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Novel Dual Method Approach for Solving Third Order Pseudo Hyperbolic PDEs Using Variational Iteration and Group Preserving Schemes

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ABSTRACT: The study explores advanced numerical techniques for solving pseudo-hyperbolic equations through an innovative computational approach integrating variational iteration methodology, symmetry-preserving numerical schemes, and a novel fictitious time discretization strategy. By transforming the original mathematical problem into an alternative formulation using fictitious time integration, we develop a robust computational framework that enhances numerical stability and convergence properties. The proposed method leverages group-preserving transformation techniques to maintain critical mathematical symmetries throughout the numerical solution process. Comprehensive numerical experiments are conducted across diverse test scenarios to validate the method's performance and computational efficiency. Rigorous analysis demonstrates the approach's significant advantages, including rapid convergence, high accuracy, and exceptional adaptability to complex pseudo-hyperbolic systems. The research contributes novel insights into advanced computational mathematics, offering a sophisticated alternative to traditional numerical solution strategies for challenging partial differential equations. Detailed numerical simulations substantiate the method's effectiveness, revealing its potential for addressing intricate mathematical modeling challenges in various scientific and engineering domains.

Key Words: Pseudo-hyperbolic PDEs, group preserving scheme, variational iteration method, Lagrange multiplier, exact solution.

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1. Introduction

Partial differential equations (PDEs) serve as powerful mathematical tools for modeling and understanding complex physical phenomena. These sophisticated mathematical representations capture intricate dynamics across various scientific domains, including wave propagation, thermal transfer, quantum interactions, and fluid mechanics [1,2,3,4,5]. High-order PDEs, particularly pseudo-hyperbolic equations with mixed time-space derivatives, represent critical mathematical models in physics and engineering research. Extensive scholarly investigations have explored numerous aspects of these equations, ranging from numerical solution methodologies [1] to stability analyses [4], solution existence theorems [2], and analytical approaches [3]. Researchers have systematically examined the theoretical and practical implications of such equations, demonstrating their significance in modeling structural vibrations, atomic physics [6,7,8,9], and complex engineering challenges [10,11,12]. Innovative computational techniques have been developed to approximate solutions, with scholars progressively refining algorithmic approaches to enhance mathematical precision. Recent scholarly work has made substantial contributions to understanding boundary value problems, such as periodic boundary conditions [13] and integral conditions for differential equation systems [14]. Investigations have also explored semi-periodic boundary value problems for non-classical differential equations [15], expanding our comprehension of complex mathematical models. The ongoing exploration of pseudo-hyperbolic and related partial differential equations

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continues to push the boundaries of mathematical physics, offering increasingly sophisticated tools for scientific and engineering applications. The authors of [16] considered the initial-boundary value problem for the nonlinear Klein-Gordon equation's inverse problem of recovering a time-dependent coefficient multiplying nonlinear term, and then the finite difference method was also used to numerically solve the inverse problem.

The Variational Iteration Method (VIM), introduced by He and colleagues, offers a robust approach for generating consecutive approximations to precise solutions across various mathematical domains. This powerful technique has demonstrated remarkable effectiveness in deriving analytical solutions for complex linear and nonlinear partial differential equations. Researchers have successfully applied VIM to a diverse range of mathematical challenges, including intricate problems such as nonlinear Voltera's Integro-differential equations, Burger's equations, autonomous ordinary differential systems, the Helmholtz equation, Telegraph Equation, nonlinear Jaulent–Miodek equations, and nonlinear pseudo-hyperbolic partial differential equations [17,18,19,20,21,22,23,24]. Complementing VIM, the Group Preserving Scheme (GPS) represents an innovative geometric methodology formulated within Minkowski space, diverging from conventional numerical approaches typically grounded in Euclidean space. GPS has emerged as a versatile computational technique, enabling researchers to tackle complex mathematical problems. Notable applications include solving the nonlinear nonlinear fractional telegraph equations in electromagnetics [25], Through numerical validation, the study demonstrated that TSM and NHPM accurately anticipate infection patterns and efficiently represent the dynamics of Zika virus transmission [26], to better understand the structural aspects of complicated order univalent functions, the study developed criteria for Mittag-Leffler-type Poisson distribution series to fall into particular subclasses [27], and developing numerical strategies for space- and time-fractional derivative Burgers equations [28]. This research integrates the Variational Iteration Method and Group Preserving Scheme to derive solutions for third-order nonhomogeneous linear pseudo-hyperbolic partial differential equations. The manuscript is structured to systematically explore these methodological approaches, with subsequent sections dedicated to detailed mathematical explanations, computational implementations, and analytical demonstrations using Maple software. The paper's organizational framework includes: a comprehensive introduction to VIM for linear pseudo-hyperbolic partial differential equations, an examination of the fictitious time integration method's role, practical implementation of computational techniques, and a concluding section synthesizing the study's primary findings and contributions.

