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Surpassing Taylor Method: Generalized Taylor Method for Solving Initial Value Problems

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ABSTRACT: This study presents an advancement in the Taylor method through the development of an enhanced, higher-order variant that accelerates series expansion, yielding a refined, implicit formulation with improved accuracy. Both stability and convergence properties are thoroughly analyzed. While this method demonstrates enhanced efficiency in comparison to conventional Taylor and fourth-order Runge-Kutta (RK4) methods, the improvements are presented as relative advancements rather than definitive superiority. This refined approach provides a valuable addition to numerical methods for solving initial value problems with greater precision and reliability.

Key Words: Generalized Taylor metod, Taylor method, Runge-Kutta method, approximations

Contents

1. Introduction

Numerical methods play a crucial role in approximating solutions to Initial Value Problems (IVPs) in differential equations. This study examines two primary approaches—the Taylor and Runge-Kutta (RK) methods—focusing on their effectiveness in balancing accuracy and computational efficiency [1,2,3,4,5,6].

Higher-order Taylor methods are known for their exceptional accuracy in solving IVPs by leveraging the Taylor series expansion. However, while increasing the order enhances precision, it also significantly raises computational costs. On the other hand, RK methods, such as the fourth-order Runge-Kutta (RK4), offer reliable accuracy with fewer derivative evaluations, making them more computationally efficient for many applications. The choice between Taylor and RK methods depends on the desired level of accuracy and the system's complexity. Readers interested in recent advancements in this field can refer to works such as [7,8,9,10,11,12,13,14], and for further extensions or generalizations, see [15,16,17,18].

In this work, we advance the Taylor method by introducing an enhanced, higher-order version that accelerates series expansion, yielding a corrected explicit method with improved accuracy. The stability and convergence properties of this method are thoroughly analyzed. While it demonstrates greater efficiency compared to conventional Taylor and RK4 methods, these improvements are emphasized as comparative rather than absolute. Numerical experiments are conducted to validate the effectiveness of this enhanced Taylor approach, with direct comparisons to traditional Taylor and RK methods. The results illustrate the method's practical advantages, reinforcing its role as an efficient and accurate alternative for solving IVPs.

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2. The Generalized Taylor Method

Throughout this work, let I be a real interval, and let $a, b \in I^{\circ}$ (the interior of I) with $a < b$. Suppose that $g^{(m+1)}$ is continuous on I. In [19, Remark 2], the first author of this paper proved that

$$g(x) = g(a) + \sum_{k=1}^{m} \frac{1}{k!} \left[(s-a)^k g^{(k)}(a) - (s-x)^k g^{(k)}(x) \right] + \frac{1}{m!} \int_a^x (s-t)^m g^{(m+1)}(t)\, dt, \tag{2.1}$$

for all $a \le s \le x$. This formulation expands g near two points instead of one. Moreover, by setting $s = x$, we recover the well-known Taylor series expansion. Alternatively, choosing $s = \frac{pa+qx}{p+q}$ for $p, q \ge 0$ (not both zero), we obtain

$$g(x) = g(a) + \sum_{k=1}^{m} \frac{(x-a)^k}{(p+q)^k k!} \left[p^k g^{(k)}(a) + (-1)^{k+1} q^k g^{(k)}(x) \right] + \frac{1}{m!} \int_a^x \left(\frac{pa+qx}{p+q} - t \right)^m g^{(m+1)}(t)\, dt. \tag{2.2}$$

For further details on this construction, see [19]. For simplicity, we rewrite (2.2) as:

$$g(x) = g(a) + \sum_{k=1}^{m} \frac{(-1)^{k-1} (x-a)^k}{(p+q)^k k!} \left[q^k g^{(k)}(x) + (-1)^{k-1} p^k g^{(k)}(a) \right] + \mathcal{R}_m(g), \tag{2.3}$$

where

$$\mathcal{R}_m(g) := \frac{(x-a)^{m+1}}{(p+q)^m m!} \int_0^1 (q - (p+q)u)^m g^{(m+1)}(a + u(x-a))\, du.$$

Clearly, by setting $p = 1$ and $q = 0$, equation (2.3) reduces to the standard m-th Taylor polynomial.

The derivation of these formulas, along with others, follows directly from (2.1), significantly extending its utility and applicability. This provides a valuable and versatile framework that addresses the growing need for comprehensive and advanced extensions of such foundational equations.

