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Cross Ratio Geometry: Advances for Multiple Collinear Points in the Desargues Affine Plane

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ABSTRACT: This paper introduces advances in the geometry of the cross ratio of multiple collinear points of a skew field in the Desargues affine plane. This paper extends earlier results for three collinear points in terms of the geometry resulting from four collinear points in the Desargues affine plane. The results given here have a clean rendition, based on Desargues affine plane axiomatic's, skew field properties and the addition and multiplication of planar collinear points.

Key Words: Collinear Points, cross Ratio, skew Field, desargues affine plane, skew-field line.

Contents

1. Introduction

This paper introduces advances in the study of multiple collinear points in the Desargues affine plane. The focus here is on four collinear points, extending earlier results for 2 and 3 collinear points in the Desargues affine plane [18], [19]. It has been shown that on each line on Desargues affine plane, we can construct a skew field simply and constructively, using elementary geometry, and basic axioms of the Desargues affine plane (see [17], [5], [13], [24]).

Earlier results germane to this work have been given relative to the association of algebraic structures in affine planes and in Desargues affine plane, and vice versa in [16,17,5,14,15,13,24,25,23,22,18]. Here, we consider dilations and translations entirely in the Desargues affine plane (see [16], [14], [13], [24]).

2. Preliminaries

This section introduce the cornerstones of the geometry of collinear points in the Desargues affine plane.

Let $\mathcal{P}$ be a nonempty space, $\mathcal{L}$ a nonempty a family of subsets of $\mathcal{P}$. The elements p of $\mathcal{P}$ are points and an element ℓ of $\mathcal{L}$ is a line.

Definition 2.1 Affine Plane.
The incidence structure $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called affine plane, where satisfies the above axioms:

1° *For each points $\{P, Q\} \in \mathcal{P}$, there is exactly one line $\ell \in \mathcal{L}$ such that $\{P, Q\} \in \ell$.*

2° *For each point $P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell$, there is exactly one line $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \emptyset$ (Playfair Parallel Axiom [10]). Put another way, if the point $P \notin \ell$, then there is a unique line ℓ' on P missing ℓ [11].*

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3° *There is a 3-subset of points $\{P, Q, R\} \in \mathcal{P}$, which is not a subset of any ℓ in the plane. Put another way, there exist three non-collinear points $\mathcal{P}$ [11].*

Desargues' Axiom, circa 1630.
Desargues' Axiom, circa 1630 [9, §3.9, pp. 60-61] [12] is given, here. Let $A, B, C, A', B', C' \in \mathcal{P}$ and let pairwise distinct lines $\ell^{AA'}, \ell^{BB'}, \ell^{CC'}, \ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ such that

$$\ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'} \text{ (Fig. 1(a)) } \mathbf{or} \ \ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P. \text{(Fig. 1(b))}$$
$$\text{and } \ell^{AB} \parallel \ell^{A'B'} \text{ and } \ell^{BC} \parallel \ell^{B'C'}.$$
$$A, B \in \ell^{AB}, A'B' \in \ell^{A'B'}, AC \in \ell^{AC}, \text{ and } A'C' \in \ell^{A'C'}, B, C \in \ell^{BC}, B'C' \in \ell^{B'C'}.$$
$$A \neq C, A' \neq C', \text{ and } \ell^{AB} \neq \ell^{A'B'}, \ell^{BC} \neq \ell^{B'C'}.$$

Then $\boldsymbol{\ell^{AC} \parallel \ell^{A'C'}}$.

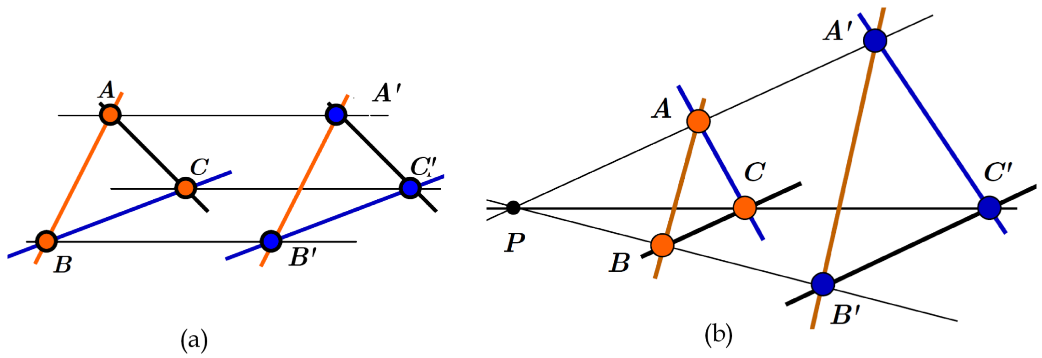

Figure 1: Desargues Axioms: (a) For parallel lines $\ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'}$; (b) For lines which are cutting in a single point P, $\ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P$.

In Euclidean plane, three vertexes ABC and $A'B'C'$, are similar (in (a) are equivalent-triangle and in (b) are homothetical-triangle) the parallel lines, $\ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ in Desargues' Axiom are represented in Fig. 1. In other words, the side AC of the triangle of $\triangle ABC$ is parallel with the side $A'C'$ of the triangle $\triangle A'B'C'$, provided the restrictions on the points and lines in Desargues' Axiom are satisfied.
A **Desargues affine plane** is an affine plane that satisfies Desargues' Axiom.

Note 1 *Three vertexes ABC and $A'B'C'$, which, fulfilling the conditions of the Desargues Axiom, we call* 'Desarguesian'.

2.1. Addition and Multiplication of collinear points

This section has two principal parts, namely, addition and multiplication of collinear points in the Desargues affine plane.