2. General Solution Form of the Proposed Problem via VIM

This part explores the general form of the Variational Iteration Method (VIM) to address third-order linear pseudo-hyperbolic partial differential equations under non-local boundary conditions. The study focuses on a generalized mathematical framework that captures the complex dynamics of resistance-related phenomena. The proposed approach considers a comprehensive model defined by the following partial differential equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^3 u}{\partial x \partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) - h(t,x) = 0, \quad (0 < x < X), \quad (0 < t < T), \tag{2.1}$$

accompanied by intricate initial and boundary conditions:

$$u(x,0) = W_0(x), \quad \frac{\partial u}{\partial t}(x,0) = W_1(x), \quad 0 \le x \le X, \tag{2.2}$$

$$u(0,t) = A(t) + \int_0^X u(x,t)dx, \quad 0 \le t \le T,$$

$$u(X,t) = B(t) + \int_0^X u(x,t)dx, \quad 0 \le t \le T,$$

where $u(t,x)$ is the resistance and $\mu > 0$.

To illustrate the basic structure of the variational iteration method [20], for Eq. (2.1), we consider the following formula:

$$L[u(x,t)] + N[u(x,t)] = h(x,t), \tag{2.3}$$

where $h(x,t)$ is given continuous function, L is a linear operator and N is a nonlinear operator. The correction functional in the t-direction can be used to write the solution of Eq. (2.1) using the variational iteration method

$$u_{n+1}(x,t) \quad = \quad u_n(x,t) + \int_0^t \lambda(\sigma)\Big[L[u(x,\sigma)] + N[u(x,\sigma)] - f(x,\sigma)\Big]\partial\sigma. \tag{2.4}$$

From Eq. (2.1), we know $L[u(x,t)] = \frac{\partial^2 u}{\partial t^2}(x,t) - \mu\frac{\partial^3 u}{\partial t\partial x^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t)$ and $N = 0$. The differential equation's solution is regarded as the fixed point when the initial term is properly chosen, making the correction functional stationary. As a result, the solution to Eq. (2.4) will be as follows:

$$\begin{aligned} u_{n+1}(x,t) \quad = \quad & u_n(x,t) + \int_0^t \lambda(\sigma)\Bigg[\frac{\partial^2 u_n(x,\sigma)}{\partial\sigma^2} - \mu\frac{\partial}{\partial\sigma}\left(\frac{\partial^2 u_n(x,\sigma)}{\partial x^2}\right) \\ & -\frac{\partial^2 u_n(x,\sigma)}{\partial x^2} - f(x,\sigma)\Bigg]\partial\sigma. \end{aligned} \tag{2.5}$$

By employing the integration by parts method, we initially establish the Lagrange $\lambda(\sigma)$ multiplier, which is systematically determined through a precise optimization process. Subsequently, the iterative approximations $u_n(x,t), n > 0$ for the target solution $u(x,t)$ can be efficiently derived by leveraging the computed Lagrange multiplier in conjunction with a carefully selected reference function $u_0(x,t)$, yielding the following result:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.6}$$

As demonstrated in the work of [15], the Variational Iteration Method (VIM) applied to Eq. (2.1) can be conclusively formulated as follows

$$\begin{cases} u(x,0) = W_0(x) + tW_1(x), \quad \text{initial guess} \\ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\sigma)\Bigg[\frac{\partial^2 u_n(x,\sigma)}{\partial\sigma^2} - \mu\frac{\partial}{\partial\sigma}\left(\frac{\partial^2 u_n(x,\sigma)}{\partial x^2}\right) \\ -\frac{\partial^2 u_n(x,\sigma)}{\partial x^2} - f(x,\sigma)\Bigg]\partial\sigma. \end{cases} \tag{2.7}$$

The Lagrange $\lambda(\sigma)$ multiplier should be defined in the next section.
In the next section, an application of a geometric method can be seen.