We now formally introduce the Generalized Taylor method, designed to approximate the solution of the well-posed initial-value problem:

$$\frac{dz}{ds} = g(s, z), \qquad a \le s \le b, \qquad z(a) = \alpha. \tag{2.4}$$

Let us assume that the solution $z(s)$ to this initial-value problem possesses $(m+1)$ continuous derivatives. Expanding $z(s)$ in terms of its m-th Generalized Taylor polynomial—where $z^{(k)}(s_{\ell+1})$ in (2.3) can be re-expressed using classical Taylor polynomials about s_ℓ and evaluated at $s_{\ell+1}$—yields

$$\begin{aligned} z(s_{\ell+1}) = z(s_\ell) &+ \sum_{k=1}^{m} \frac{(-1)^{k-1} (s_{\ell+1} - s_\ell)^k}{(p+q)^k k!} \left[q^k z^{(k)}(s_{\ell+1}) + (-1)^{k-1} p^k z^{(k)}(s_\ell) \right] \\ &+ \frac{(s_{\ell+1} - s_\ell)^{m+1}}{(p+q)^m m!} \cdot z^{(m+1)}(s_\ell + \zeta_\ell (s_{\ell+1} - s_\ell)) \int_0^1 (q - (p+q)u)^m\, du \end{aligned}$$

for some $\zeta_\ell \in (s_\ell, s_{\ell+1})$. To ensure an even distribution of mesh points across the interval $[a, b]$, we choose a positive integer N and define the mesh points as:

$$s_\ell = a + \ell h, \qquad \text{for } \ell = 0, 1, 2, \ldots, N.$$

Here, $h = \frac{b-a}{N} = s_{\ell+1} - s_\ell$ represents the step size, ensuring uniform spacing between consecutive points.

Suppose that the unique solution to (2.4) possesses $(m+1)$ continuous derivatives on the interval $[a, b]$. For each $\ell = 0, 1, 2, \ldots, N-1$, it follows that

$$\begin{aligned} z(s_{\ell+1}) = z(s_\ell) &+ \sum_{k=1}^{m} \frac{(-1)^{k-1} h^k}{(p+q)^k k!} \left[q^k z^{(k)}(s_{\ell+1}) + (-1)^{k-1} p^k z^{(k)}(s_\ell) \right] \\ &+ \frac{h^{m+1}}{(p+q)^m m!} \cdot z^{(m+1)}(s_\ell + \zeta_\ell h) \int_0^1 (q - (p+q)u)^m\, du \end{aligned} \tag{2.5}$$

for some $\zeta_\ell \in (s_\ell, s_{\ell+1})$. Since $z(s)$ satisfies the differential equation (2.5), it can be successively differentiated, yielding

$$z'(s) = g(s, z(s)), \quad z''(s) = g'(s, z(s)), \quad \ldots, \quad z^{(k)}(s) = g^{(k-1)}(s, z(s)).$$

Substituting these derivatives into (2.5) gives

$$\begin{aligned} z(s_{\ell+1}) = z(s_\ell) &+ \sum_{k=1}^{m} \frac{(-1)^{k-1}h^k}{(p+q)^k k!} \left[q^k g^{(k-1)}(s_{\ell+1}, z(s_{\ell+1})) + (-1)^{k-1} p^k g^{(k-1)}(s_\ell, z(s_\ell)) \right] \\ &+ \frac{h^{m+1}}{(p+q)^m m!} \cdot z^{(m+1)}(s_\ell + \zeta_\ell h) \int_0^1 (q - (p+q)u)^m du, \end{aligned} \tag{2.6}$$

for some $\zeta_\ell \in (s_\ell, s_{\ell+1})$. The corresponding difference equation derived from (2.6) is obtained by neglecting the remainder term involving ζ_ℓ:

$$\begin{aligned} \psi_0 &= \alpha, \\ \psi_{\ell+1} &= \psi_\ell + h\, \mathrm{G}^{(m)}(\psi_\ell, \psi_{\ell+1}), \end{aligned} \tag{2.7}$$

for each $\ell = 0, 1, 2, \ldots, N-1$, where

$$\mathrm{G}^{(m)}(\psi_\ell, \psi_{\ell+1}) := \sum_{k=1}^{m} \frac{(-1)^{k-1}h^{k-1}}{(p+q)^k k!} \left[q^k g^{(k-1)}(s_{\ell+1}, \psi_{\ell+1}) + (-1)^{k-1} p^k g^{(k-1)}(s_\ell, \psi_\ell) \right].$$

In particular, the case where $m = 4$ is of interest, with the corresponding difference equation given by:

$$\begin{aligned} \psi_0 &= \alpha \\ \psi_{\ell+1} &= \psi_\ell + h\, \mathrm{G}^{(4)}(\psi_\ell, \psi_{\ell+1}) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathrm{G}^{(4)}(\psi_\ell, \psi_{\ell+1}) := &\frac{1}{p+q} \left[qg(s_{\ell+1}, \psi_{\ell+1}) + pg(s_\ell, \psi_\ell) \right] - \frac{h}{2(p+q)^2} \left[q^2 g'(s_{\ell+1}, \psi_{\ell+1}) - p^2 g'(s_\ell, \psi_\ell) \right] \\ &+ \frac{h^2}{6(p+q)^3} \left[q^3 g''(s_{\ell+1}, \psi_{\ell+1}) + p^3 g''(s_\ell, \psi_\ell) \right] \\ &- \frac{h^3}{24(p+q)^4} \left[q^4 g'''(s_{\ell+1}, \psi_{\ell+1}) - p^4 g'''(s_\ell, \psi_\ell) \right]. \end{aligned}$$

Theorem 2.1 *The Generalized Taylor method* (2.8) *is of order* 4.