Addition of collinear points in an affine plane
In a Desargues affine plane $\mathcal{A_D} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ we fix two different points $O, I \in \mathcal{P}$, which, according to Axiom 1, determine a line $\ell^{OI} \in \mathcal{L}$. Let A and B be two arbitrary points of a line ℓ^{OI}. In the plane $\mathcal{A_D}$ we choose a point B_1 not incident with ℓ^{OI}: $B_1 \notin \ell^{OI}$ (we call it the auxiliary point). Construct the line $\ell_{OI}^{B_1}$, which is unique according to the Axiom 2. Then construct the line $\ell_{OB_1}^{A}$, which also is unique according to the Axiom 2. Let us denote their intersection with $P_1 = \ell_{OI}^{B_1} \cap \ell_{OB_1}^{A}$. Finally construct the line $\ell_{BB_1}^{P_1}$. For as much as ℓ^{BB_1} cuts the line ℓ^{OI} in point B, then this line, parallel with ℓ^{BB_1}, cuts the line ℓ^{OI} in a single point C, we have called this point the addition of points A with point B (Figure 2.

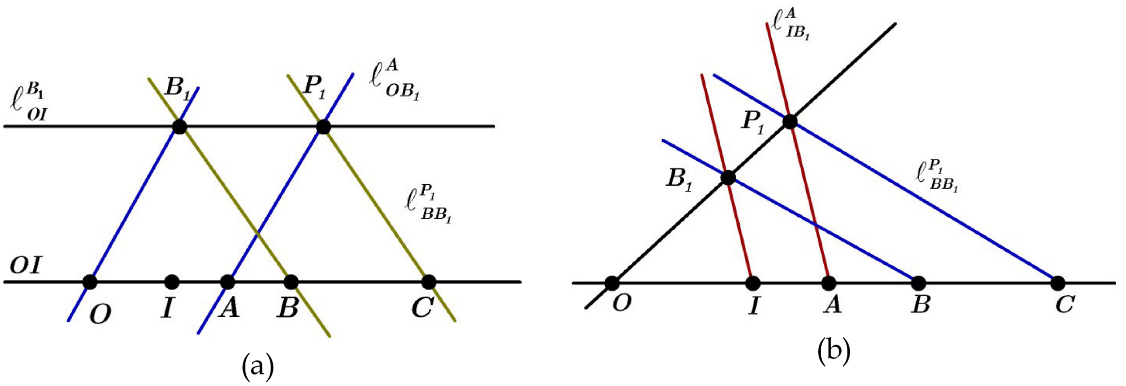

Figure 2: (a) Addition of points in a line in affine plane, (b) Multiplication of points in a line in affine plane

Multiplication of collinear points in an affine plane. Choose in the plane $\mathcal{A}_{\mathcal{D}}$ one point B_1 not incident with lines ℓ^{OI}, and construct the line ℓ^{IB_1}. Construct the line $\ell^{A}_{IB_1}$, which is unique according to the Axiom 2 and cutting the line ℓ^{OB_1}. Let us denote their intersection with $P_1 = \ell^{A}_{IB_1} \cap OB_1$. Finally, construct the line $\ell^{P_1}_{BB_1}$. Since ℓ^{BB_1} cuts the line ℓ^{OI} in a single point B, then this line, parallel with ℓ^{BB_1}, cuts the line ℓ^{OI} in one single point C, we have called this point the multiplication of points A with point B (Figure 2 (b)).

Multiplication & Addition of Collinear Points.
The process to construct the points C for addition and multiplication of points in a $\ell^{OI}-$line in an affine plane, is presented here in the two algorithms. In [13], [5], we prove that $(\ell^{OI}, +, \cdot)$ is a skew field in

Addition Algorithm(Fig.2(a))	**Multiplication Algorithm**(Fig.2(b))
Step.1: $B_1 \notin \ell^{OI}$	**Step.1:** $B_1 \notin \ell^{OI}$
Step.2: $\ell^{B_1}_{OI} \cap \ell^{A}_{OB_1} = P_1$	**Step.2:** $\ell^{A}_{IB_1} \cap \ell^{OB_1} = P_1$
Step.3: $\ell^{P_1}_{BB_1} \cap \ell^{OI} = C(= A + B)$	**Step.3:** $\ell^{P_1}_{BB_1} \cap \ell^{OI} = C(= A \cdot B)$

the Desargues affine plane, and is a commutative skew field in the Papus affine plane. In this paper we consider the Desargues affine plane as one *"non-Papian"* plane, since we have decided that multiplication of points is non-commutative.

2.2. Some algebraic properties of Skew Fields

In this section K will denote a skew field [27] and $z[K]$ its center, where is the set K such that

$$z[K] = \{k \in K \quad | \quad ak = ka, \quad \forall a \in K\}.$$

Proposition 2.1 *$z[K]$ is a commutative subfield of a skew field K.*

Let now $p \in K$ be a fixed element of the skew field K. We will denote by $z_K(p)$ the centralizer in K of the element p, where is the set,

$$z_K(p) = \{k \in K | pk = kp, \}.$$

$z_K(p)$ is sub skew field of K, but, in general, it is not commutative.

Let K be a skew field, $p \in K$, and let us denote by $[p_K]$ the conjugacy class of p:

$$[p_K] = \left\{q^{-1}pq \quad | \quad q \in K \setminus \{0\}\right\}.$$

If, $p \in z[K]$, for all $q \in K$ we have that $q^{-1}pq = p$.

2.3. Ratio of two and three collinear points

In the paper [18], we introduce a detailed study concerning the ratio of two and three collinear points in the Desargues affine plane. In this sectionm, we briefly present some of the results for ratio of two and three collinear points.

Definition 2.2 *[18] Lets have two different points $A, B \in \ell^{OI}-$line, and $B \neq O$, in Desargues affine plane. We define as ratio of these tow points, a point $R \in \ell^{OI}$, such that,*

$$R = B^{-1}A, \qquad \textit{we mark this, with,} \qquad R = r(A : B) = B^{-1}A.$$

For a 'ratio-point' $R \in \ell^{OI}$, and for point $B \neq O$ in line ℓ^{OI}, is a unique defined point, $A \in \ell^{OI}$, such that $R = B^{-1}A = r(A : B)$.