3. The Fictitious time integration method(FTIM) and Group Preserving Scheme(GPS)

This section is devoted to showing the usage of FTIM and GPS. First, using FTIM we convert the main equation into a new equation with one more dimension. After that, GPS is applied to solve the obtained equation.
Thus, we reach:

$$\zeta\frac{\partial^2 u}{\partial t^2}(t,x) - \zeta\mu\frac{\partial^3 u}{\partial x\partial t^2}(t,x) - \zeta\frac{\partial^2 u}{\partial x^2}(t,x) = \zeta h(x,t). \tag{3.1}$$

Completing the later alteration on Eq. (3.1)

$$\Sigma(x,t,\sigma) = (1+\sigma)^\kappa u(x,t),\ 0 < \kappa \le 1, \tag{3.2}$$

presents us:

$$\frac{\zeta}{(1+\sigma)^{\kappa}}\left[\frac{\partial^2\Sigma}{\partial t^2}(x,t,\sigma)-\mu\frac{\partial^3\Sigma}{\partial t\partial x^2}(x,t,\sigma)-\frac{\partial^2\Sigma}{\partial x^2}(x,t,\sigma)\right]-\zeta h(x,t)=0, \tag{3.3}$$

with regard to

$$\frac{\partial\Sigma}{\partial\sigma}=\kappa(1+\sigma)^{\kappa-1}u(x,t). \tag{3.4}$$

For Eq. (3.3), we reach:

$$\frac{\partial\Sigma}{\partial\sigma}=\frac{\zeta}{(1+\sigma)^{\kappa}}\Big[\Sigma_{tt}(x,t,\sigma)-\mu\Sigma_{txx}(x,t,\sigma)-\Sigma_{xx}(x,t,\sigma)\Big]-\zeta h(x,t) \tag{3.5}$$
$$+\kappa(1+\sigma)^{\kappa-1}u.$$

By taking $u=\Sigma/(1+\sigma)^{\kappa}$, we get:

$$\frac{\partial\Sigma}{\partial\sigma}=\frac{\zeta}{(1+\sigma)^{\kappa}}\Big[\Sigma_{tt}(x,t,\sigma)-\mu\Sigma_{txx}(x,t,\sigma)-\Sigma_{xx}(x,t,\sigma)\Big]-\zeta h(x,t) \tag{3.6}$$
$$+\frac{\kappa\Sigma}{\kappa+\sigma}.$$

Using

$$\frac{\partial}{\partial\sigma}\left(\frac{\Sigma}{(1+\sigma)^{\kappa}}\right)=\frac{\Sigma_{\sigma}}{(1+\sigma)^{\kappa}}-\frac{\kappa\Sigma}{(1+\sigma)^{1+\kappa}}, \tag{3.7}$$

and by using $1/(1+\sigma)^{\kappa}$ on Eq. (3.6), one gets

$$\frac{\partial}{\partial\sigma}\left(\frac{\Sigma}{(1+\sigma)^{\kappa}}\right)=\frac{\zeta}{(1+\sigma)^{\kappa}}\Big[\Sigma_{tt}(x,t,\sigma)-\mu\Sigma_{txx}(x,t,\sigma)-\Sigma_{xx}(x,t,\sigma)\Big] \tag{3.8}$$
$$-\zeta h(x,t).$$

By using $u=\frac{\Sigma}{(1+\sigma)^{\kappa}}$, we have:

$$u_{\sigma}=\frac{\zeta}{(1+\sigma)^{\kappa}}\Big[u_{tt}(x,t,\sigma)-\mu u_{txx}(x,t,\sigma)-u_{xx}(x,t,\sigma)\Big]-\zeta h(x,t). \tag{3.9}$$