Proof: Substituting the exact solution into the Taylor series expansion yields

$$\begin{aligned} &z(s_{\ell+1}) - \frac{h}{p+q} \left[qf(s_{\ell+1}, z_{\ell+1}) + pf(s_\ell, z_\ell) \right] + \frac{h^2}{2(p+q)^2} \left[q^2 f'(s_{\ell+1}, z_{\ell+1}) - p^2 f'(s_\ell, z_\ell) \right] \\ &\quad - \frac{h^3}{6(p+q)^3} \left[q^3 f''(s_{\ell+1}, z_{\ell+1}) + p^3 f''(s_\ell, z_\ell) \right] + \frac{h^4}{24(p+q)^4} \left[q^4 f'''(s_{\ell+1}, z_{\ell+1}) - p^4 f'''(s_\ell, z_\ell) \right] \\ &= z(s_\ell) + hz'(s_\ell) + \frac{h^2}{2} z''(s_\ell) + \frac{h^3}{6} z'''(s_\ell) + O(h^4) \\ &\quad - z(s_\ell) - \frac{h}{p+q} \left[qz'(s_\ell) + qhz''(s_\ell) + \frac{qh^2}{2} z'''(s_\ell) + pz'(s_\ell) \right] \\ &\quad + \frac{h^2}{2(p+q)^2} \left[q^2 z''(s_\ell) + q^2 h z'''(s_\ell) + O(h^2) - p^2 z''(s_\ell) \right] \\ &\quad - \frac{h^3}{6(p+q)^3} \left[q^3 z'''(s_\ell) + O(h) + p^3 z'''(s_\ell) \right] - \frac{h^4}{24(p+q)^4} \left[q^4 z^{(4)}(s_\ell) - p^4 z^{(4)}(s_\ell) \right] \\ &= O(h^4), \end{aligned}$$

which confirms that (2.8) is of order 4. □

Corollary 2.1 *The Generalized Taylor method* (2.7) *is of order* m.

Proof: The proof follows the same reasoning as in Theorem 2.1; thus, the details are left to the reader. □

3. Convergence and Stability of Generalized Taylor Method

To establish the convergence and derive an error bound for the Generalized Taylor method presented in (2.7), we first state the following key lemma [20, Lemma 5.8, p.270].

Lemma 3.1 *Let* s *and* t *be positive real numbers, and let* $\{\beta_\ell\}_{\ell=0}^{k}$ *be a sequence such that* $\beta_0 \geq -t/s$ *and*

$$\beta_{\ell+1} \leq (1+s)\beta_\ell + t, \quad \text{for each } \ell = 0, 1, 2, \ldots, k-1.$$

Then, the following inequality holds:

$$\beta_{\ell+1} \leq \exp\left((1+\ell)s\right)\left(\beta_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Next, we establish that the Generalized Taylor method of order n is convergent and derive its error bound.

Theorem 3.1 *Assume that the functions* $g^{(n)}$ *for* $0 \leq n \leq m-1$ *are continuous and satisfy a Lipschitz condition with constant* L_n *on*

$$D := \{(s,z) : a \leq s \leq b, -\infty < z < \infty\},$$

and that there exists a constant M *such that*

$$|g^{(m)}(s, z(s))| \leq M, \quad \text{for all } s \in [a,b],$$

where $z(s)$ *is the unique solution to the initial value problem*

$$z' = g(s,z), \quad a \leq s \leq b, \quad z(a) = \alpha.$$

Let $\psi_0, \psi_1, \ldots, \psi_N$ *denote the approximations generated by the Generalized Taylor method* (2.7) *for some positive integer* N. *Then, the Generalized Taylor method described in* (2.7) *is convergent.*

Proof: When $\ell = 0$, the assertion is correct, as it holds that $z(s_0) = \psi_0 = \alpha$. Otherwise, from (2.6), we have

$$\begin{aligned} z\left(s_{\ell+1}\right) = z\left(s_\ell\right) &+ \sum_{k=1}^{m} \frac{(-1)^{k-1} h^k}{(p+q)^k k!} \left[q^k g^{(k-1)}\left(s_{\ell+1}, z\left(s_{\ell+1}\right)\right) + (-1)^{k-1} p^k g^{(k-1)}\left(s_\ell, z\left(s_\ell\right)\right)\right] \\ &+ \frac{h^{m+1}}{(p+q)^m m!} \cdot g^{(m)}\left(\zeta_\ell, z\left(\zeta_\ell\right)\right) \int_0^1 \left(q-(p+q)u\right)^m du \end{aligned}$$

for $i = 0, 1, \cdots, N-1$ From (2.7), we have

$$\psi_{\ell+1} = \psi_\ell + \sum_{k=1}^{m} \frac{(-1)^{k-1} h^k}{(p+q)^k k!} \left[q^k g^{(k-1)}\left(s_{\ell+1}, \psi_{\ell+1}\right) + (-1)^{k-1} p^k g^{(k-1)}\left(s_\ell, \psi_\ell\right)\right],$$

