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A new and efficient numerical algorithm to solve fractional boundary value problems

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ABSTRACT: In this paper, we developed a new numerical technique based on Lagrange Interpolation Polynomials (LIPs) to obtain the solutions of fractional higher order nonlinear boundary value problems. The newly established method is called the Lagrange Interpolation Transform Method (LITM). The fractional derivatives are represented by the Caputo operator. The validity of the proposed technique is confirmed with the help of illustrative examples. The exact and (LITM) solutions are compared by using graphs and tables, which show the closed contact between the actual and (LITM) solutions. The results of the suggested technique are compared with the solutions of the Chebyshev Wavelet Method (CWM) and the Optimal Homotopy Asymptotic Method (OHAM). The comparison has shown that the (LITM) has better accuracy as compared to the (CWM) and (OHAM) solutions. The fractional order solutions are investigated, which are convergent towards integer order solutions of the targeted problems. Moreover, the present technique has a straightforward and simple procedure to solve both fractional and integer order problems. The computational work is correctly done with the help of MAPLE software and requires less CPU time. The present method can be used directly to solve the problems expressed in tabular form, which confirms the novelty of (LITM).

Key Words: Lagrange Interpolation Polynomials (LIPs), Lagrange Interpolation Transform Method (LITM), fractional nonlinear boundary value problems.

Contents

1. Introduction

Nowadays, the fractional calculus (FC), which is the generalisation of the ordinary calculus, plays a vital role in modelling various phenomena in science and engineering. The origin of (FC) can be traced back to 1695, when Leibniz and L'Hospital conversed about how we can find the derivative of fractional order i-e $\frac{d^{1/2}y}{dx^{1/2}}$. Later on, (FC) was considered as an apparent paradox (prediction given by Leibniz and L'Hospital) and later on gained popularity among researchers. Then, in 1819, Lacroix [1] published a paper in which (FC) was introduced for the first time, and in 1823, Abel [2] applied (FC) to tautron problems. Later, (FC) applications are found to be very useful in modelling other physical problems, such as fluid dynamics [3], blood alcohol model [4], Coronavirus model [5], fluid traffic model [6], financial model [7], air foil model [8], and Poisson-Nerst-Planck diffusion model [9].

Many researchers have analysed that fractional order derivatives are very useful for modelling various phenomena in nature, and the dynamics of the damping laws and diffusion processes are discussed in terms of fractional order derivatives. The newly developed technique is called the Lagrange Interpolation Transform method (LITM), because many physical phenomena in nature have been found to have derivatives of fractional order that are very important for modelling various physical problems, such as damping laws, diffusion processes, etc. Moreover, the subject of (FC) has numerous applications in different areas

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of applied sciences, such as ,astrophysics [10], the narrow convecting layers bounded by stable layers which are believed to surround A-type stars [11]. For example, Hassan et al. presented the solutions of some nonlinear fractional differential equations (FDEs) in [12,13,14,15,16,17], and investigated the solutions of systems of linear and nonlinear (FDEs) in [18].
Numerical and analytical methods are frequently used to obtain the solutions of important mathematical models of physical processes in nature. Mathematicians have devised a number of techniques for solving (FDEs) and their systems. The obtained results of both fractional and integer order problems support the actual dynamics of the problems and thus become a prominent area of research. Thus, the researchers have made their best efforts and established valuable techniques at regular intervals of their time. In this connection, important and efficient techniques have been used for the solutions of (FDEs) and their systems, such as the Chebyshev Wavelet Method (CWM) [19], the Finite Difference Method (FDM) [20], the Adomian Decomposition Method (ADM) [21], the Homotopy Perturbation Transform Method (HPTM) [22], the Haar Wavelet Method (HWM) [23], the Variational Iteration Transform Method (VIM) [24], the Differential Transform Method (DTM) [25], Homotopy Analysis Sumudu Transform Method (HASTM) [26], Local Fractional Natural Homotopy Analysis Method (LFNHAM) [27], etc.
In this research work, the solutions of boundary value problems of fractional orders are investigated by using a newly established method, which is known as the Lagrange Interpolation Transform Method (LITM). The (LITM) provides the approximation with a sufficient degree of accuracy. The solutions obtained by the proposed technique are compared with the results of (CWM) and (OHAM) [28,29], which confirmed the higher accuracy of the (LITM). Some numerical examples related to fractional boundary value problems are considered for the numerical solutions and compared with the results of (CWM). (LITM) solutions are represented by using graphs and tables. The tabular and graphical representations have been done for both fractional and integer order solutions. The (LITM) and actual solutions are observed to be very close to each other. The fractional solutions confirmed the actual dynamics of the suggested problems. Our new technique is very useful if the problem is described in tabular form. In these kinds of problems, other existing methods do not work, and thus the present method has the novelty to handle the situation. Moreover, the fractional solutions are calculated, which explains the valuable information about the targeted problems. It analysed that the fractional solutions are convergent to the integer solution of each problem and thus confirmed the validity of the new method.

