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A note on simple, 4-dimensional, ternary Filippov algebras *

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ABSTRACT: Properties of simple, 4-dimensional, ternary Filippov algebras are presented. More concretely, 1-identities and 2-identities, conservativeness and some equations are studied.

Key Words: Ternary Filippov algebra, 3-Lie algebra, identity, conservative algebra, equation.

Contents

1. Introduction

Over the years, the concept of n-ary Filippov algebra – a generalization of the notion of Lie algebra – has attracted a lot of attention. As revealed by Takhtajan in [15], where these algebras are called Nambu-Lie "gebras" of order n, the referred notion is the implicit algebraic concept underlying the Nambu Mechanics, [12]. In addition, as mentioned by Azcárraga and Izquierdo in [1], the interest in the applications of generalizations of the Lie algebra structure to Theoretical Physics problems has peaked in recent years. In particular, ternary Filippov algebras have risen to the surface in multi-brane theory in the context of the Bagger-Lambert-Gustavsson model. Last but not least, according to Bai and Wu who call n-Lie algebras to n-ary Filippov algebras, 3-Lie algebras are closely related to many important fields in mathematics and mathematical physics. For instance, metric 3-Lie algebras are used to describe a world volume of multiple M2-branes, [2].

There were several attempts to generalize the concept of Lie algebra to the n-ary case, but the most relevant generalization was presented in [6] by Filippov. He defined the notion of n-Lie algebra, nowadays also known, namely following Beites, Nicolás, Pozhidaev and Saraiva in [4], as n-ary Filippov algebra. The main attribute of Filippov's generalization is the fact that the right multiplication operators are derivations of the considered n-ary algebra, [11], in much the same way as in the case of Lie algebras (that is, 2-Lie algebras or 2-ary Fillipov algebras). Properties of n-ary Filippov algebras, or related to these, were studied by Kasymov in [8], Ling in [11], Saraiva in [13,14], Beites, Nicolás, Pozhidaev and Saraiva in [4], Kaygorodov in [9], Bai and Wu in [2], Beites and Nicolás in [3]. The aim of the present work is to study some more properties of simple, 4-dimensional, ternary Filippov algebras over an algebraically closed field with characteristic 0.

In section 2, definitions needed in the following sections are recalled – (binary, ternary) conservative algebra, ternary terminal algebra, s-identity of a ternary algebra, and (simple) ternary Filippov algebra. In addition, it is also recalled a classification result of the simple, 4-dimensional, ternary Filippov algebras: over an algebraically closed field with characteristic 0 and up to isomorphism, there is only one denoted

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by A_4. By hand and applying computational linear algebra on matrices, 1-identities and 2-identities of A_4 are studied in section 3. In particular, the random vectors method (for more details, see for instance [5]) is applied: the space of identities is given by the nullspace of an adequate matrix, linear-algebraic data which is translated back into the sought identities. In section 4, A_4 is proved to be a conservative algebra, a consequence of the validity of the mentioned property for ternary Filippov algebras. Finally, some equations in A_4 are solved in section 5.

2. Preliminaries

In what follows, unless stated otherwise, let F be a field with $\text{char}(F) \neq 2$.

The two subsequent definitions are related to the conservativeness of an algebra, property introduced by Kantor as a generalization of Jordan algebras.

Definition 2.1 ([10]) *Let A be a binary algebra over F, with binary multiplication denoted by juxtaposition. A is a conservative algebra if a new binary multiplication $(\cdot,\cdot)_n$ can be defined on the underlying vector space of A in such a way that*

$$\begin{aligned} & b(a(xy)-(ax)y-x(ay))-a((bx)y)+(a(bx))y \\ & +(bx)(ay)-a(x(by))+(ax)(by)+x(a(by)) \\ = \quad & -(a,b)_n(xy)+((a,b)_nx)y+x((a,b)_ny) \end{aligned}$$

is an identity of A.