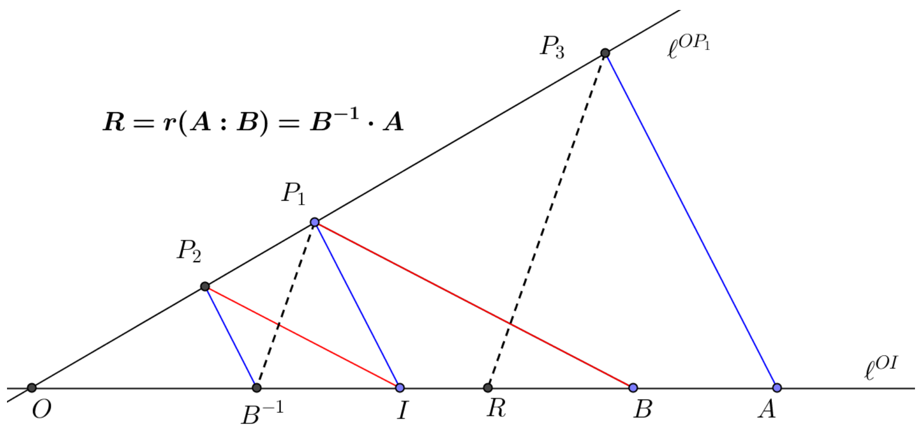

Figure 3: Ilustrate the Ratio-Point, of 2-Points in a line of Desargues affine plane $R = r(A : B) = B^{-1}A$.

Some results for Ratio of 2-points in Desargues affine plane (see [18]).

- If have two different points $A, B \in \ell^{OI}-$line, and $B \neq O$, in Desargues affine plane, then, $r^{-1}(A : B) = r(B : A)$.
- For three collinear points A, B, C and $C \neq O$, in $\ell^{OI}-$line, have,

$$r(A + B : C) = r(A : C) + r(B : C).$$

- For three collinear point A, B, C and $C \neq O$, in $\ell^{OI}-$line, have,
  1. $r(A \cdot B : C) = r(A : C) \cdot B$.
  2. $r(A : B \cdot C) = C^{-1} r(A : C)$.
- Let's have the points $A, B \in \ell^{OI}-$line where $B \neq O$. Then have that,

$$r(A : B) = r(B : A) \Leftrightarrow A = B.$$

- This ratio-map, $r_B : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_2 = \{r_B(X) | \forall X \in \ell^{OI}\}$, for a fixed point B in $\ell^{OI}-$line, forms a skew-field with 'addition and multiplication' of points.

  This, skew field $(\mathcal{R}_2, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

Ratio of three points in a line on Desargues affine plane. (see [18])

Definition 2.3 *If A, B, C are three points on a line ℓ^{OI} (collinear) in Desargues affine plane, then we define their* ***ratio*** *to be a point $R \in \ell^{OI}$, such that:*

$$(B - C) \cdot R = A - C, \quad \textit{concisely} \quad R = (B - C)^{-1}(A - C),$$

and we mark this with $r(A, B; C) = (B - C)^{-1}(A - C)$.

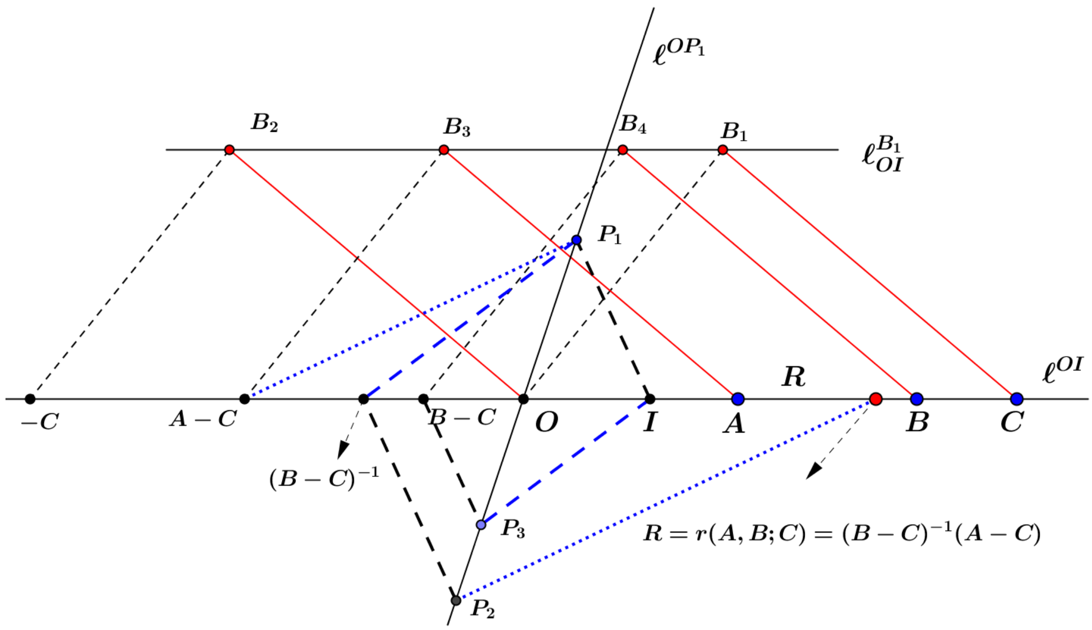

Figure 4: Ratio of 3-Points in a line of Desargues affine plane $R = r(A, B; C)$.

Some Results for Ratio of 3-points in Desargues affine plane ([18]).

- For 3-points A, B, C in a line ℓ^{OI} of Desargues affine plane, we have that,

$$r(-A, -B; -C) = r(A, B; C).$$

- For 3-points A, B, C in a line ℓ^{OI} in the Desargues affine plane, have

$$r^{-1}(A, B; C) = r(B, A; C).$$

- If A, B, C, are three different points, and different from point O, in a line ℓ^{OI} on Desargues affine plane, then

$$r(A^{-1}, B^{-1}; C^{-1}) = B[r(A, B; C)]A^{-1}.$$

- In the Pappus affine plane, for three point different from point O, in $\ell^{OI}-$line, we have $r(A^{-1}, B^{-1}; C^{-1}) = r(A, B; C) \cdot r(B, A; O)$.
- This ratio-map, $r_{BC} : \ell^{OI} \to \ell^{OI}$ is a bijection in $\ell^{OI}-$line in Desargues affine plane.
- The ratio-maps-set $\mathcal{R}_3 = \{r_{BC}(X) | \forall X \in \ell^{OI}\}$, for a different fixed points B, C in $\ell^{OI}-$line, forms a skew-field with 'addition and multiplication' of points in $\ell^{OI}-$line. This, skew field $(\mathcal{R}_3, +, \cdot)$ is sub-skew field of the skew field $(\ell^{OI}, +, \cdot)$.