2. PRELIMINARIES

In this section of the paper, we included some basic definitions and preliminary concepts about fractional calculus, which is the foundation for this paper. These definitions and preliminary concepts are necessary to complete the present research task.

Definition 2.1 *The Lagrange nth Lagrange interpolating polynomial is given by, [30]*

$$P_n(x) = f(x_0)K_{n,0}(x) + f(x_1)K_{n,1}(x) + \cdots + f(x_n)K_{n,n}(x) = \Sigma_{k=0}^{n} f(x_k)K_{n,k}(x) \tag{2.1}$$

where

$$K_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}, \ k = 0, 1, \cdots, n. \tag{2.2}$$

Definition 2.2 The Riemann-Liouville fractional differential operator of order $\vartheta > 0$, denoted by D_x^{ϑ}, for a function $f(x)$ in the interval (a, b) is expressed as [28]

$$D_x^{\vartheta} f(x) = \frac{1}{\Gamma(m-\vartheta)} \frac{d^m}{dx^m} \int_a^x (x-t)^{\vartheta-1} f(t)dt, \quad m-1 < \vartheta \leq m, \quad m = 1, 2, \cdots \tag{2.3}$$

Definition 2.3 The Riemann-Liouville fractional integral of order $\vartheta > 0$, denoted by $D_x^{-\vartheta}$, for a function $f(x)$ in the interval (a, b) is defined as [28]

$$D_x^{-\vartheta} f(x) = \frac{1}{\Gamma(\vartheta)} \int_a^x (x-t)^{\vartheta-1} f(t)dt, \ m-1 < \vartheta \leq m, \ m = 1, 2, \cdots \tag{2.4}$$

Definition 2.4 The fractional differential operator defined by Caputo derivative of order $\vartheta > 0$, is expressed as, [32]

$$D_{\tau}^{\vartheta} f(x) = \frac{\partial^{\vartheta}}{\partial t^{\vartheta}} v(x,t) = \begin{cases} \frac{1}{\Gamma(\hbar-\vartheta)} \int_0^x (x-\tau)^{\hbar-\vartheta-1} f^{(\hbar)}(\tau) d\tau, & \hbar - 1 < \vartheta \leq \hbar \\ \frac{\partial^{\hbar}}{\partial t^{\hbar}} v(x,t), & \vartheta = \hbar \in N \end{cases} \tag{2.5}$$

Lemma 2.1 *For $m-1 < \gamma \leq m$ with $m \in \mathbb{N}$ and $h \in \mathbb{C}_\tau$ with $\tau \geq 0$, then [28]*

$$\begin{cases} I^{\gamma} I^{\vartheta} h(\xi) = I^{\gamma+\vartheta} h(\xi), & \vartheta,\ \gamma \geq 0, \\ I^{\gamma} \kappa^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\gamma+\vartheta+1)} \kappa^{\gamma+\vartheta}, & \gamma > 0,\ \vartheta > -1,\ \kappa > 0 \\ I^{\gamma} D^{\gamma} h(\kappa) = h(\kappa) - \sum_{k=0}^{m-1} h^k(0^+) \frac{\kappa^k}{k!}, \end{cases} \tag{2.6}$$

where $\kappa > 0$, $m-1 < \gamma \leq m$.