Definition 2.2 ([7]) *Let B be a ternary algebra over F, with ternary multiplication denoted by $(\cdot,\cdot,\cdot)$. B is a conservative algebra if a new ternary multiplication $(\cdot,\cdot,\cdot)_n$ can be defined on the underlying vector space of B in such a way that*

$$((a,b,c)_n,x,y)=(a,b,(c,x,y))-(c,(a,b,x),y)-(c,x,(a,b,y))$$

is an identity of B. B is a terminal algebra if B is a conservative algebra and

$$(a,b,c)_n=\frac{1}{2}((a,b,c)+(b,a,c)+(c,b,a)-(c,a,b))$$

is an identity of B.

Next definition concerns a certain type of identities.

Definition 2.3 ([3,4]) *Let B be a ternary algebra over F. An identity of B is an s-identity, where $s\in\mathbb{N}$, if the ternary multiplication of B appears s times in each term of the identity.*

In addition to the definition of ternary Filippov algebra due to Filippov, we recall a classification result of the simple, 4-dimensional, ternary Filippov algebras.

Definition 2.4 ([6]) *Let $\mathcal{S}_3$ denote the symmetric group of degree 3. Let L be a vector space over F equipped with a ternary multiplication denoted by $[\cdot,\cdot,\cdot]$. L is a ternary Filippov algebra or a 3-Lie algebra over F if, for all $x_1,x_2,x_3,y_2,y_3\in L$ and for all $\sigma\in\mathcal{S}_3$,*

$$[x_1,x_2,x_3]=sgn(\sigma)[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}], \tag{2.1}$$

and

$$[[x_1,x_2,x_3],y_2,y_3]=[[x_1,y_2,y_3],x_2,x_3]+[x_1,[x_2,y_2,y_3],x_3]+[x_1,x_2,[x_3,y_2,y_3]]. \tag{2.2}$$

When the former identity holds, L and $[\cdot,\cdot,\cdot]$ are said to be, respectively, anti-commutative and skewsymmetric. The latter identity is usually called the ternary Jacobi identity.

Definition 2.5 ([6]) *Let L be a ternary Filippov algebra. A subspace I of L is an ideal of L if $[I,L,L]\subseteq I$. L is simple if $[L,L,L]\neq\{0\}$ and L has no proper ideals.*

Theorem 2.1 ([11]) *Up to isomorphism, there is a unique simple, 4-dimensional, ternary Filippov algebra A_4 over an algebraically closed field F with $char(F) = 0$. With $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ a basis of the underlying vector space of A_4, its multiplication table is given by*

$$[e_1, \ldots, \hat{e}_i, \ldots, e_4] = (-1)^i e_i, i \in \{1, 2, 3, 4\},$$

where $\hat{e}_i$ means that e_i is omitted. The remaining products are either 0 or obtained by skewsymmetry.

Let us call *canonical* to the basis $\mathcal{B}$ in Theorem 2.1.

3. Identities

The present section, where $\mathcal{S}_n$ denotes the symmetric group of degree n with $n \in \{3, 5\}$, is devoted to the 1-identities and the 2-identities of A_4.

Theorem 3.1 *Let F be an algebraically closed field with $char(F) = 0$. The 1-identities of A_4 are consequences of (2.1), that is,*

$$[x_1, x_2, x_3] = sgn(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}].$$

Proof: The 1-identities of A_4 are of the form

$$\sum_{\sigma \in \mathcal{S}_3} \alpha_\sigma [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 0, \tag{3.1}$$

with $\alpha_\sigma \in F$. Considering valuations for x_i in $\mathcal{B}$, from (3.1), we obtain a system of linear equations on α_σ that leads to

$$\begin{gathered}\alpha([x_1, x_2, x_3] + [x_1, x_3, x_2]) + \beta([x_1, x_2, x_3] + [x_2, x_1, x_3]) \\ +\gamma(-[x_1, x_2, x_3] + [x_2, x_3, x_1]) + \tau(-[x_1, x_2, x_3] + [x_3, x_1, x_2]) \\ +\eta([x_1, x_2, x_3] + [x_3, x_2, x_1]) = 0,\end{gathered}$$

where, for the sake of simplicity, the scalars were denoted by $\alpha, \beta, \gamma, \tau, \eta$. Hence, the result follows from (2.1). □