3. Main Results for the cross-ratio for four collinear points in the Desargues affine plane

This section gives the main results for this paper. We consider the cross-ratio of collinear points in Desargues affine planes, utilizing a method that is naive and direct without requiring planar coordinates. We define the cross-ratio of four collinear points in a line on Desargues affine plane *as a point in this line*. This work carries forward earlier results that reveal the close connection between lines in the Desargues affine planes and corresponding skew fields. Mainly, we rely on our results regarding the addition and multiplication of co-linear points in the Desargues affine plane, and the fact that a line (set of points), with addition and multiplication, forms a skew field (for more about this, see [13], [17], [5], [15], [14], [16], [23], [22], [21], [24], [25], [19]).

The classical definition of the cross-ratio (see [30,7,2,3]) for 4-points, is given as a product of tow ratio of lengths. So, for example, for three co-linear points A, B, C,

$$c_r(A, B; C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD},$$

where AC, BC, BD, AD are the lengths of segments $[AB], [BC], [BD].[AD]$, respectively.

Since we will not use coordinates and metrics, our definitions are rely solely on the algebra and axiomatics for the Desargues affine plane.

Let us have the line ℓ^{OI} in Desargues affine plane $\mathcal{A}_{\mathcal{D}}$, and four points, $A, B, C, D \in \ell^{OI}$

Definition 3.1 *If A, B, C, D are four points in a line ℓ^{OI} on Desargues affine plane $\mathcal{A}_{\mathcal{D}}$, no three of them equal, then we define their cross ratio to be a point:*

$$c_r(A, B; C, D) = \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right].$$

Remark 3.1 Similar to 'ratio', we can define it, the cross-ratio, also as

$$c_r(A, B; C, D) = [(B-D)(A-D)^{-1}][(A-C)(B-C)^{-1}],$$

or

$$c_r(A, B; C, D) = [(B-D)(A-C)][(A-D)^{-1}(B-C)^{-1}],$$

(or all combination of product of this 4-factors) the results would be similar, *but the obtained point will always be different for each case.* In $\ell^{OI}-$line, in Desargues affine planes (we consider the affine plane non-Papian plane, since we have decided that multiplication of points is non-commutative), these are a different point from that of our definition, since:

$$[(B-D)(A-D)^{-1}][(A-C)(B-C)^{-1}] \neq \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right],$$

and

$$[(B-D)(A-C)][(A-D)^{-1}(B-C)^{-1}] \neq \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right].$$

also for the other cases, we would have a difference for each pair, found for the cross ratio, according to any definition we take. We are keeping our definition.

Definition 3.2 *If the line ℓ^{OI} on Desargues affine plane, is an infinite line (number of points in this line is $+\infty$), we define as follows:*

$$\begin{aligned} c_r(\infty, B; C, D) &= (B-D)(B-C)^{-1}, \\ c_r(A, \infty; C, D) &= (A-D)^{-1}(A-C), \\ c_r(A, B; \infty, D) &= (A-D)^{-1}(B-D), \\ c_r(A, B; C, \infty) &= (B-C)^{-1}(A-C). \end{aligned}$$

From this definition and from ratio definition 2.3 we have that,

- $c_r(A,B;C,D) = \big[(A-D)^{-1}(B-D)\big]\,\big[(B-C)^{-1}(A-C)\big] = r(B,A;D)\cdot r(A,B;C)$.
- $c_r(\infty,B;C,D) = (B-D)(B-C)^{-1} = [(D-B)^{-1}(C-B)]^{-1} = r^{-1}(C,D;B)$.
- $c_r(A,\infty;C,D) = (A-D)^{-1}(A-C) = (D-A)^{-1}(C-A) = r(C,D;A)$.
- $c_r(A,B;\infty,D) = (A-D)^{-1}(B-D) = r(A,B;D)$.
- $c_r(A,B;C,\infty) = (B-C)^{-1}(A-C) = r(A,B;C)$.

Some simple properties of Cross-Ratios, which derive directly from the definition, related to the position of the points A,B,C,D in $\ell^{OI}-$line on Desargues affine plane.

- If $A=B$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,A;C,D) \\ &= [(A-D)^{-1}(A-D)][(A-C)^{-1}(A-C)] \\ &= [I][I] \\ &= I. \end{aligned}$$
- If $A=C$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,B;A,D) \\ &= [(A-D)^{-1}(B-D)][(B-A)^{-1}(A-A)] \\ &= [(A-D)^{-1}(B-D)][(B-A)^{-1}\cdot O] \\ &= O. \end{aligned}$$
- If $A=D$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,B;C,A) \\ &= [(A-A)^{-1}(B-A)][(B-C)^{-1}(A-C)] \\ &= [O^{-1}(B-A)][(B-C)^{-1}(A-C)] \\ &\text{(think that } O^{-1}=\infty\text{(point in infinity))} \\ &= \infty. \end{aligned}$$
- If $B=C$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,B;B,D) \\ &= [(A-D)^{-1}(B-D)][(B-B)^{-1}(A-B)] \\ &= [(A-D)^{-1}(B-D)][O^{-1}(A-B)] \\ &\text{(think that } O^{-1}=\infty\text{(point in infinity))} \\ &= \infty. \end{aligned}$$
- If $B=D$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,B;C,B) \\ &= [(A-B)^{-1}(B-B)][(B-C)^{-1}(A-C)] \\ &= [(A-B)^{-1}\cdot O][(B-C)^{-1}(A-C)] \\ &= O. \end{aligned}$$
- If $C=D$, then
$$\begin{aligned} c_r(A,B;C,D) &= c_r(A,B;C,C) \\ &= [(A-C)^{-1}(B-C)][(B-C)^{-1}(A-C)] \\ &= (A-C)^{-1}[(B-C)(B-C)^{-1}](A-C) \\ &= (A-C)^{-1}\cdot I\cdot(A-C) \\ &= (A-C)^{-1}(A-C) \\ &= I. \end{aligned}$$