3. Methodology

In this section, we consider the following fractional boundary value problem:

$$D^{\alpha} y(x) = g(x) + f(y), \quad 0 < x \leq b, \quad n-1 < \alpha \leq n, \tag{3.1}$$

with the boundary conditions:

$$\begin{aligned} y(0) = \alpha_0, \ \ y^{'}(0) = \alpha_1, \ \ y^{''}(0) = \alpha_2, \cdots \ y^{n}(0) = \alpha_n, \\ y(b) = \beta_0, \ \ y^{'}(b) = \beta_1, \ \ y^{''}(b) = \beta_2, \cdots y^{n}(b) = \beta_n, \end{aligned} \tag{3.2}$$

$f(y)$ is a linear or nonlinear function and $g(x)$ is the source term.
Now, using the procedure of (LITM), we approximate the solution of Eq. (3.1), by using the Lagrange interpolation formula.
That is,

$$y(x) \simeq P_n(x) = f(x_0) K_{n,0}(x) + f(x_1) K_{n,1}(x) + \cdots + f(x_n) K_{n,n}(x) = \sum_{n=1}^{2^{k-1}} \sum_{k=0}^{M-1} f(x_k) K_{n,k}(x), \tag{3.3}$$

where $K_{n,0}(x)$, $K_{n,1}(x)$, $K_{n,2}(x)$, $\cdots$, $K_{n,n}(x)$ are Lagrange polynomials defined in Eq. (2.2).
To determine $2^{k-1}M$ coefficients, we will use $2^{k-1}(M-1)$ conditions.

For this, n conditions are given by the following boundary conditions:

$$\begin{aligned}
u_{k,M}(0) &= \sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(0) = \alpha_0,\\
\frac{d}{dx}u_{k,M}(0) &= \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(0) = \alpha_1,\\
\frac{d^2}{dx^2}u_{k,M}(0) &= \frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(0) = \alpha_2,\\
&\vdots\\
\frac{d^n}{dx^n}u_{k,M}(0) &= \frac{d^n}{dx^n}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(0) = \alpha_n,\\
u_{k,M}(b) &= \sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(b) = \beta_0,\\
\frac{d}{dx}u_{k,M}(b) &= \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(b) = \beta_1,\\
\frac{d^2}{dx^2}u_{k,M}(b) &= \frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(b) = \beta_2\\
&\vdots\\
\frac{d^n}{dx^n}u_{k,M}(b) &= \frac{d^n}{dx^n}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(b) = \beta_n,
\end{aligned} \tag{3.4}$$

Now using these n boundary conditions, we need $2^{k-1}(M-1)$, extra conditions to calculate the unknown's coefficients $K_{n,k}$. These conditions can be obtained by putting Eq. (3.3) in Eq. (3.1) as

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(x) = g(x) + f\left(\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-n} f(x_k)K_{n,k}(x)\right). \tag{3.5}$$

Assume that Equation (3.5) is exact at $2^{k-1}(M-n)$ points which we consider as x_i, then

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(x_i) = g(x_i) + f\left(\sum_{n=1}^{2^{k-1}}\sum_{k=0}^{M-1} f(x_k)K_{n,k}(x_i)\right). \tag{3.6}$$

For the choice of x_i, the points are the zeros of the shifted Chebyshev polynomials of degree $2^{k-1}(M-n)$ in the interval $[0,1]$ that is
$x_i = \frac{s_i+1}{2}$, where $x_i = cos\left(\frac{(2i-1)\pi}{2^{k-1}(M-1)}\right)$, $i = 1, 2, \cdots 2^{k-1}(M-n)$.
Eq. (3.4) and Eq. (3.6) give $2^{k-1}M$ linear system or the nonlinear equations as the case may be occur for the problem. The fractional derivative in Eq. (3.1), is expressed in terms of the Caputo operator given in Eq. (2.3). Then Eq. (3.1), is converted in to a system of linear or nonlinear algebraic equations. Some of the equations are obtained by evaluating ordinary differential equations (ODEs) in Eq. (3.1), at the points where $P_n(x)$ equals $y(x)$.
Finally, we solve the system to obtain the arbitrary constant $f(x_i)$, $i = 0, 1, \cdots, n$, which we then plug into Eq. (3.1), to obtain the (LITM) solution for the given problems.