Suppose $u_i^j(\sigma):=u(x_i,t_j,\sigma)$, $\Delta x=\frac{b-a}{m}$, $\Delta t=\frac{T}{n}$, $x_i=a+i\Delta x$ and $t_j=j\Delta t$. After that, we have Eq.(3.9) as:

$$\begin{aligned}\frac{d}{d\sigma}u_i^j(\sigma)=&\ \frac{\zeta}{(1+\sigma)^{\kappa}}\Bigg[\frac{u_i^{j+1}(\sigma)-2u_i^j(\sigma)+u_i^{j-1}(\sigma)}{\Delta t^2}-\frac{u_{i+1}^j(\sigma)-2u_i^j(\sigma)+u_{i-1}^j(\sigma)}{\Delta x^2}\\&-\frac{u_{i+1}^{j+1}(\sigma)-u_{i+1}^j(\sigma)-2u_i^{j+1}(\sigma)+2u_i^j(\sigma)+u_{i-1}^{j+1}(\sigma)-u_{i-1}^j(\sigma)}{2\Delta t\Delta x^2}\Bigg]\\&-\zeta h(x_i,t_j).\end{aligned} \tag{3.10}$$

If we take

$$\mathbf{u}=(u_1^1,u_1^2,...,u_m^n)^T, \tag{3.11}$$

and

$$\begin{aligned}\mathbf{E}(\mathbf{u},\sigma)=&\ \frac{\zeta}{(1+\sigma)^{\kappa}}\Bigg[\frac{u_i^{j+1}(\sigma)-2u_i^j(\sigma)+u_i^{j-1}(\sigma)}{\Delta t^2}-\frac{u_{i+1}^j(\sigma)-2u_i^j(\sigma)+u_{i-1}^j(\sigma)}{\Delta x^2}\\&-\frac{u_{i+1}^{j+1}(\sigma)-u_{i+1}^j(\sigma)-2u_i^{j+1}(\sigma)+2u_i^j(\sigma)+u_{i-1}^{j+1}(\sigma)-u_{i-1}^j(\sigma)}{2\Delta t\Delta x^2}\Bigg]\\&-\zeta h(x_i,t_j).\end{aligned} \tag{3.12}$$

Then eq.(3.10) can be writen as :

$$\mathbf{u}' = \mathbf{E}(\mathbf{u}, \sigma), \ \mathbf{u} \in \mathbb{R}^N, \sigma \in \mathbb{R}, \tag{3.13}$$

where $\mathbf{E}$ symbolizes a vector including ij-components holding the right-hand side of Eq. (3.10) and $N = m \times n$ is the quantity of the entire point.
Now, we can use a geometric numerical method called Group-Preserving Scheme (GPS) introduced by Liu [9] for solving Eq. (3.9):

$$\mathbf{u}_{l+1} = \mathbf{u}_l + \frac{(\Upsilon_l - 1)\mathbf{E}_l.\mathbf{u}_l + \Lambda_l \|\mathbf{u}_l\| \|\mathbf{E}_l\|}{\|\mathbf{E}_l\|^2} \mathbf{E}_l = \mathbf{u}_l + \Pi_l \mathbf{E}_l, \tag{3.14}$$

where

$$\Pi_l = \frac{(\Upsilon_l - 1)\mathbf{E}_l.\mathbf{u}_l + \Lambda_l \|\mathbf{u}_l\| \|\mathbf{E}_l\|}{\|\mathbf{E}_l\|^2}, \tag{3.15}$$

$$\begin{aligned} \Upsilon_l &= \cosh\left(\frac{\Delta\sigma \|\mathbf{E}_l\|}{\|\mathbf{u}_l\|}\right), \\ \Lambda_l &= \sinh\left(\frac{\Delta\sigma \|\mathbf{E}_l\|}{\|\mathbf{u}_l\|}\right). \end{aligned} \tag{3.16}$$

We can check the convergence of u_i^j at the l and $l+1$ measures by the next rule:

$$\sqrt{\sum_{i,j=1}^{m,n} [\mathbf{u}_i^j(l+1) - \mathbf{u}_i^j(l)]^2} \le \varepsilon, \tag{3.17}$$

where ε is the convergence criterion. Then, we acquire:

$$\mathbf{u}_i^j = \frac{\mathbf{u}_i^j(\sigma_0)}{(1+\sigma_0)^l}, \tag{3.18}$$

where $\sigma_0(\le \sigma_h)$ contents at Eq. (3.18).