for each $i = 0, 1, 2, \cdots N-1$. Utilizing the notations $z_\ell = y(s_\ell)$ and $z_{\ell+1} = y(s_{\ell+1})$, we deduce the following by subtracting these two equations:

$$\begin{aligned} z_{\ell+1} - \psi_{\ell+1} = z_\ell - \psi_\ell &+ \sum_{k=1}^{m} \frac{(-1)^{k-1} h^k}{(p+q)^k k!} \cdot p^k \cdot \left[g^{(k-1)}\left(s_\ell, z_\ell\right) - g^{(k-1)}\left(s_\ell, \psi_\ell\right)\right] \\ &+ \sum_{k=1}^{m} \frac{(-1)^{k-1} h^k}{(p+q)^k k!} \cdot q^k \cdot \left[g^{(k-1)}\left(s_{\ell+1}, z_{\ell+1}\right) - g^{(k-1)}\left(s_{\ell+1}, \psi_{\ell+1}\right)\right] \\ &+ \frac{h^{m+1}}{(p+q)^m m!} \cdot g^{(m)}\left(\zeta_\ell, z\left(\zeta_\ell\right)\right) \int_0^1 \left(q-(p+q)u\right)^m du. \end{aligned}$$

Employing the triangle inequality, we have

$$\begin{aligned}|z_{\ell+1}-\psi_{\ell+1}| \le |z_\ell-\psi_\ell| &+ \sum_{k=1}^{m}\frac{p^k h^k}{(p+q)^k\,k!}\left|g^{(k-1)}(s_\ell,z_\ell)-g^{(k-1)}(s_\ell,\psi_\ell)\right| \\ &+ \sum_{k=1}^{m}\frac{q^k h^k}{(p+q)^k\,k!}\left|g^{(k-1)}(s_{\ell+1},z_{\ell+1})-g^{(k-1)}(s_{\ell+1},\psi_{\ell+1})\right| \\ &+ \frac{h^{m+1}}{(p+q)^m\,m!}\left|g^{(m)}(\zeta_\ell,z(\zeta_\ell))\right|\int_0^1 |q-(p+q)u|^m\,du.\end{aligned}$$

for some $\zeta\in(0,1)$.

Consider the function $g^{(n-1)}$ for $n=1,2,\dots,m$, which satisfies the Lipschitz condition with respect to the second variable, with a constant denoted as $L:=\max\limits_{1\le n\le m}\{L_{n-1}\}$, and $\left|g^{(m)}(s,z(t))\right|\le M$, so

$$\begin{aligned}|z_{\ell+1}-\psi_{\ell+1}| \le |z_\ell-\psi_\ell| + L\cdot\sum_{k=1}^{m}\frac{p^k h^k}{(p+q)^k\,k!}|z_\ell-\psi_\ell| + L\cdot\sum_{k=1}^{m}\frac{q^k h^k}{(p+q)^k\,k!}|z_{\ell+1}-\psi_{\ell+1}| \\ + \frac{h^{m+1}\left(q^{m+1}+(-1)^{m+1}p^{m+1}\right)}{(p+q)^{m+1}(m+1)!}\cdot M.\end{aligned}$$

For simplicity, let us define

$$\mathfrak{S}_m(h):=1+L\cdot\sum_{k=1}^{m}\frac{p^k h^k}{(p+q)^k\,k!},\qquad \mathfrak{W}_m(h):=1-L\cdot\sum_{k=1}^{m}\frac{q^k h^k}{(p+q)^k\,k!},$$

and

$$\mathfrak{F}_m(h):=\sum_{k=1}^{m}\frac{\left(p^k+q^k\right)h^{k-1}}{(p+q)^k\,k!}.$$

Before we go further, we need to remark that

$$Lh\mathfrak{F}_m(h)=L\sum_{k=1}^{m}\frac{\left(p^k+q^k\right)h^k}{(p+q)^k\,k!}=L\cdot\left[\frac{e^{\frac{ph}{p+q}}\Gamma\left(m+1,\frac{ph}{p+q}\right)-m!}{m!}+\frac{e^{\frac{qh}{p+q}}\Gamma\left(m+1,\frac{qh}{p+q}\right)-m!}{m!}\right]$$

where $\Gamma(\cdot)$ is the Euler Gamma function and $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function, where we used Maple Software to evaluate the previous sum. Given our primary interest in letting $h\to 0^+$, it is straightforward to observe that

$$Lh\mathfrak{F}_m(h)\to 0,\qquad \text{as long as}\qquad h\to 0^+$$

Moreover, $\mathfrak{W}_m(h)\neq 0$, for all m and h. In fact, $\mathfrak{W}_m(h)\to 1$, as long as $h\to 0^+$, without any adverse consequences. Consequently, we can infer that