Theorem 3.1 *Let $R \in \ell^{OI}$, such that $R \neq O$ and $R \neq I$. If $A, B, C \in \ell^{OI}$ are three different points, then there exists a single point $D \in \ell^{OI}$, such that $c_r(A, B; C, D) = R$.*

Proof: We prove first, the **existence** of point D. Let us denote,

$$D = [B(B-C)^{-1}(A-C) - AR][(B-C)^{-1}(A-C) - R]^{-1},$$

we will prove that this point D is precisely the fourth point of the cross-ratio, indeed

$$\begin{aligned} D[(B-C)^{-1}(A-C) - R] &= [B(B-C)^{-1}(A-C) - AR] \\ D(B-C)^{-1}(A-C) - DR &= B(B-C)^{-1}(A-C) - AR \\ AR - DR &= B(B-C)^{-1}(A-C) - D(B-C)^{-1}(A-C) \\ (A-D)R &= (B-D)(B-C)^{-1}(A-C), \end{aligned}$$

so, have

$$\begin{aligned} R &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ R &= c_r(A, B : C, D). \end{aligned}$$

Unicity of point D: Suppose that exist tow different points D an D' in $\ell^{OI}-$line, such that

$$c_r(A, B; C, D) = c_r(A, B; C, D').$$

We rewrite them, cross ratios, as products of 'ratios', and we have,

$$c_r(A, B; C, D) = \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right] = r(B, A; D) \cdot r(A, B; C),$$

and

$$c_r(A, B; C, D') = \left[(A-D')^{-1}(B-D')\right]\left[(B-C)^{-1}(A-C)\right] = r(B, A; D') \cdot r(A, B; C).$$

So, have,

$$r(B, A; D) \cdot r(A, B; C) = r(B, A; D') \cdot r(A, B; C),$$

we mark $r(B, A; D) = R_1; r(A, B; C) = R_2, r(B, A; D') = R_3$, remember that these are points of the line ℓ^{OI}, so they are elements of the skew-fields $(\ell^{OI}, +, \cdot)$, and have

$$R = R_1 \cdot R_2 \quad \text{and} \quad R = R_3 \cdot R_2.$$

Thus, for it, we have

$$R_1 \cdot R_2 = R_3 \cdot R_2 \Rightarrow R_1 \cdot R_2 - R_3 \cdot R_2 = O \Rightarrow (R_1 - R_3) \cdot R_2 = O.$$

But the points, R_1, R_2, R_3, are points of $\ell^{OI}-$line in Desargues affine plane, therefore, they are elements of skew-fields $K = (\ell^{OI}, +, \cdot)$. We also know the fact that 'a skew field does not have a divisor of zero' (more on skew fields, see [26], [27], [28], [29])

$$R_1 - R_3 = O \quad \text{or} \quad R_2 = O, \quad \text{but} \quad R_2 \neq O \Rightarrow R_1 - R_3 = O,$$

so,

$$R_1 = R_3 \Rightarrow r(B, A; D) = r(B, A; D'),$$

and from the uniqueness of the definition for 'ratio', we have,

$$D = D'.$$

□

Theorem 3.2 *If A, B, C, D are distinct points in a $\ell^{OI}-$line, in Desargues affine plane, then*

$$c_r(-A,-B;-C,-D) = c_r(A,B;D,C).$$

Proof: From cross-ratio definition 3.1, we have

$$\begin{aligned} c_r(-A,-B;-C,-D) &= [(-A-(-C))^{-1}(-B-(-C))][(-B-(-D))^{-1}(-A-(-D))] \\ &= [(-A+C)^{-1}(-B+C)][(-B+D)^{-1}(-A+D)] \\ &= [([-I](A-C))^{-1}[-I](B-C)][([-I](B-D))^{-1}[-I](A-D)] \\ &= [(A-C)^{-1}[-I]^{-1}[-I](B-C)][(B-D)^{-1}[-I]^{-1}[-I](A-D)] \\ &= [(A-C)^{-1}[-I][-I](B-C)][(B-D)^{-1}[-I][-I](A-D)] \\ &= [(A-C)^{-1}(B-C)][(B-D)^{-1}(A-D)] \\ &= c_r(A,B;D,C). \end{aligned}$$

From skew fields properties we have that $(ab)^{-1} = b^{-1}a^{-1}$and $ab \neq ba$, $[-I]^{-1} = -I$, and $[-I][-I] = I$. □

Theorem 3.3 *If A, B, C, D are distinct points in a line, of the Desargues affine plane, then*

$$c_r^{-1}(A,B;C,D) = c_r(A,B;D,C).$$

Proof: From cross-ratio Definition 3.1, have

$$\begin{aligned} c_r^{-1}(A,B;C,D) &= \left\{\left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right]\right\}^{-1} \\ &= \left[(B-C)^{-1}(A-C)\right]^{-1}\left[(A-D)^{-1}(B-D)\right]^{-1} \\ &= [(A-C)^{-1}(B-C)][(B-D)^{-1}(A-D)] \\ &= c_r(A,B;D,C). \end{aligned}$$

□

Theorem 3.4 *For 4 collinear points A, B, C, D in a line ℓ^{OI} of the Desargues affine plane, the cross-ratio satisfies the equation,*

$$c_r(A,B;C,D) = \left[(A-B)^{-1} - (A-D)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}.$$

Proof: From the definition 3.1 we have $c_r(A,B;C,D) = \left[(A-D)^{-1}(B-D)\right]\left[(B-C)^{-1}(A-C)\right]$, and the points of this line form a skew-field, therefore, we have association property:

$$\begin{aligned} R &= c_r(A,B;C,D) \\ &= [(A-D)^{-1}(B-D)][(B-C)^{-1}(A-C)] \\ &\text{(since from skew field-associative property, have)} \\ &= \left[(A-D)^{-1}(B-D)(B-C)^{-1}\right](A-C). \end{aligned}$$