4. Applications of the methods

In this section, we delve into the practical implementation and comprehensive exploration of advanced analytical techniques specifically tailored to examining and resolving complex third-order linear pseudo-hyperbolic partial differential equations. Our approach encompasses a systematic methodology for applying sophisticated mathematical methods to unravel the intricate dynamics and structural characteristics of these challenging mathematical models.

Example. Consider the non-homogeneous linear pseudo-hyperbolic partial differential equations [3] as follow:

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^3 u}{\partial x \partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) - x^3 e^t + 18xe^t = 0, \quad x \in (0,1), \quad t \in (0,2), \tag{4.1}$$

with the initial and boundary conditions

$$u(x,0) = x^3, \quad \frac{\partial u}{\partial t}(x,0) = x^3, \quad x \in [0,1], \tag{4.2}$$

$$u(0,t) = \int_0^1 u(x,t)dx - \frac{1}{4}e^t, \quad t \in [0,2], \tag{4.3}$$

$$u(1,t) = \frac{3}{4}e^t + \int_0^1 u(x,t)dx, \quad t \in [0,2]. \tag{4.4}$$

The corrective function of a given problem via the VIM is approximately obtained by the following formulas

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\sigma)\left[\frac{\partial^2 u_n(x,\sigma)}{\partial\sigma^2} - \mu\frac{\partial}{\partial\sigma}\left(\frac{\partial^2 u_n(x,\sigma)}{\partial x^2}\right) - \frac{\partial^2 u_n(x,\sigma)}{\partial x^2} - f(x,\sigma)\right]\partial\sigma. \tag{4.5}$$

Within this framework, $\lambda(\sigma)$ serves as the Lagrange multiplier parameter. The precise value of this multiplier emerges naturally from variational calculus methods, enabling us to find the optimal solution that satisfies both our objective function and constraints simultaneously.

$$\lambda(\sigma) = \sigma - t.$$

Putting the Lagrange multiplier value in (30), we get

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\sigma - t)\left[\frac{\partial^2 u_n(x,\sigma)}{\partial\sigma^2} - 2\frac{\partial}{\partial\sigma}\left(\frac{\partial^2 u_n(x,\sigma)}{\partial x^2}\right) - \frac{\partial^2 u_n(x,\sigma)}{\partial x^2} - x^3e^t + 18xe^t\right]\partial\sigma.$$

The initial approximation $u_0(x,t)$ can be derived directly from the specified initial condition presented in Eq. (2.7), as demonstrated in the Variational Iteration Method (VIM) framework [21]. This serves as the foundation for constructing the iterative sequence. The subsequent approximations in the series can be systematically generated through the following iterative process, where each new term builds upon the previous estimates while incorporating the method's correction functional.

$$u_0(x,t) = (1+t)x^3, \tag{4.6}$$

$$u_1(x,t) = (1+t)x^3 + \int_0^t (\sigma - t)\left[\frac{\partial^2 u_n(x,\sigma)}{\partial\sigma^2} - 2\frac{\partial}{\partial\sigma}\left(\frac{\partial^2 u_n(x,\sigma)}{\partial x^2}\right) - \frac{\partial^2 u_n(x,\sigma)}{\partial x^2} - x^3e^t + 18xe^t\right]\partial\sigma,$$

simplify the above equation, we get

$$u_1(x,t) = x^3e^t + xt^3 + 9xt^2 - 18xe^t + 18xt + 18x, \tag{4.7}$$

using this procedure for $u_2(x,t)$, $u_3(x,t)$, $u_4(x,t)$,..., we get

$$u_n(x,t) = x^3e^t, \quad \text{for} \quad n \geq 2. \tag{4.8}$$

By using Eq. (2.6), we get the exact solution

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x^3e^t. \tag{4.9}$$

The graphical representation of the solution at $x \in [0,1]$, and $t \in [0,2]$ are as follow:

As can be seen from the graphical representation of the solutions, figures (1) and (2), provided the exact solutions to Eq. (4.1), at $0 \leq t \leq 2$, $0 \leq x \leq 1$, and $0 \leq t \leq 100$, $-10 \leq x \leq 10$, respectively.