$$\begin{aligned}|z_{\ell+1}-\psi_{\ell+1}| &\le \frac{\mathfrak{S}_m(h)}{\mathfrak{W}_m(h)}\cdot|z_\ell-\psi_\ell| + \frac{h^{m+1}\left(q^{m+1}+(-1)^{m+1}p^{m+1}\right)}{(p+q)^{m+1}(m+1)!\,\mathfrak{W}_m(h)}\cdot M \\ &= \left(1+\frac{\mathfrak{S}_m(h)-\mathfrak{W}_m(h)}{\mathfrak{W}_m(h)}\right)\cdot|z_\ell-\psi_\ell| + \frac{h^{m+1}\left(q^{m+1}+(-1)^{m+1}p^{m+1}\right)}{(p+q)^{m+1}(m+1)!\,\mathfrak{W}_m(h)}\cdot M \\ &= \left(1+\frac{L\cdot h\cdot\mathfrak{F}_m(h)}{\mathfrak{W}_m(h)}\right)\cdot|z_\ell-\psi_\ell| + \frac{h^{m+1}\left(q^{m+1}+(-1)^{m+1}p^{m+1}\right)}{(p+q)^{m+1}(m+1)!\,\mathfrak{W}_m(h)}\cdot M\end{aligned}$$

By applying Lemma 3.1, with $U(h)=\frac{L\cdot h\cdot \mathfrak{F}_m(h)}{\mathfrak{W}_m(h)}$, $V(h)=\frac{h^{m+1}\left(q^{m+1}+(-1)^{m+1}p^{m+1}\right)}{(p+q)^{m+1}(m+1)!\mathfrak{W}_m(h)}\cdot M$, and $a_j=|z_j-\psi_j|$, for each $j=0,1,2,\cdots,N$, we observe that

$$|z_{\ell+1}-\psi_{\ell+1}|\le \exp\left((\ell+1)\cdot\frac{L\cdot h\cdot \mathfrak{F}_m(h)}{\mathfrak{W}_m(h)}\right)\left(|z_0-\psi_0|+\frac{V(h)}{U(h)}\right)-\frac{V(h)}{U(h)}.$$

Since $|z_0-\psi_0|=0$,

$$\lim_{h\to 0^+}\frac{L\cdot h\cdot \mathfrak{F}_m(h)}{\mathfrak{W}_m(h)}=0,\qquad \text{and}\qquad \lim_{h\to 0^+}\frac{V(h)}{U(h)}=0.$$

then $\lim\limits_{h\to 0^+}\max\limits_{1\le\ell\le N}|z_{\ell+1}-\psi_{\ell+1}|=0$, which means that $\psi_{\ell+1}$ converges to $z_{\ell+1}$, and thus the Generalized Taylor method of order m is converge as required. □

Theorem 3.2 *Assuming the conditions of Theorem 3.1 are satisfied, it is established that*

$$|z_{\ell+1}-\psi_{\ell+1}|\le \frac{V(h)}{U(h)}\cdot\left(\exp\left((s_{\ell+1}-a)\frac{L\cdot \mathfrak{F}_m(h)}{\mathfrak{W}_m(h)}\right)-1\right) \tag{3.1}$$

for all $\ell=0,1,2,\ldots,N-1$.

Proof: The inequality follows directly from the final inequality in the proof of Theorem 3.1. Since $(\ell+1)h=s_{\ell+1}-t_0=s_{\ell+1}-a$, the error bound of the method is inferred from this relation. □

Remark 3.1 *In accordance with the general stability theorem for well-posed initial value problems, Theorem 3.1 demonstrates that the Generalized Taylor method, as specified in (2.7), exhibits stability and consistency.*

The error bound derived in Theorem 3.2 is significant due to its explicit dependence on the step size h. This relationship indicates that reducing the step size will lead to an improvement in the accuracy of the approximation.

4. Perturbations of the Generalized Taylor Method

The analyses in Theorem 3.1 and Theorem 3.2 do not take into account the influence of round-off errors that arise from the choice of step size. As h decreases, the number of computations increases, thus heightening the possibility of round-off errors. In practical implementations, the difference equation in (2.7) is not employed to compute the approximation z_ℓ at the mesh point s_ℓ. Instead, the following form is used:

$$\begin{aligned} v_0&=\alpha+\delta_0\\ v_{\ell+1}&=v_\ell+h\mathrm{G}^{(m)}(v_\ell,v_{\ell+1})+\delta_{\ell+1}, \end{aligned} \tag{4.1}$$

for each $\ell=0,1,2,\ldots,N-1$, where

$$\mathrm{G}^{(m)}(v_\ell,v_{\ell+1}):=\sum_{k=1}^{m}\frac{(-1)^{k-1}h^{k-1}}{(p+q)^k k!}\left[q^k g^{(k-1)}(s_{\ell+1},v_{\ell+1})+(-1)^{k-1}p^k g^{(k-1)}(s_\ell,v_\ell)\right],$$

for each $\ell=0,1,2,\ldots,N-1$. Let δ_ℓ represent the round-off error associated with v_ℓ. By applying techniques analogous to those used in the proof of Theorem 2.1, an error bound for the finite-precision approximations z_ℓ produced by the Generalized Taylor method can be derived. This leads to the following result.