4. Numerical Examples

Example 4.1 *The fourth order fractional nonlinear boundary value problem is [29]*

$$\frac{d^{\alpha}y}{dx^{\alpha}} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,\ 3 < \alpha \leq 4, \tag{4.1}$$

with the following boundary conditions:

$$y(0) = 0, \quad y(1) = 1, \quad y^{'}(0) = 0, \quad y^{'}(1) = 1, \tag{4.2}$$

the exact solution is

$$y(x) = x^5 - 2x^4 + 2x^2. \tag{4.3}$$

Table 1: Fractional order (LITM) solutions of Example 4.1 at different values of α

x_i	$Exact(\alpha = 4)$	$LITM(\alpha = 4)$	$LITM(\alpha = 3.85)$	$LITM(\alpha = 3.75)$	$LITM(\alpha = 3.50)$	$LITM(\alpha = 3.25)$	$CWM(M = 8)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0198100	0.0198099	0.0221148	0.0240054	0.0397288	0.0397288	0.0198100
0.2	0.0771200	0.0771199	0.0850947	0.0915821	0.1442169	0.1442169	0.0771200
0.3	0.1662300	0.1662299	0.1809884	0.1928808	0.2865756	0.2865756	0.1662300
0.4	0.2790400	0.2790400	0.2994550	0.3157387	0.4400794	0.4400794	0.2790400
0.5	0.4062500	0.4062500	0.4294537	0.4477703	0.5834084	0.5834084	0.4062500
0.6	0.5385600	0.5385600	0.5607700	0.5781251	0.7031761	0.7031761	0.5385600
0.7	0.6678700	0.6678700	0.6854155	0.6989969	0.7948268	0.7948268	0.6678700
0.8	0.7884800	0.7884800	0.7989376	0.8069657	0.8629800	0.8629800	0.7684800
0.9	0.8982900	0.8982900	0.9016754	0.9042573	0.9223043	0.9223043	0.8982900
1.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

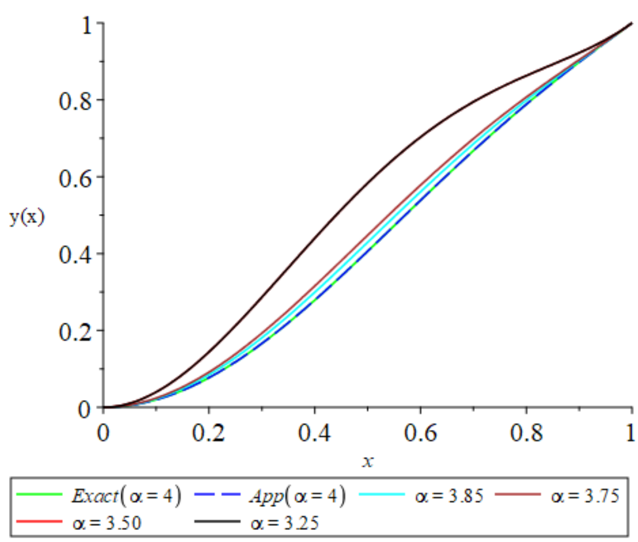

Figure 1: (LITM) fractional solution graphs of Example 4.1.

Table 2: Error Analysis of the fractional solutions of Example 4.1 at different value of α

x_i	A.E($\alpha = 4$)	A.E($\alpha = 3.85$)	A.E($\alpha = 3.75$)	A.E($\alpha = 3.50$)	A.E($\alpha = 3.25$)	$OHAM Error$
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	$7.00E-10$	$2.30E-3$	$4.19E-3$	$1.99E-2$	$1.99E-2$	$7.27E-8$
0.2	$1.40E-10$	$7.97E-3$	$1.44E-2$	$6.70E-2$	$6.70E-2$	$2.45E-7$
0.3	$6.00E-10$	$1.47E-2$	$2.66E-2$	$1.20E-1$	$1.20E-1$	$4.48E-7$
0.4	$8.00E-10$	$2.04E-2$	$3.66E-2$	$1.61E-1$	$1.61E-1$	$6.18E-7$
0.5	$2.10E-9$	$2.32E-2$	$4.15E-2$	$1.77E-1$	$1.77E-1$	$6.99E-7$
0.6	$2.30E-9$	$2.22E-2$	$3.95E-2$	$1.64E-1$	$1.64E-1$	$6.61E-7$
0.7	$1.90E-9$	$1.75E-2$	$3.11E-2$	$1.26E-1$	$1.26E-1$	$5.07E-7$
0.8	$1.10E-10$	$1.04E-2$	$1.84E-2$	$7.45E-2$	$7.45E-2$	$2.87E-7$
0.9	0.0000000	$3.38E-3$	$5.96E-3$	$2.40E-2$	$2.40E-2$	$2.87E-7$
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