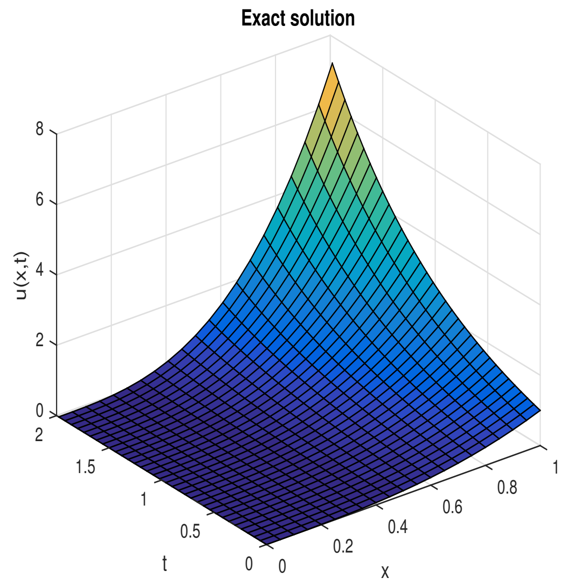


Figure 1: Illustrate the exact solution of pseudo-hyperbolic equations at $0 \leq t \leq 2$ and $0 \leq x \leq 1$

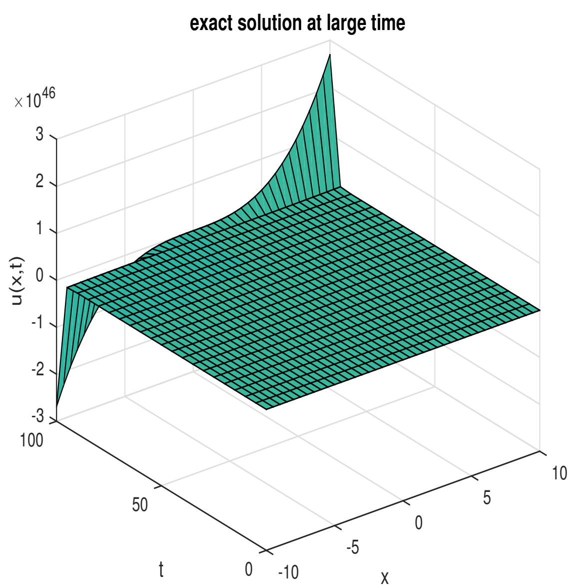
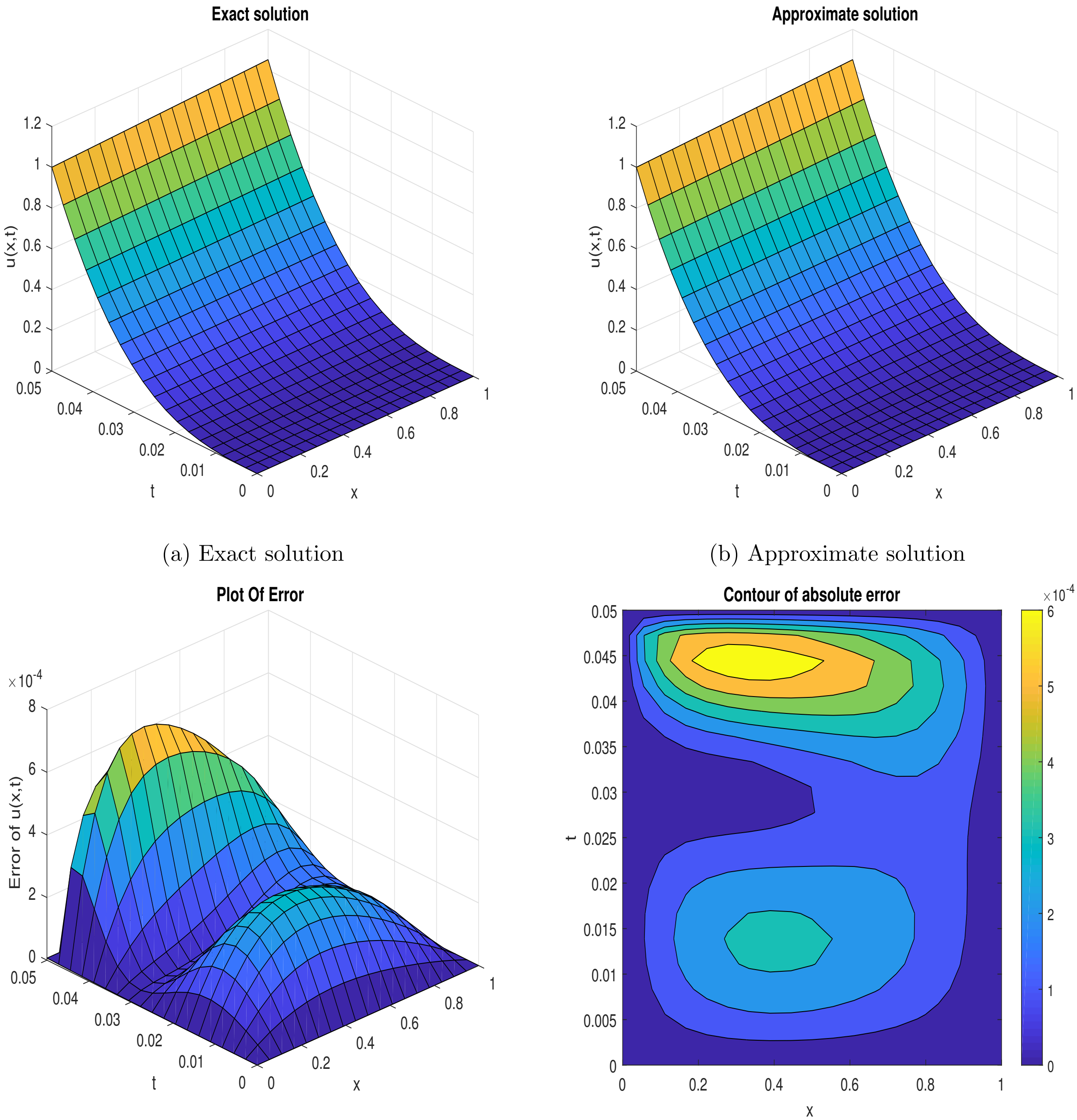