Theorem 4.1 *Let $z(s)$ denote the unique solution to the initial-value problem*

$$z' = g\left(s, z\right), \qquad a \leq s \leq b, \qquad z(a) = \alpha. \tag{4.2}$$

Let $v_0, v_1, \ldots, v_N$ represent the approximations generated by the Generalized Taylor method in (4.1) for some positive integer N. If $|\delta_\ell| < \delta$ for each $\ell = 0, 1, 2, \ldots, N$, and the conditions of Theorem 3.1 are satisfied for (4.2), then it follows that

$$|z_\ell - v_\ell| \leq \left(\frac{V\left(h\right)}{U\left(h\right)} + \frac{\delta \mathfrak{W}_m\left(h\right)}{Lh\mathfrak{F}_m\left(h\right)}\right) \cdot \left(\exp\left(\left(s_\ell - a\right)\frac{L \cdot \mathfrak{F}_m\left(h\right)}{\mathfrak{W}_m\left(h\right)}\right) - 1\right) + |\delta_0| \exp\left(\left(s_\ell - a\right)\frac{L \cdot \mathfrak{F}_m\left(h\right)}{\mathfrak{W}_m\left(h\right)}\right) \tag{4.3}$$

for each $\ell = 0, 1, 2, \ldots, N$.

Proof: The proof follows the same steps as that of Theorem 3.1, applied to the difference equation in (2.7). □

It should be noted that the error bound in (4.3) is no longer linear in h. In fact,

$$\lim_{h\to 0^+}\left(\frac{V\left(h\right)}{U\left(h\right)} + \frac{\delta \mathfrak{W}_m\left(h\right)}{Lh\mathfrak{F}_m\left(h\right)}\right) \to \infty.$$

As the step size h decreases to very small values, one might expect an increase in error. Specifically, when h falls below a certain critical level, the total approximation error tends to rise. However, under typical conditions, the error—denoted as δ—remains within acceptable bounds. This suggests that setting a minimum threshold for h has little effect on the accuracy or efficiency of computations when using the Generalized Taylor method. Furthermore, the approximation improves as more terms are incorporated into the Generalized Taylor series expansion.

5. Numerical Experiments

In this section, we apply the Generalized Taylor method of order 4 with various step sizes to several initial value problems (IVPs).

Example 5.1 The Generalized Taylor method (2.8) with $p = 2.9$ and $q = 3.4$ is used to approximate the solution of the initial-value problem:

$$z'(s) = z - s^2 + 1, \qquad 0 \leq s \leq 2, \qquad z(0) = 0.5. \tag{5.1}$$

The parameters are set as $N = 10$, $h = 0.2$, $s_\ell = 0.2\ell$, and $\psi_0 = 0.5$. The resulting approximation is compared to the exact solution given by

$$z(s) = (s+1)^2 - 0.5\,\mathrm{e}^s.$$

Additionally, a comparison is made between our approximation and several well-known classical methods.

Figure 1 illustrates the exact solution alongside the numerical approximations for a step size of $h = 0.2$. Furthermore, Figure 2 and Table 1 present the absolute errors for the Generalized Taylor method and other methods using the same step size.

It is evident that the Generalized Taylor method (2.8) provides significantly more accurate approximations than the other methods evaluated. Figures 1 and 2 illustrate a comparison between the approximate solutions produced by the Generalized Taylor method (2.8) and those from other methods, alongside their respective absolute errors for a step size of $h = 0.2$.

Example 5.2 The Generalized Taylor method (2.8) with $p = 3.5$ and $q = 3.5$ is applied to approximate the solution of the initial-value problem

$$z'\left(s\right) = \exp\left(s - z\right), \qquad 0 \leq s \leq 1, \qquad z\left(0\right) = 1, \tag{5.2}$$

Table 1: Comparison of absolute errors for the Modified Euler, Trapezoid, Midpoint, Taylor, and Runge-Kutta (RK) methods of order 4 with the Generalized Taylor method (GT) using $p = 2.9$ and $q = 3.4$ in Example 5.1, with a step size of $h = 0.2$.

s_ℓ	Modified Euler	Trapezoid	Midpoint	Taylor $\times 10^{-4}$	RK $\times 10^{-3}$	GT $\times 10^{-5}$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.003298	0.003298	0.001298	0.013790	0.005287	0.010449
0.4	0.007167	0.007167	0.002727	0.033688	0.011440	0.025526
0.6	0.011698	0.011698	0.004281	0.061720	0.018582	0.046767
0.8	0.016993	0.016993	0.005945	0.100513	0.026850	0.076161
1.0	0.023171	0.023171	0.007692	0.153459	0.036393	0.116280
1.2	0.030362	0.030362	0.009478	0.224922	0.047368	0.170430
1.4	0.038713	0.038713	0.011234	0.320507	0.059943	0.242857
1.6	0.048386	0.048386	0.012862	0.447392	0.074289	0.339002
1.8	0.059557	0.059557	0.014217	0.614751	0.090573	0.465815
2.0	0.072417	0.072417	0.015102	0.834286	0.108949	0.632165

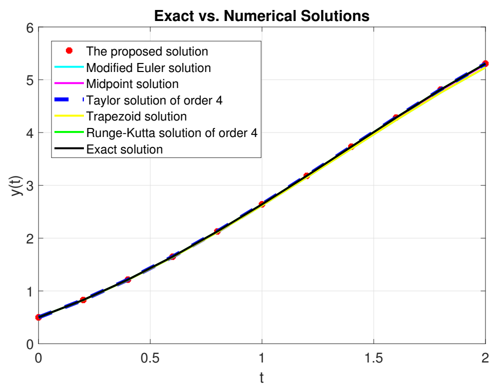

Figure 1: The exact solution compared with the Generalized Taylor method and other methods for a step size of $h = 0.2$ in Example 5.1.