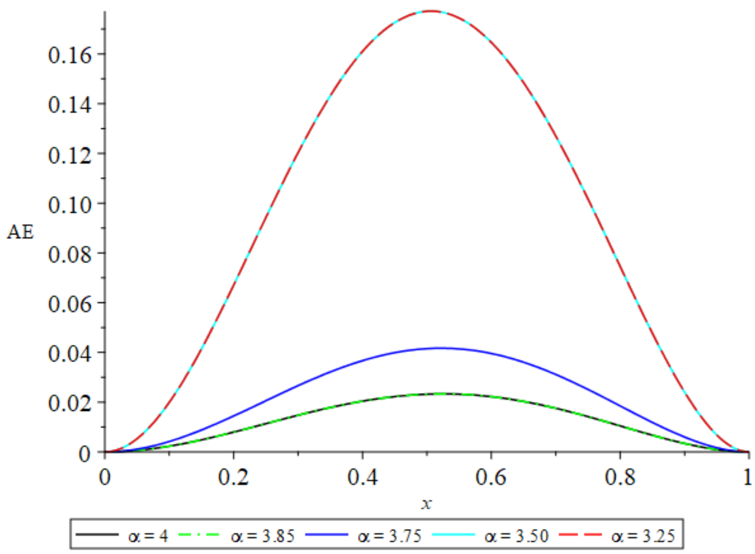

Figure 2: Error Analysis graph of Example 4.1

Example 4.2 *The sixth order fractional nonlinear boundary value problem is [31]*

$$\frac{d^{\alpha}y}{dx^{\alpha}} = -6e^{x} + y, \ 5 < \alpha \leq 6, \tag{4.4}$$

with the following boundary conditions:

$$y(0) = 1, \quad y^{''}(0) = -1, \quad y^{iv}(0) = -3, \quad y(1) = 0, \quad y^{''}(1) = -2e, \quad y^{iv}(1) = -4e, \tag{4.5}$$

The exact solution is

$$y(x) = (1 - x)e^{x}. \tag{4.6}$$

Table 3: Fractional order (LITM) solutions of Example 4.2 at different value of α at $(M = 7)$

x_i	$Exact(\alpha = 6)$	$LITM(\alpha = 6)$	$LITM(\alpha = 5.85)$	$LITM(\alpha = 5.75)$	$LITM(\alpha = 5.50)$	$LITM(\alpha = 5.25)$	$CWM(M = 20)$
0.0	1.0000000	0.9999999	0.9999999	0.9999999	0.9999999	0.9999999	1.0000000
0.1	0.9946538	0.9981162	0.9991913	0.9987123	0.9911231	0.9802283	0.9946538
0.2	0.9771222	0.9836474	0.9856090	0.9846223	0.9700383	0.9493174	0.9771222
0.3	0.9449011	0.9537581	0.9562853	0.9547702	0.9343848	0.9058649	0.9449011
0.4	0.8950948	0.9053298	0.9080524	0.9060441	0.8816230	0.8480865	0.8950948
0.5	0.8243606	0.8349220	0.8375003	0.8351271	0.8089162	0.7736319	0.8243606
0.6	0.7288475	0.7387043	0.7408867	0.7383800	0.7129369	0.6793478	0.7288475
0.7	0.6041258	0.6123664	0.6140135	0.6116863	0.5896359	0.5610302	0.6041258
0.8	0.4451081	0.4510112	0.4520842	0.4502776	0.4340125	0.4132051	0.4451081
0.9	0.2459603	0.2490346	0.2495557	0.2485655	0.2399281	0.2289793	0.2459603
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