Remark 3.1 *An alternative proof for Theorem 3.1 is achieved applying the random vectors method using Maple in characteristic 0. A 10×6 matrix M initialized to zero, a 6×6 square matrix on top of a 4×6 matrix, is constructed. Its columns are labeled by the 3! monomials $[\cdot, \cdot, \cdot]$ in (3.1). We generate three pseudo-random vectors, each with 4 components, and store the components of the evaluation of the jth monomial $[\cdot, \cdot, \cdot]$ in column j of M, in rows 7 through 10. We then compute the row canonical form of the obtained matrix. This fill and reduce process is repeated till we arrive at the stabilization of the rank, in this case 1. The 1-identities satisfied by A_4 lie in the nullspace of this matrix. We now construct a 12×6 matrix initialized to zero and store, in its last 6 rows, the first of the two subsequent identities*

$$[a, b, c] = -[c, b, a], \quad [a, b, c] = [c, a, b], \tag{3.2}$$

and the 1-identities obtained from it under the action of $\mathcal{S}_3$. We calculate the row canonical form of this matrix and, after doing the same for the second identity in (3.2), we get a matrix of rank 5. Thus, the dimension of the $\mathcal{S}_3$-module generated by the considered identities is equal to the dimension of the nullspace of M.

Theorem 3.2 *Let F be an algebraically closed field with $char(F) = 0$. The 2-identities of A_4 are consequences of (2.1), that is,*

$$[x_1, x_2, x_3] = sgn(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}],$$

and of (2.2), that is,

$$[[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3] + [x_1, [x_2, y_2, y_3], x_3] + [x_1, x_2, [x_3, y_2, y_3]].$$

Proof: We apply the random vectors method using Maple in characteristic 0. The 2-identities of A_4 are of the form

$$\sum_{\sigma\in\mathcal{S}_5} \begin{array}{l}\big(\alpha_\sigma[[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}],x_{\sigma(4)},x_{\sigma(5)}]] \\ +\beta_\sigma[x_{\sigma(1)},[x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}],x_{\sigma(5)}]] \\ +\gamma_\sigma[x_{\sigma(1)},x_{\sigma(2)},[x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)}]]\big)=0,\end{array} \tag{3.3}$$

with $\alpha_\sigma,\beta_\sigma,\gamma_\sigma\in F$. From (2.1), all 2-identities of A_4 can be written using only the first association type $[[,,],,]$. We construct a 124×120 matrix M initialized to zero, and we think of M as consisting of a 120×120 square matrix on top of a 4×120 matrix. We generate five pseudo-random vectors, each with 4 components, and store the 4 components of the evaluation of the jth monomial $[[\cdot,\cdot,\cdot],\cdot,\cdot]$ in column j of M, $j\in\{1,\ldots,120\}$, in rows 121 through 124. The computation of the row canonical form of the obtained matrix completes the first iteration of the algorithm. We repeat this fill and reduce process until the stabilization of the rank of the matrix is reached, in this case 5. The 2-identities of A_4 lie in the nullspace, with dimension 115, of this matrix. Now observe that the 1-identity (2.1) is a trilinear identity of the form $I_1=I_1(x_1,x_2,x_3)=0$. Its six liftings, obtained from embedding the identity in a triple, and also from replacing one variable by a triple in the identity (for more details, see for instance [3]), are

$$\begin{array}{l}[I_1(x_1,x_2,x_3),d,e]=0,[d,I_1(x_1,x_2,x_3),e]=0,[d,e,I_1(x_1,x_2,x_3)]=0,\\ I_1([x_1,x_2,x_3],d,e)=0,I_1(d,[x_1,x_2,x_3],e)=0,I_1(d,e,[x_1,x_2,x_3])=0.\end{array}$$

Invoking again the 1-identity (2.1), all the listed liftings represent the 2-identity $[[x_1,x_2,x_3]-sgn(\sigma)[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}],d,e]=0$. Thus, under the action of $\mathcal{S}_5$,

$$[[x_1,x_2,x_3],d,e]+[[x_3,x_2,x_1],d,e]=0$$

is the only distinct way to lift the 1-identity (2.1) to a 2-identity. The result follows from the dimension – 115 – of the $\mathcal{S}_5$-module generated by $[[x_1,x_2,x_3],d,e]+[[x_3,x_2,x_1],d,e]=0$ and by the ternary Jacobi identity. □

4. Conservativeness

The present section is devoted to the conservativeness of A_4, and of some related algebras.

