n - k$. Conversely, if $D_e(G) = n - k$, then claim $cd^e(G) = k$. Suppose $cd^e(G) < k$. Let S be a cd^e-set of G. Then $|V - S| > n - k$. That is, $V - S$ is degree equitable and $|V - S| > D_e(G)$, a contradiction. Therefore, $cd^e(G) \geq k$. Suppose $cd^e(G) > k$. Let S be a cd^e-set of G. Then $|V - S| < n - k$. Therefore, $|V - S| < D_e(G)$. Let T be a D_e-set of G. Then $V - T$ is complementary degree equitable and $|V - T| = n - k$. Therefore, $cd^e(G) \leq k$, a contradiction. Therefore, $cd^e(G) = k$. Therefore, $cd^e(G) = n - D_e(G)$.

Theorem 2.3 *Let G be a graph. Then $CD^e(G) = n - d_e(G)$.*

Proof. Suppose $CD^e(G) = k$. Let D be a CD^e-set of G. Then $V - D$ is degree equitable. $CD^e(G) = \max\{|\text{minimal complementary degree equitable set of } G|\}$. Therefore, $d_e(G) \leq |V - D| = n - k$. Suppose $d_e(G) < n - k$. Let T be a d_e-set of G. Consider $V - T$. $V - T$ is complementary degree equitable. $|V - T| = n - |T| \geq n - d_e(G) > k$ (since $n - d_e(G) > k$). Now, T is a subset of $V(G)$ such that $V - T$ is complementary degree equitable and $|V - T| > k$. That is, $|V - T| > CD^e(G)$, a contradiction. Therefore, $d_e(G) = n - k = n - CD^e(G)$. Conversely, let $d_e(G) = n - k$. Then claim $CD^e(G) = k$. Suppose $CD^e(G) > k$. Let S be a CD^e-set of G. Then $|V - S| < n - k$. That is, $V - S$ is degree equitable and $|V - S| < d_e(G)$, a contradiction. Therefore, $CD^e(G) \leq k$. Suppose $CD^e(G) < k$. Let S be a CD^e-set of G. Then $|V - S| > n - k$. Therefore, $|V - S| > d_e(G)$. Let T be a d_e-set of G. Then $V - T$ is complementary degree equitable and $|V - T| = n - k$. Therefore, $CD^e(G) \geq k$, a contradiction. Therefore, $CD^e(G) = k$. Therefore, $CD^e(G) = n - d_e(G)$.

Theorem 2.4 *Let G be a non trivial graph on n vertices. $cd^e(G) = n - 2$ if and only if $G = K_2$ or $\overline{K_2}$.*

Proof. Suppose $cd^e(G) = n - 2$. Then $D_e(G) = 2$. By theorem 2.8 [2], $G = K_2$ or $\overline{K_2}$. The converse is obvious.

Theorem 2.5 *[2] For a tree T, $cd^e(T) = n - |S_1(T)|$ where $S_1(T) = \{v \in V(G) : deg(v) = 1 \text{ or } 2\}$.*

Proof. By Proposition 2.10 [2], $D_e(T) = |S_1(T)|$. But $cd^e(T) = n - D_e(T) = n - |S_1(T)|$.

Theorem 2.6 *Let a and b be positive integers with $a \leq b$. Then there exists a graph G with $cd^e(G) = a$ and $CD^e(G) = b$.*

Proof. In Theorem 2.9 [2], it is proved that if a and b are positive integers such that $a \le b$, then there exists a graph G with $d_e(G) = a$ and $D_e(G) = b$ except when $a = 1$ and $b = 2$. Since $a \le b$, $n - b \le n - a$. Therefore, there exists a graph G with $d_e(G) = n - b$ and $D_e(G) = n - a$ except when $n - b = 1$ and $n - a = 2$. $CD^e(G) = n - d_e(G) = n - (n - b) = b$. Also, $cd^e(G) = n - D_e(G) = n - (n - a) = a$. Thus, there exists a graph G with $cd^e(G) = a$ and $CD^e(G) = b$ except when $n - b = 1$ and $n - a = 2$. Consider $n - b = 1$ and $n - a = 2$. Therefore, $a = n - 2$ and $b = n - 1$. Therefore, $d_e(G) = 1$ and $D_e(G) = 2$. Suppose $D_e(G) = 2$. Then $G = K_2$ or $\overline{K_2}$. Therefore, there exists no graph with $d_e(G) = 1$ and $D_e(G) = 2$. Therefore, there exists no graph with $n - b = 1$ and $n - a = 2$. That is, $a = n - 2$ and $b = n - 1$ (Since $cd^e(G) = n - 2$ if and only if $G = K_2$ or $\overline{K_2}$, $cd^e(G) = CD^e(G) = n - 2$. That is, there exists no graph with $cd^e(G) = n - 2$ and $CD^e(G) = n - 2$).

Further study:

(i) A study of coloring in which the color classes are independent and complementary equitable.
(ii) Secure complementary equitable sets.
(iii) Fair complementary equitable sets.

3. Conclusion

In this study, complementary degree equitable sets(*cde*-sets) in graphs is introduced and studied. The minimum and maximum cardinality of a minimal *cde*-sets are defined and found the value of these two parameters for standard networks. The bounds on these two parameters are derived in this work. Further, we may extend on these concepts in fuzzy networks and apply in communication networks to identify the stability of the network based on these parameters.

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L. Muthusubramanian,

Department of Mathematics Thiagarajar College of Engineering, Madurai, Tamilnadu, India.

E-mail address: lmsmat@tce.edu

and

R. Sundareswaran,

Department of Mathematics ,Sri Sivasubramaniya Nadar College of Engineering, Chennai, Tamilnadu, India.

E-mail address: sundareswaranr@ssn.edu.in

and

V. Swaminathan,
Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, Tamilnadu, India.
E-mail address: swaminathan.sulanesri@gmail.com