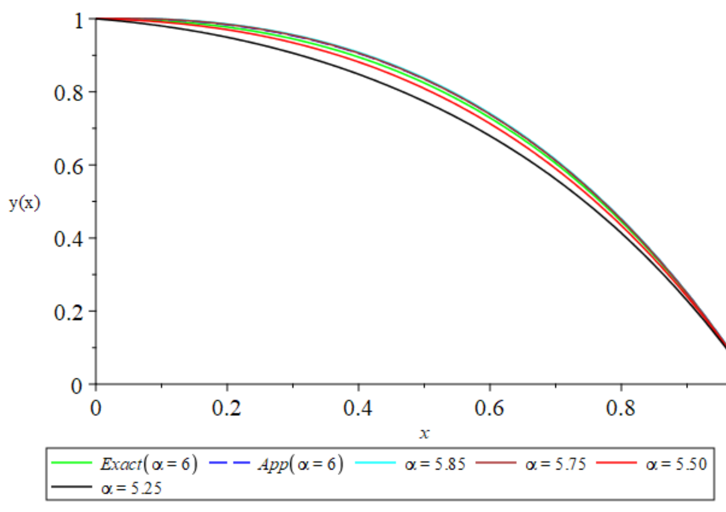

Figure 3: (LITM) fractional solution graphs of Example 4.2.

Table 4: Error Analysis of the fractional solutions of Example 4.2 at different value of α at $(M = 7)$

x_i	A.E($\alpha = 6$)	A.E($\alpha = 5.85$)	A.E($\alpha = 5.75$)	A.E($\alpha = 5.50$)	A.E($\alpha = 5.25$)	$CWMError(M = 20)$
0.0	$2.00E-10$	$2.00E-10$	$2.00E-10$	$2.00E-10$	$2.00E-10$	$2.00E-1000$
0.1	$3.46E-3$	$4.53E-3$	$4.05E-3$	$3.53E-3$	$1.44E-2$	$3.53E-14$
0.2	$6.52E-3$	$8.48E-3$	$7.50E-3$	$7.08E-3$	$2.78E-2$	$6.77E-14$
0.3	$8.85E-3$	$1.13E-2$	$9.86E-3$	$1.05E-2$	$3.90E-2$	$9.41E-14$
0.4	$1.02E-2$	$1.29E-2$	$1.09E-2$	$1.34E-2$	$4.70E-2$	$1.12E-14$
0.5	$1.05E-2$	$1.31E-2$	$1.07E-2$	$1.54E-2$	$5.07E-2$	$1.19E-14$
0.6	$9.85E-3$	$1.20E-2$	$9.53E-3$	$1.59E-2$	$4.94E-2$	$1.16E-14$
0.7	$8.24E-3$	$9.88E-3$	$7.56E-3$	$1.44E-2$	$4.30E-2$	$1.01E-14$
0.8	$5.90E-3$	$6.97E-3$	$5.16E-3$	$1.10E-2$	$3.19E-2$	$7.43E-14$
0.9	$3.07E-3$	$3.59E-3$	$2.60E-3$	$6.03E-3$	$1.69E-2$	$3.95E-14$
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	$4.20E-99$

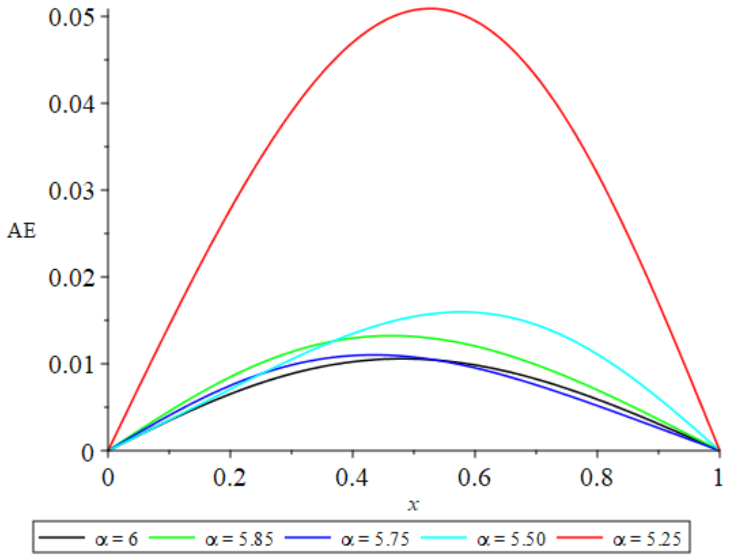

Figure 4: Error Analysis graph of Example 4.2.