Theorem 4.1 *Let F be a field with $char(F)\neq 2$. Every ternary Filippov algebra is a conservative algebra which is terminal if and only if it is trivial.*

Proof: Let us prove that

$$[[a,b,c]_n,x,y]=[a,b,[c,x,y]]-[c,[a,b,x],y]-[c,x,[a,b,y]],$$

where $[\cdot,\cdot,\cdot]_n=[\cdot,\cdot,\cdot]$. On the one hand, by (2.2), we get

$$[[a,b,c],x,y]=[[a,x,y],b,c]+[a,[b,x,y],c]+[a,b,[c,x,y]].$$

On the other hand, again from (2.2), and also by (2.1), we have

$$\begin{array}{l}[[a,x,y],b,c]+[a,[b,x,y],c]+[c,[a,b,x],y]+[c,x,[a,b,y]]\\ =[[a,b,c],x,y]-[a,b,[c,x,y]]+[c,[x,a,b],y]+[c,x,[y,a,b]]\\ =[[a,b,c],x,y]-[a,b,[c,x,y]]+[[c,x,y],a,b]-[[c,a,b],x,y]\\ =[[a,b,c],x,y]-[a,b,[c,x,y]]+[a,b,[c,x,y]]-[[a,b,c],x,y]\\ =0.\end{array}$$

However, $[a,b,c]=\frac{1}{2}([a,b,c]+[b,a,c]+[c,b,a]-[c,a,b])$ does not hold for non-trivial ternary Filippov algebras since, from (2.1), the right-hand side of this equality is equal to $-[a,b,c]$. □

Corollary 4.1 *Let F be an algebraically closed field with char$(F) = 0$. The ternary Filippov algebra A_4 is a conservative algebra which is not terminal.*

Proof: A straightforward consequence of Theorem 4.1. □

Theorem 4.2 *Let F be a field with char$(F) \neq 2$. Every reduced algebra of a ternary Filippov algebra is a conservative algebra.*

Proof: From [11, Remark 1.1.1] and [13], every reduced algebra of a ternary Filippov algebra is a Lie algebra. As recalled in [10], every Lie algebra is a conservative algebra. □

5. Equations

The present section is devoted to some equations in A_4.

Theorem 5.1 *Let F be an algebraically closed field with char$(F) = 0$. If $a, b \in A_4$ are linearly dependent then the set of solutions of the equation $[a, b, x] = 0$ is the F-linear span $\langle e_1, e_2, e_3, e_4\rangle$ of the canonical basis $\mathcal{B}$ of A_4. If $a, b \in A_4$ are linearly independent then the set of solutions of the equation $[a, b, x] = 0$ is the F-linear span $\langle a, b\rangle$ of a and b.*

Proof: The first part of the result is straightforward since, taking into account (2.1), if $b = \eta a$ for some $\eta \in F$ then

$$[a, \eta a, x] = \eta[a, a, x] = 0.$$

For the second part of the result now observe that, for all $\alpha, \beta \in F$ and again by (2.1),

$$[a, b, \alpha a + \beta b] = \alpha[a, b, a] + \beta[a, b, b] = 0.$$

Thus, for all $\alpha, \beta \in F$, $\alpha a + \beta b$ is a solution of $[a, b, x] = 0$. If $u \in A_4$ is another solution of this equation then $[a, b, u] = 0$, and

$$[a, b, u - \alpha a - \beta b] = 0.$$

Let $s_1, s_2, s_3 \in F$ such that $s_1 a + s_2 b + s_3(u - \alpha a - \beta b) = 0$, that is,

$$(s_1 - s_3\alpha)a + (s_2 - s_3\beta)b + s_3 u = 0.$$

If $s_3 = 0$ then $s_1 a + s_2 b = 0$ and, as a and b are linearly independent, $s_1 = s_2 = 0$. Thus, if $s_3 = 0$ then a, b and $u - \alpha a - \beta b$ are linearly independent. But taking $u = a$ leads to a contradiction since a, b and $a - \alpha a - \beta b$ are linearly dependent. We conclude that $s_3 \neq 0$, and

$$u = s_3^{-1}(-s_1 + s_3\alpha)a + s_3^{-1}(-s_2 + s_3\beta)b \in \langle a, b\rangle.$$