So, the point R, is,

$$R = \left[(A-D)^{-1}(B-D)(B-C)^{-1}\right](A-C),$$

multiply in the right side by side with $(A-C)^{-1}$, which leads to

$$R \cdot (A-C)^{-1} = (A-D)^{-1}(B-D)(B-C)^{-1},$$

now multiply in the right side by side with $(B-C)$, which leads to

$$(A-D)^{-1}(B-D) = \left(R\cdot(A-C)^{-1}\right)(B-C),$$

multiply side by side with $(A-B)^{-1}$, which leads to,

$$\left[(A-D)^{-1}(B-D)\right](A-B)^{-1} = \left[R\cdot(A-C)^{-1}(B-C)\right](A-B)^{-1}.$$

Transform the left side of this equation as

$$\begin{aligned}\left[(A-D)^{-1}(B-D)\right](A-B)^{-1} &= \left[(A-D)^{-1}(B+A-A-D)\right](A-B)^{-1}\\ &= \left[(A-D)^{-1}([A-D]-[A-B])\right](A-B)^{-1},\end{aligned}$$

rewrite it, which gives

$$\begin{aligned}\left[(A-D)^{-1}(B-D)\right](A-B)^{-1} &= [(A-D)^{-1}([A-D]-[A-B])](A-B)^{-1}\\ &= [(A-D)^{-1}[A-D]-(A-D)^{-1}[A-B]](A-B)^{-1}\\ &= [I-(A-D)^{-1}[A-B]](A-B)^{-1}\\ &= (A-B)^{-1}-(A-D)^{-1}[A-B](A-B)^{-1}\\ &= (A-B)^{-1}-(A-D)^{-1}.\end{aligned}$$

So, have

$$(A-B)^{-1}-(A-D)^{-1} = \left[R(A-C)^{-1}(B-C)\right](A-B)^{-1}.$$

In the same way as above, (always bearing in mind that the points of a line of Desargues affine planes form a skew-field related to the addition and multiplication of the points, on this line, and the properties that satisfy a skew-field) we do the following transformations.

First we have the associative property for the multiplication of points on a line,

$$\left[R(A-C)^{-1}(B-C)\right](A-B)^{-1} = R\left[(A-C)^{-1}(B-C)(A-B)^{-1}\right].$$

Now we transform the expression

$$\begin{aligned}\left[(A-C)^{-1}(B-C)(A-B)^{-1}\right] &= \left[(A-C)^{-1}(B+A-A-C)(A-B)^{-1}\right]\\ &= \left[(A-C)^{-1}([A-C]-[A-B])(A-B)^{-1}\right]\\ &= [I-(A-C)^{-1}[A-B]](A-B)^{-1}\\ &= (A-B)^{-1}-(A-C)^{-1}[A-B](A-B)^{-1}\\ &= (A-B)^{-1}-(A-C)^{-1}.\end{aligned}$$

So, have

$$(A-B)^{-1}-(A-D)^{-1} = R\left[(A-B)^{-1}-(A-C)^{-1}\right].$$

Hence

$$R = \left[(A-B)^{-1}-(A-D)^{-1}\right]\left[(A-B)^{-1}-(A-C)^{-1}\right]^{-1},$$

so,

$$c_r(A,B;C,D) = \left[(A-B)^{-1}-(A-D)^{-1}\right]\left[(A-B)^{-1}-(A-C)^{-1}\right]^{-1}.$$

□

Theorem 3.5 *If A,B,C,D are distinct points in a line, of Desargues affine plane and I is unital point for the multiplications of points in same line, then*

$$I-c_r(A,B;C,D) = c_r(A,C;B,D).$$

Proof: Let's start the calculations, by theorem 3.4, have

$$I - \left[(A-B)^{-1} - (A-D)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1} =$$

$$= \left\{(A-C)^{-1} - (A-D)^{-1}\right\}\left[\left\{(A-C)^{-1} - (A-B)^{-1}\right\}\right]^{-1},$$

write,

$$I = \left[(A-B)^{-1} - (A-C)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1},$$

so,

$$\begin{aligned}
I - c_r(A,B;C,D) &= I - \left[(A-B)^{-1} - (A-D)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}\\
&= \left[(A-B)^{-1} - (A-C)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}\\
&- \left[(A-B)^{-1} - (A-D)^{-1}\right]\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}\\
&= \left\{\left[(A-B)^{-1} - (A-C)^{-1}\right] - \left[(A-B)^{-1} - (A-D)^{-1}\right]\right\}\\
&\cdot \left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}\\
&= \left\{-(A-C)^{-1} + (A-D)^{-1}\right\}\left[(A-B)^{-1} - (A-C)^{-1}\right]^{-1}\\
&= (-I)\left\{(A-C)^{-1} - (A-D)^{-1}\right\}\left[(-I)\left\{(A-C)^{-1} - (A-B)^{-1}\right\}\right]^{-1}\\
&= (-I)\left\{(A-C)^{-1} - (A-D)^{-1}\right\}\left[\left\{(A-C)^{-1} - (A-B)^{-1}\right\}\right]^{-1}(-I)^{-1}\\
&= \left\{(A-C)^{-1} - (A-D)^{-1}\right\}\left[\left\{(A-C)^{-1} - (A-B)^{-1}\right\}\right]^{-1}\\
&= c_r(A,C;B,D).
\end{aligned}$$

from skew-field properties, we have $(-I)^{-1} = -I$ and $(-I)(-I) = I$. □

Theorem 3.6 *If A,B,C,D are distinct points in a line, of the Desargues affine plane and I is unitary point for multiplications of points in same line, then,*

(a) $c_r(A,D;B,C) = I - c_r^{-1}(A,B;C,D)$;

(b) $c_r(A,C;D,B) = [I - c_r(A,B;C,D)]^{-1}$;

(c) $c_r(A,D;C,B) = [c_r(A,B;C,D) - I]^{-1}c_r(A,B;C,D)$.