Example 4.3 *The fifth order fractional nonlinear boundary value problem is [31]*

$$\frac{d^\alpha y}{dx^\alpha} = y - 15e^x - 10xe^x, \; 4 < \alpha \leq 5, \tag{4.7}$$

with the following boundary conditions:

$$y(0) = 0, \quad y^{'}(0) = 1, \quad y^{''}(0) = 0, \quad y(1) = 0, \quad y^{'}(1) = -e, \tag{4.8}$$

The exact solution is

$$y(x) = x(1-x)e^x. \tag{4.9}$$

Table 5: Fractional order (LITM) solutions of Example 4.3 at different value of α at $(M = 8)$

x_i	$Exact(\alpha = 5)$	$LITM(\alpha = 5)$	$LITM(\alpha = 4.85)$	$LITM(\alpha = 4.75)$	$LITM(\alpha = 4.50)$	$LITM(\alpha = 4.25)$	$CWM(M = 50)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0994653	0.0994654	0.0994376	0.0994179	0.0993197	0.0993197	0.0994653
0.2	0.1954244	0.1954245	0.1952517	0.1951303	0.1945569	0.1945569	0.1954244
0.3	0.2834703	0.2834705	0.2830292	0.2827222	0.2813314	0.2813314	0.2834703
0.4	0.3580379	0.3580382	0.3572767	0.3567512	0.3544427	0.3544427	0.3580379
0.5	0.4121803	0.4121806	0.4111549	0.4104517	0.4074266	0.4074266	0.4121803
0.6	0.4373085	0.4373088	0.4361798	0.4354100	0.4321432	0.4321432	0.4373085
0.7	0.4228880	0.4228883	0.4218825	0.4212002	0.4183308	0.4183308	0.4228880
0.8	0.3560865	0.3560867	0.3554201	0.3549699	0.3530926	0.3530926	0.3560865
0.9	0.2213642	0.2213643	0.2211269	0.2209674	0.2203094	0.2203094	0.2213642
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

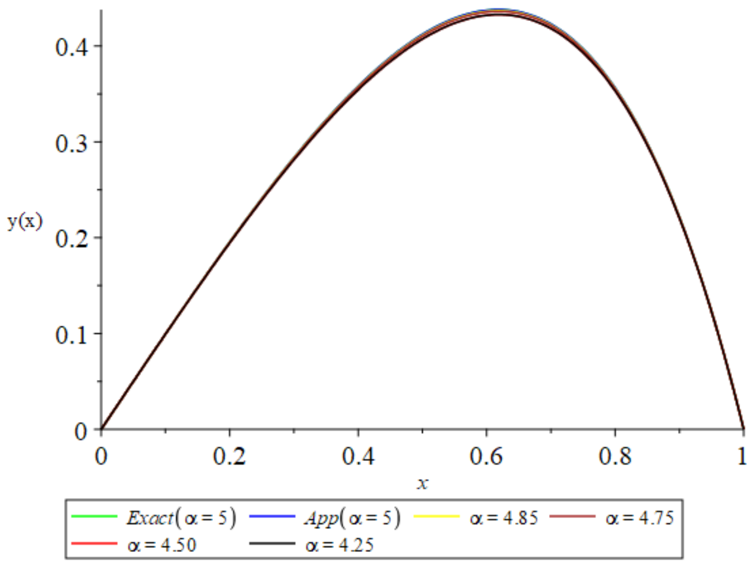

Figure 5: (LITM) fractional solution graphs of Example 4.3.

Table 6: Error Analysis of the fractional solutions of Example 4.3 at different value of α at $(M = 8)$

x_i	A.E($\alpha = 5$)	A.E($\alpha = 4.85$)	A.E($\alpha = 4.75$)	A.E($\alpha = 4.50$)	A.E($\alpha = 4.25$)	$CWMError(M = 50)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	$1.91E-100$
0.1	$3.03E-8$	$2.77E-5$	$4.74E-5$	$1.45E-4$	$1.45E-4$	$5.04E-69$
0.2	$1.33E-7$	$1.72E-4$	$2.94E-4$	$8.67E-4$	$8.67E-4$	$3.58E-68$
0.3	$2.37E-7$	$4.41E-4$	$7.48E-4$	$2.13E-3$	$2.13E-3$	$1.05E-67$
0.4	$3.11E-7$	$7.61E-4$	$1.28E-3$	$3.59E-3$	$3.59E-3$	$2.14E-67$
0.5	$3.46E-7$	$1.02E-3$	$1.72E-3$	$4.75E-3$	$4.75E-3$	$3.48E-67$
0.6	$3.46E-7$	$1.12E-3$	$1.89E-3$	$5.16E-3$	$5.16E-3$	$4.79E-67$
0.7	$2.56E-7$	$1.01E-3$	$1.68E-3$	$4.55E-3$	$4.55E-3$	$5.68E-67$
0.8	$1.59E-7$	$6.66E-4$	$1.11E-3$	$2.99E-3$	$2.99E-3$	$5.58E-67$
0.9	$6.63E-8$	$2.37E-4$	$3.96E-4$	$1.05E-3$	$1.05E-3$	$3.83E-67$
1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	$3.26E-100$