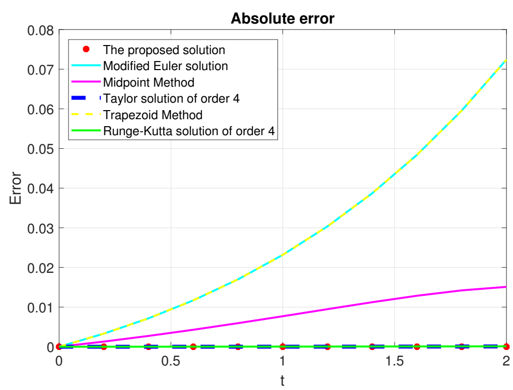

Figure 2: Absolute errors of the Generalized Taylor method and other methods for a step size of $h = 0.2$ in Example 5.1.

with parameters set to $N = 10$, $h = 0.1$, $s_\ell = 0.1\ell$, and $\psi_0 = 1$. This approximation is compared to the exact solution $z(s) = \ln(\exp(s) + e - 1)$. Additionally, the approximation results are evaluated against some well-known classical methods. Specifically, Figure 3 shows the exact solution in comparison with these methods for a step size of $h = 0.1$, while Figure 4 and Table 2 display the absolute errors for the Generalized Taylor method and other methods, as indicated in the table and figures, using the same step size.

Table 2: Comparison of absolute errors for the Modified Euler (MEM), Trapezoid (TM), Midpoint (MM), Taylor, and Runge-Kutta (RKM) methods of order 4, alongside the Generalized Taylor method (GTM) using $p = 3.5$ and $q = 3.5$ in Example 5.2, with a step size of $h = 0.1$.

s_ℓ	MEM $\times 10^{-3}$	TM $\times 10^{-3}$	MM $\times 10^{-3}$	Taylor $\times 10^{-7}$	RKM $\times 10^{-7}$	GTM $\times 10^{-8}$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.027686	0.027686	0.008720	0.089951	0.044205	0.052733
0.2	0.055392	0.055392	0.018494	0.164243	0.091928	0.095288
0.3	0.082890	0.082890	0.029265	0.220118	0.143059	0.126127
0.4	0.109938	0.109938	0.040945	0.255526	0.197384	0.144185
0.5	0.136292	0.136292	0.053418	0.269295	0.254577	0.148962
0.6	0.161707	0.161707	0.066538	0.261239	0.314194	0.140569
0.7	0.185946	0.185946	0.080135	0.232199	0.375680	0.119739
0.8	0.208782	0.208782	0.094021	0.184002	0.438375	0.087783
0.9	0.230012	0.230012	0.107991	0.119349	0.501537	0.046504
1.0	0.249454	0.249454	0.121836	0.041639	0.564359	0.001919

It is clear that the Generalized Taylor method (2.8) achieves significantly improved accuracy compared to the other methods considered. Figures 3 and 4 display the comparison of approximate solutions generated by the Generalized Taylor method (2.8) and the other methods, along with their respective absolute errors for a step size of $h = 0.1$.

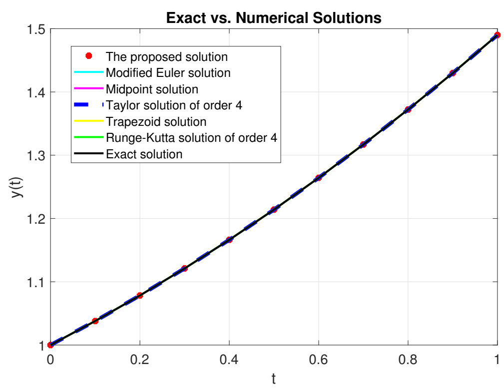

Figure 3: The exact solution compared with the Generalized Taylor and other methods with step size $h = 0.1$ applied in Example 5.2.

Example 5.3 The Generalized Taylor method (2.8) with $p = 2.5$ and $q = 2.7$ is used to approximate the solution of the initial-value problem:

$$z'(s) = z^2, \qquad 0 \le s \le 0.9, \qquad z(0) = 1. \tag{5.3}$$

The parameters are set as $N = 10$, $h = 0.09$, $s_\ell = 0.09\ell$, and $\psi_0 = 1$. The resulting approximation is compared to the exact solution given by $z(s) = \frac{1}{1-s}$. Additionally, a comparison is made between the classical Taylor method and our proposed approximation.