Figure 2: Simulation of the $u(x,t)$ at $0 \leq t \leq 100$



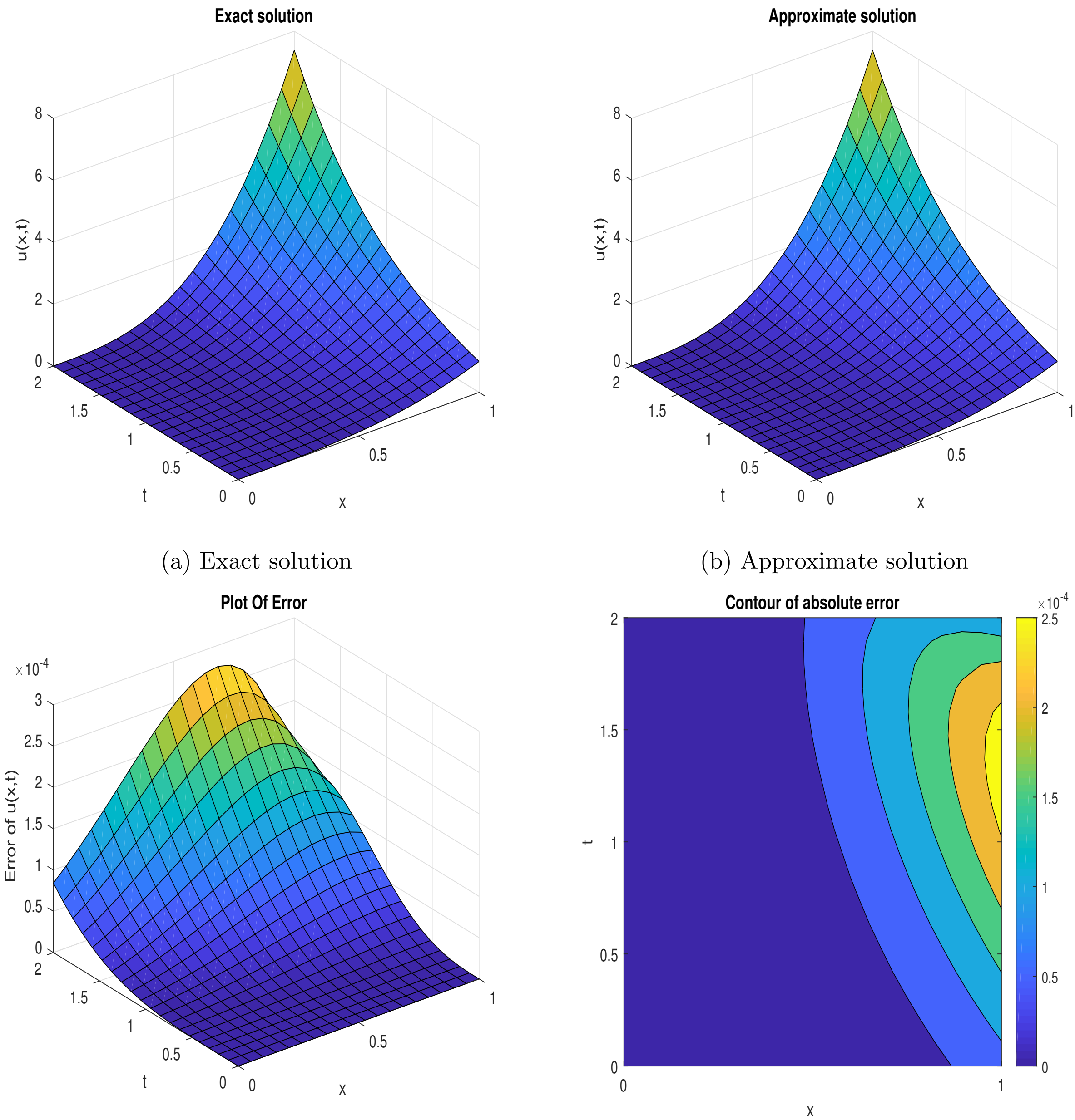
(a) Exact solution

(b) Approximate solution

(c) Error distribution

(d) Convergence error

Figure 3: Numerical results for $\zeta = 220$ and $u_i^j(0) = 1e - 7$ of Eq. (4.1).



(a) Exact solution

(b) Approximate solution

(c) Error distribution

(d) Convergence error

Figure 4: Numerical results for $\zeta = 150$ and $u_i^j(0) = 1e - 5$ of Eq. (4.1).

Figure 3 shows the error, exact and numerical solutions of the problem (4.1) by GPS choosing $T = 0.05, m = n = 19, \zeta = 220$, $\kappa = 1$ and $u_i^j(0) = 1e-7$ and $\Delta\sigma = 5 \times 1e-9$. Also, Figure 4 is responsible to carry out the solutions and error using GPS under $T = 1, m = n = 20, \zeta = 150$, $\kappa = 0.011$ and $u_i^j(0) = 1e-3$ and $\Delta\sigma = 5 \times 1e-3$ in different domain.

The graphics that are displayed offer a thorough examination of pseudo-hyperbolic partial differential equations by means of meticulously crafted visuals. In figure (1), the precise solution is shown throughout the normalized domain $0 \le t \le 2$ and $0 \le x \le 1$, demonstrating the complex behavior of the solution at various computational stages. Figure (2) reveals the long-term dynamics and possible asymptotic properties of the equation by expanding the simulation of the function $u(x,t)$ over a wider time span $0 \le t \le 100$. Parametric investigations are the subject of figures (3) and figures (4), which examine how the equation responds to different values of ζ (220 and 150, respectively) and beginning conditions $u_i^j(0) = 1e-7$ and $1e-5$. Important information about the numerical stability, convergence characteristics, and overall behavior of the suggested solution methodology for pseudo-hyperbolic partial differential equations is provided by these figures, which also show how sensitive the method is to initial parameters.

5. Conclusion

In this work, the VIM method has been successfully used to obtain the solutions of nonhomogeneous third-order linear pseudo-hyperbolic partial differential equations. The obtained solution via the present method is an exact solution. The results offered in this contribution show that the VIM method is extremely robust and effective in finding solutions for the class of linear pseudo-hyperbolic equations. Also, a combination of GPS and FTIM is proposed to get the approximate solutions to this problem. Figures of solutions are provided successfully during this study. The results can be cleared that the suggested methods have instant convergence. The proposed methods can be used and expanded to solve the fractional order pseudo-hyperbolic partial differential equations.

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