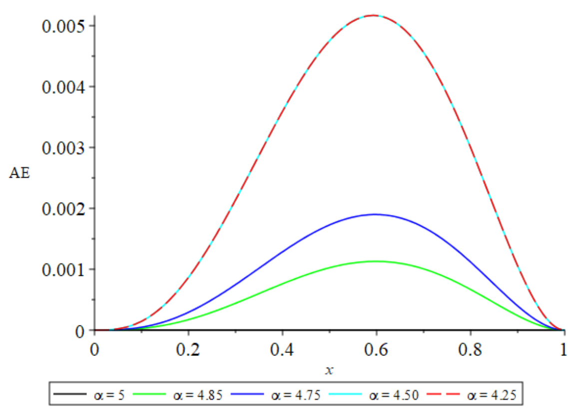

Figure 6: Error Analysis graph of Example 4.3.

5. Results and Discussion

In this section, the (LITM) solutions for Examples 4.1, 4.2 and 4.3 are discussed. The Tables and Figures are constructed to highlight the obtained results in a sophisticated manner. In this connection, Table 1 is constructed for (LITM) and exact solutions at different values of the fractional order α of Example 4.1. In Figure 1, the graphs of (LITM) results at various α are plotted. The fractional order solutions have shown the closed relationship with the actual dynamics of the problems and verified by using different plots of Example 4.1. The error analysis of the suggested technique is shown in Table 2. The tables have justified the increase in accuracy at various α levels towards integer order solutions. Similarly, the error graphs are plotted for various α values, and convergence has been confirmed for fractional order solutions. Table 3 represent the (LITM) solutions at various fractional orders α, as well as the exact solution to Example 4.2. The same results are shown by using Figure 3, which has the same convergence rate as given for Example 4.2. The error analysis of problem 4.2 is given in Table 4 which has confirmed the sufficient degree of accuracy for Example 4.2. The error graph for Example 4.2 is plotted in Figure 4 and has shown the valuable dynamics at different values of α. In Table 5 the (LITM) and exact results are displayed at different fractional orders of Example 4.3. In Figure 5 the fractional order results are displayed and analyzed for the closed relation. Table 6 included the (LITM) error analysis where the error varies with α and converges to the error at the integer order derivative. Figure 6 shows the error graphs for Example 4.3 at different fractional orders of α. The error changes at different α and converges to the error at integer order derivative. Again, a very consistent dynamical behavioral is observed for Example 4.3.

6. Conclusion

In this work, a new numerical technique based on the Lagrange Interpolation Polynomial is introduced. The new method is used for the solution of different fractional boundary value problems and is found to be an accurate and effective technique for the solution of problems. The solutions, graphs, and tables are presented for the validation of (LITM). The (LITM) solutions are compared with other numerical techniques such as (CWM) and (OHAM) and found to be very efficient and effective. The problems that are described in tabular form can be addressed in a better way by using (LITM). Furthermore, the present method can be modified for other nonlinear fractional problems in nature by using various interpolation techniques.

Table 7: Nomenclature

FPDEs	Fractional partial differential equations
LITM	Lagrange Interpolation Transform method
FC	Fractional calculus
VIM	Variational iteration method
FDEs	Fractional differential equations
HPTM	Homotopy perturbation transform method
OHAM	Optimal Homotopy asymptotic method
ADM	Adomian decomposition method
FDM	Finite difference method
DTM	Differential transform method
BVPs	Boundary value problems
CWM	Chebyshev wavelete method
BCs	boundary conditions
AE	Absolute Error

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