Figure 5 illustrates the exact solution alongside the numerical approximations for a step size of $h = 0.09$. Furthermore, Figure 6 and Table 3 present the absolute errors for the Generalized Taylor method and other methods using the same step size.

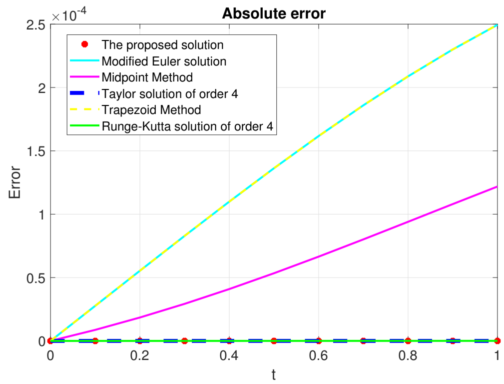

Figure 4: Absolute errors of the Generalized Taylor's and other methods with step size $h = 0.1$ applied in Example 5.2.

Table 3: Comparison of absolute errors for the Modified Euler, Trapezoid, Midpoint, Taylor, and Runge-Kutta (RK) methods of order 4, along with the Generalized Taylor method (GT) using $p = 2.5$ and $q = 2.7$ in Example 5.3, with a step size of $h = 0.09$.

s_ℓ	Modified Euler	Trapezoid	Midpoint	Taylor	RK	GT
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.09	0.0004365	0.0004365	0.0006188	0.0000064	0.0000005	0.0000003
0.18	0.0011848	0.0011848	0.0016739	0.0000195	0.0000010	0.0000001
0.27	0.0024956	0.0024956	0.0035118	0.0000464	0.0000040	0.0000020
0.36	0.0048779	0.0048779	0.0068292	0.0001049	0.0000100	0.0000060
0.45	0.0094467	0.0231710	0.0131379	0.0002420	0.0000250	0.0000140
0.54	0.0094467	0.0303620	0.0260605	0.0006008	0.0000690	0.0000370
0.63	0.0408101	0.0387130	0.0555173	0.0017021	0.0002260	0.0001150
0.72	0.0408101	0.0483860	0.1343522	0.0060008	0.0010190	0.0004590
0.81	0.3168519	0.3168519	0.4085411	0.0308843	0.0085620	0.0029210
0.9	1.6533461	1.6533461	1.9963781	0.3398992	0.3069350	0.0508930

It is evident that the Generalized Taylor method (2.8) delivers significantly more accurate approximations compared to the other methods considered. Figures 5 and 6 illustrate a comparison of the approximate solutions generated by the Generalized Taylor method (2.8) and other methods, along with their corresponding absolute errors for a step size of $h = 0.09$.

Additionally, it is noteworthy that the Generalized Taylor method (2.8) performs particularly well near discontinuities, such as at $s = 1$, outperforming the Runge-Kutta method and other approaches in this region.

6. Recommendation

The Generalized Taylor method (2.8) shows promising results in comparison to classical Taylor and Runge-Kutta methods. Numerical evaluations indicate that this approach offers higher accuracy when approximating solutions for both linear and nonlinear initial value problems (IVPs). It exhibits improved stability and faster convergence, highlighting its robustness and efficiency for a range of mathematical modeling applications.

In the long term, the Generalized Taylor method (2.7) of order m consistently surpasses the Taylor and Runge-Kutta methods of the same order in terms of analytic solution accuracy. Additionally, this method performs well across various cases, as seen in Example 5.3, where method (2.8) demonstrates exceptional accuracy near points of discontinuity.

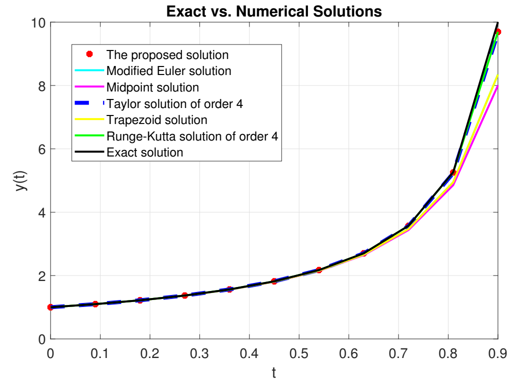

Figure 5: The exact solution compared with the Generalized Taylor and other methods for a step size of $h = 0.09$ in Example 5.3.

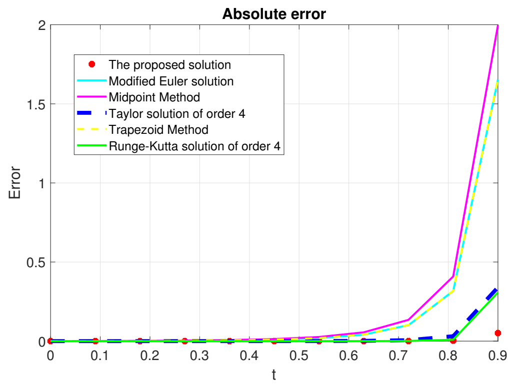

Figure 6: Absolute errors of the Generalized Taylor method and other methods for a step size of $h = 0.09$ in Example 5.3.

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