□

Theorem 5.2 *Let F be an algebraically closed field with char$(F) = 0$. Let $a, b \in A_4$ be linearly independent. Let $c \in A_4\backslash\{0\}$. The set of solutions of the equation $[a, b, x] = c$ is $\{-[a, b, c] + \alpha a + \beta b : \alpha, \beta \in F\}$.*

Proof: Observe that if u is a solution of $[a, b, x] = c$ then, for all $\alpha, \beta \in F$, $u + \alpha a + \beta b$ is a solution too. In fact, by (2.1), we have

$$[a, b, u + \alpha a + \beta b] = [a, b, u] + \alpha[a, b, a] + \beta[a, b, b] = c.$$

Now suppose that v is another solution. Then, we get $[a, b, v - u - \alpha a - \beta b] = 0$ from $[a, b, u + \alpha a + \beta b] = c = [a, b, v]$. Hence, from Theorem 5.1, $v - u - \alpha a - \beta = \gamma a + \delta b$ for some $\gamma, \delta \in F$, and

$$v = u + (\alpha + \gamma)a + (\beta + \delta)b.$$

Lastly, $-[a,b,c]$ is a solution of $[a,b,x]=c$ as, again by (2.1),

$$[a,b,-[a,b,c]] = [-[c,a,b],a,b] = c.$$

In fact, for $w,y,z \in \mathcal{B}$ – the canonical basis of A_4 – with y,z linearly independent elements, the spectrum $\sigma_{R_{y,z}}$ of the right multiplication operator $R_{y,z}$ by y and z, defined by $d \mapsto [d,y,z]$, is $\{-i,i\}$ and, thus,

$$[[w,y,z],y,z] = (R_{y,z} \circ R_{y,z})(w) = -w.$$

□

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References

1. Azcárraga, J. A., Izquierdo, J. M., *n-Ary algebras: A review with applications,* J. Phys. A. 43, 1-117, (2010).
2. Bai, R.-P., Wu, Y., *Constructions of 3-Lie algebras,* Linear Multilinear Algebra 63, 2171-2186, (2015).
3. Beites, P. D., Nicolás, A. P., *An associative triple system of the second kind,* Comm. Algebra 44, 5027-5043, (2016).
4. Beites, P. D., Nicolás, A. P., Pozhidaev, A. P., Saraiva, P., *On identities of a ternary quaternion algebra,* Comm. Algebra 39, 830-842, (2011).
5. Bremner, M. R., Hentzel, I., *Invariant nonassociative algebra structures on irreducible representations of simple Lie algebras,* Exp. Math. 13, 231-256, (2003).
6. Filippov, V. T., *n-Lie algebras,* Sib. Math. J. 26, 879-891, (1985).
7. Kantor, I. L., *Terminal trilinear operations,* Algebra Logic 28, 25-40, (1989).
8. Kasymov, S. M., *Theory of n-Lie algebras,* Algebra Logic 26, 155-166, (1987).
9. Kaygorodov, I., *(n + 1)-Ary derivations of semisimple Filippov algebras,* Math. Notes 96, 208-216, (2014).
10. Kaygorodov, I., Lopatin, A., Popov, Y., *Conservative algebras of 2-dimensional algebras,* Linear Algebra Appl. 486, 255-274, (2015).
11. Ling, W., *On the structure of n-Lie algebras,* PhD Thesis, (1993).
12. Nambu, Y., *Generalized Hamiltonian dynamics,* Phys. Rev. D 7, 2405-2412, (1973).
13. Saraiva, P., *Reduced n-Lie algebras,* Comm. Algebra 30, 2057-2074, (2002).
14. Saraiva, P., *On some generalizations of Malcev algebras,* Int. J. Math. Game Theory Algebra 13, 89-108, (2002).
15. Takhtajan, L., *On foundation of the generalized Nambu mechanics,* Comm. Math. Phys. 160, 295-315, (1994).

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