Proof: (a) In theorem 3.3 we have prove that $c_r^{-1}(A,B;C,D) = c_r(A,B;D,C)$, and from theorem 3.5, have that $I - c_r(A,B;D,C) = c_r(A,D;B,C)$. So, we have prove that

$$I - c_r^{-1}(A,B;C,D) = I - c_r(A,B;D,C) = c_r(A,D;B,C).$$

(b) From theorem 3.5, we have that, $I - c_r(A,B;C,D) = c_r(A,C;B,D)$, and from theorem 3.3 have that $[c_r(A,C;B,D)]^{-1} = c_r(A,C;D,B)$, so have that

$$c_r(A,C;D,B) = [c_r(A,C;B,D)]^{-1} = [I - c_r(A,B;C,D)]^{-1}.$$

(c) At this point we will prove that: $c_r(A,D;C,B) = [c_r(A,B;C,D) - I]^{-1}c_r(A,B;C,D)$.

From point (a), we prove that $c_r(A,D;C,B) = I - c_r^{-1}(A,B;C,D)$, and from theorem 3.3 have that $c_r(A,D;C,B) = c_r^{-1}(A,D;B,C)$. So, we have that

$$c_r(A,D;C,B) = [I - c_r^{-1}(A,B;C,D)]^{-1}.$$

Mark the cross-ratios point $R = c_r(A,B;C,D)$, and rewrite. So we have to prove that the equation holds,

$$[I - R^{-1}]^{-1} = [R - I]^{-1}R,$$

remember that the points are points of $\ell^{OI}-$line, in Desargues affine planes, and can also be thought of as elements of skew-fields $K = (\ell^{OI}, +, \cdot)$, therefore, we can make algebraic transformations, allowed for skew-fields, and we have

$$\begin{aligned}
\left[I - R^{-1}\right]^{-1} &= [R - I]^{-1}R \quad \text{(multiply from the right with } R^{-1}\text{)} \\
[I - R^{-1}]^{-1} \cdot R^{-1} &= [R - I]^{-1}R \cdot R^{-1} \quad \text{(from skew field property we have that } p^{-1}q^{-1} = (qp)^{-1}\text{)} \\
[R(I - R^{-1})]^{-1} &= [R - I]^{-1}[R \cdot R^{-1}] \\
[R \cdot I - R \cdot R^{-1}]^{-1} &= [R - I]^{-1} \cdot I \\
[R - I]^{-1} &= [R - I]^{-1}.
\end{aligned}$$

□

Theorem 3.7 *If A, B, C, D are distinct points, and different from zero-point O, in a line, of the Desargues affine plane and I is unitary point for multiplications of points in same line, have,*

$$c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = A \cdot c_r(A, B; C, D) \cdot A^{-1}.$$

Proof: From cross-ratio definition 3.1, we have,

$$c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = [(A^{-1} - D^{-1})^{-1}(B^{-1} - D^{-1})][(B^{-1} - C^{-1})(A^{-1} - C^{-1})].$$

Points A, B, C, D and $A^{-1}, B^{-1}, C^{-1}, D^{-1}$, are points of $\ell^{OI}-$line in Desargues affine plane, so they are also elements of the skew field $K = (\ell^{OI}, +, \cdot)$. First we prove that, for tow elements X, Y in a skew field K, we have that $X^{-1} - Y^{-1} = Y^{-1}(Y - X)X^{-1}$. Indeed

$$\begin{aligned}
Y^{-1}(Y - X)X^{-1} &= [Y^{-1}(Y - X)]X^{-1} \\
&= (Y^{-1}Y - Y^{-1}X)X^{-1} \\
&= (I - Y^{-1}X)X^{-1} \\
&= IX^{-1} - Y^{-1}(XX^{-1}) \\
&= X^{-1} - Y^{-1}I \\
&= X^{-1} - Y^{-1}.
\end{aligned}$$

We use this result in the calculation of $c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1})$, and have

$$\begin{aligned}
c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) &= [(A^{-1} - D^{-1})^{-1}(B^{-1} - D^{-1})] \\
&\cdot [(B^{-1} - C^{-1})(A^{-1} - C^{-1})] \\
&= [(D^{-1}(D - A)A^{-1})^{-1}(D^{-1}(D - B)B^{-1})] \\
&\cdot [(C^{-1}(C - B)B^{-1})(C^{-1}(C - A)A^{-1})] \\
&= [(A(D - A)^{-1}D)(D^{-1}(D - B)B^{-1})] \\
&\cdot [(B(C - B)^{-1}C)(C^{-1}(C - A)A^{-1})] \\
&\text{(from skew field properties } (abc)^{-1} = c^{-1}b^{-1}a^{-1}\text{)} \\
&= [A(D - A)^{-1}(DD^{-1})(D - B)B^{-1}] \\
&\cdot [B(C - B)^{-1}(CC^{-1})(C - A)A^{-1}] \\
&\text{(from associative properties for multiplication)} \\
&= [A(D - A)^{-1}(I)(D - B)B^{-1}][B(C - B)^{-1}(I)(C - A)A^{-1}] \\
&= [A(D - A)^{-1}(D - B)B^{-1}][B(C - B)^{-1}(C - A)A^{-1}] \\
&= A[(D - A)^{-1}(D - B)B^{-1}][B(C - B)^{-1}(C - A)]A^{-1} \\
&= A\left\{[(D - A)^{-1}(D - B)B^{-1}][B(C - B)^{-1}(C - A)]\right\}A^{-1} \\
&= A \cdot c_r(A, C; B, D) \cdot A^{-1}.
\end{aligned}$$

therefore, we can say that the points, $c_r(A, C; B, D)$ and $c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1})$, are *conjugatet-points* in a line of Desargues affine plane. □

Corollary 3.1 *If A is the point of $z[K]$ (center of skew field $K = (\ell^{OI}, +, \cdot)$), then,*

$$c_r(A, C; B, D) = c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}).$$

Proof: If $A \in z[K]$ then, $AX = XA, \forall X \in K$, so $AXA^{-1} = X, \forall X \in K$. So, for $A \in z[K]$, we have that,

$$A \cdot c_r(A, C; B, D) \cdot A^{-1} = c_r(A, C; B, D).$$

Hence

$$c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = c_r(A, C; B, D) \Leftrightarrow \text{if } A \in z[K].$$

□

Corollary 3.2 *In Papus affine plane, $c_r(A, C; B, D) = c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1})$.*

Theorem 3.8 *If A, B, C, D are distinct points in a line, of the Desargues affine plane and I is unitary point for multiplications of points in same line, have,*

$$c_r(A, B; C, D) \neq c_r(B, A; D, C),$$

so, $c_r(A, B; C, D)$ is different point from $c_r(B, A; D, C)$.

Proof: From Definition of Cross-Ratio we have,

$$c_r(A, B; C, D) = \left[(A - D)^{-1}(B - D)\right]\left[(B - C)^{-1}(A - C)\right] = r(B, A; D) \cdot r(A, B; C),$$

and

$$c_r(B, A; D, C) = \left[(B - C)^{-1}(A - C)\right]\left[(A - D)^{-1}(B - D)\right] = r(A, B; C) \cdot r(B, A; D),$$

We mark the points, like below

$$R_1 = r(A, B; C) \quad \text{and} \quad R_2 = r(B, A; D),$$

so

$$c_r(A, B; C, D) = R_2 \cdot R_1 \quad \text{and} \quad c_r(B, A; D, C) = R_1 \cdot R_2.$$

This points are in $\ell^{OI}--$line in Desargues affine plane, so are elements of the skew fields $K = (\ell^{OI}, +, \cdot)$, which are constructet over this line, so $E, F, G, H \in K$. So we have,

$$R_2 \cdot R_1 \neq R_1 \cdot R_2 \Rightarrow c_r(A, B; C, D) \neq c_r(B, A; D, C).$$

□

Corollary 3.3 *If $A, B, C, D \in \ell^{OI}$ are distinct points in a line, of the Pappus affine plane and I is unital point for multiplications, then*

$$c_r(A, B; C, D) = c_r(B, A; D, C).$$

Proof: If affine plane is Pappian plane, then the skew-field $(\ell^{OI}, +, \cdot)$ is commutative, then is a Field. □

We marked with $K = (\ell^{OI}, +, \cdot)$ the skew field over $\ell^{OI}-$line in Desargues affine plane, we know that the center of the skew field $z[K]$, is a sub-skew field of K, moreover, $z[K]$ it is also commutative.

Theorem 3.9 *If $A, B, C, D \in \ell^{OI}$ are distinct points in a line, of the Desargues affine plane and I is unital point for multiplications of points in same line, then equation*

$$c_r(A, B; C, D) = c_r(B, A; D, C).$$

Is true, if

(a) *points A, B, C, D are in 'center of skew-field' $z[K]$;*

(b) *ratio-points $r(A, B; C)$ are in 'center of skew-field';*

(c) *ratio-point $r(B, A; D)$ are in 'center of skew-field';*

(d) *ratio-point $r(A, B; D)$ is in centaralizer of point $r(A, B; C)$, or vice versa.*

Proof: (a) If points $A, B, C, D \in z[K]$, we have that,

$$A - D, B - D, B - C, A - C \in z[K],$$

and

$$(A - D)^{-1}, (B - D)^{-1}, (B - C)^{-1}, (A - C)^{-1} \in z[K],$$

also the production is commutative. Hence,

$$[(A - D)^{-1}(B - D)] \cdot [(B - C)^{-1}(A - C)] = [(B - C)^{-1}(A - C)] \cdot [(A - D)^{-1}(B - D)],$$

so,

$$c_r(A, B; C, D) = c_r(B, A; D, C).$$

(b) If ratio-points $r(A, B; C)$ are in 'center of skew-field', we have that,

$$X \cdot r(A, B; C) = r(A, B; C) \cdot X, \quad \forall X \in K (\text{so for all points } X \in \ell^{OI}),$$

so the equation is also true for the ratio-point $r(B, A; D)$, and have,

$$r(A, B; C) \cdot r(B, A; D) = r(B, A; D) \cdot r(A, B; C)$$
$$[(B - C)^{-1}(A - C)] \cdot [(A - D)^{-1}(B - D)] = [(A - D)^{-1}(B - D)] \cdot [(B - C)^{-1}(A - C)]$$
$$c_r(B, A; D, C) = c_r(A, B; C, D).$$

(c) in the same way, as in case (b).

(d) The Centralizer $\mathcal{C}_K(r(A, B; C)) = \{Y \in K | Y \cdot r(A, B; C) = r(A, B; C) \cdot Y\}$, and we have that, $r(B, A; D) \in \mathcal{C}_K(r(A, B; C))$, so we have,

$$r(B, A; D) \cdot r(A, B; C) = r(A, B; C) \cdot r(B, A; D),$$

so,

$$c_r(A, B; C, D) = c_r(B, A; D, C),$$

in the same way, it is proved that if $r(A, B; C) \in \mathcal{C}_K(r(B, A; D))$, then $c_r(A, B; C, D) = c_r(B, A; D, C)$. □

After presenting the new idea, related to the ratio of 2 and 3 points in a line on Desargues affine plane (see [18]), in this paper we extended and studied the cross-ratio of 4-points in a line on Desargues affine plane, also in paper [20], we studied some invariant transforms and preserving transforms for ratio of 4-points. Also a very nice interpretation is presented in papers [18], [21] related to Dyck Free Group presentation and Dyck Fundamental Group.

In theorem 3.1 the existence and uniqueness of the point obtained from the cross-ratio of 4-different points in a line on Desargues affine plane is shown and in theorem 3.4 another representation is given that makes it easy for us to do some further calculations related to the cross-ratio of 4-points. We believe that in this article we have completely exhausted the connection and relationship of every 4 different points in a line on Desargues affine plane, related to the cross-ratio. A summary of our main results is presented in Theorems: 3.6, 3.7, 3.8, 3.9. We hope and believe that the results in this direction will increase significantly and will continue to emerge for a long time.

Declaration of Interests:

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There is no conflict of interest with any funder.

Authors' contributions

The authors' contribution is as follows: O. Zaka introduced the geometry of ratio for two and for three points and cross-ratio for 4-points in a line on Desargues affine plane, discovered and proved the results in this paper. He made a wide and detailed discussion regarding the possible situations for cross-ratio of 4-points and their connections with "line-skew field" on a Desargues affine plane. J.F. Peters reviewd and refined and clarified various aspects of the presented geometry and results in this paper.

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