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On the nonlinear quadratically perturbed fractional differential systems via complex order derivative

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ABSTRACT: The aim of this work is to investigate the existence of solutions for a nonlinear quadratically perturbed system involving the ψ-Hilfer fractional derivative of complex order $\beta \in \mathbb{C}$, where $\beta = \alpha + i\gamma$, $(\alpha, \gamma, \in \mathbb{R})$ and $0 < \alpha < 1$. The existence of solutions is established using Dhage's well-known fixed-point theorem. Finally, an example is provided to illustrate the results of this study.

Key Words: Quadratically perturbed system, Complex order derivative, ψ-Hilfer fractional derivative, Fixed point theorem.

Contents

1. Introduction

Traditional calculus is based on differentiation and integration of integer order. The concept of fractional calculus has enormous potential to change the way we perceive and model the nature around us. With growing interest from both physicists and engineers, significant research efforts have been dedicated to fractional calculus. See [1,16,3,4]. Researchers have confirmed that the use of fractional differentiation and integration operators is desirable for describing the properties of various materials.

Fractional order equations play an important role in the study of phenomena in physics, biology, viscoelasticity, and population dynamics. See [18,21,2,22]. The thing that calls us to think about generalizing the order of derivation to a complex order. The complex fractional order theory was developed with the research of several authors among whom, E.R. Love who studied the derivative of imaginary order in [15], the authors of [17] studied the existence of solutions to boundary value problems with imaginary order, and in [20] Vivek et al studied Ulam stability of integro-differential equations with imaginary order.

The fixed point approach is used to demonstrate the existence of a solution to a class of ordinary differential equations, in particular equations of fractional order. These theorems have evolved over time according to the presentation of the nonlinearity and the domain in which we work. Among these theorems the fixed point theorem is due to Dhage allows us to study the existence of the solution of quadratically perturbed equations. See [5,6].

Quadratically perturbed equation theory is highly useful for studying nonlinear dynamical systems that are challenging to solve or analyze directly. The nonlinearity of such systems is often non-smooth, complicating the study of solution existence and other properties. However, by introducing perturbations, these problems can be approached using existing methods to explore various aspects of their solutions.

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Dynamical systems modified in this manner are known as hybrid differential equations. For more details on quadratically perturbed equations, we refer to [7], [9], [10], [23], [11].

The authors of [23] investigated the existence of solutions, for the following nonlinear quadratically perturbed fractional differential equation with variable order derivative:

$$\begin{cases} {}^{H}D_{0_+}^{\theta(t),\sigma,\psi}\left(\frac{u(t)}{\mathcal{F}(t,{}^{H}D_{0_+}^{\lambda(t),\sigma,\psi}u(t))}\right) = \mathcal{H}(t,{}^{H}D_{0_+}^{\lambda(t),\sigma,\psi}u(t)), \quad t\in[0,T] \\ (\psi(t)-\psi(0))^{1-\zeta(t)}u(t)|_{t=0}=u_0, \quad u_0\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $0<\lambda(t)<\theta(t)<1$, $0<\sigma<1$, $\zeta(t)=\theta(t)+\sigma(1-\theta(t))$, and ${}^{H}D_{0_+}^{\theta(t),\sigma,\psi}(.)$ is the ψ-Hilfer fractional derivative of variable order $\theta(t)$ and type σ.
In [24], the authors studied the existence results for the following nonlinear quadratically perturbed fractional differential equation involving the p-Laplacian operator:

$$\begin{cases} {}_{\delta}^{C}D_{0_+}^{\lambda,g}\Phi_p\left({}_{\delta}^{C}D_{0_+}^{\nu,g}\left(\frac{w(t)}{\mathcal{Y}(t,w(t))}\right)\right)=\mathcal{Z}(t,w(t)), \quad t\in\Sigma=[0,b], \\ \left(\frac{w(t)}{\mathcal{Y}(t,w(t))}\right)_{t=0}=w_0, \quad \left(\frac{w(t)}{\mathcal{Y}(t,w(t))}\right)'_{t=0}=0, \quad w_0\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $0<\lambda<1$, $1<\nu<2$, ${}_{\delta}^{C}D_{0^+}^{\lambda,g}(.)$ is the generalized Caputo proportional fractional derivative of order ϑ, and Φ_p is the p-Laplacian operator.

Inspired by the above works, in this paper, we examine the existence of solutions for the following nonlinear hybrid fractional differential equation involving the ψ-Hilfer fractional derivative of complex order $\beta\in\mathbb{C}$:

$$\begin{cases} {}^{H}D_{a_+}^{\beta,\sigma,\psi}\left(\frac{x(t)}{\mathcal{W}(t,I_{a^+}^{\beta,\psi}x(t))}\right)=\mathcal{Q}(t,I_{a^+}^{\beta,\psi}x(t)), \quad t\in\mathcal{T}=[a,b] \\ (\psi(t)-\psi(a))^{1-\zeta}x(t)|_{t=a}=\mu, \ \mu\in\mathbb{C}, \end{cases} \tag{1.3}$$

where ${}^{H}D_{a_+}^{\beta,\sigma,\psi}$ is the ψ-Hilfer fractional derivative of complex order $\beta=\alpha+i\gamma$ and complex type σ, $\zeta=\beta+\sigma(1-\beta)=\rho+i\eta$, such that $0<\alpha<1$, $0<\rho<1$, $0\le\Re(\sigma)\le1$, $(\alpha,\gamma,\rho,\eta\in\mathbb{R})$, $\mathcal{W}\in C([a,b]\times\mathbb{C},\mathbb{C}\setminus\{0\})$ and $\mathcal{Q}\in Car([a,b]\times\mathbb{C},\mathbb{C})$. ($Car([a,b]\times\mathbb{C},\mathbb{C})$ is called the Caratheodory class of functions).

2. Preliminaires

In this section, we present several definitions and notations that will be used consistently throughout the subsequent sections of this work.

- Let $\mathcal{T}=[a,b]$ be a finite interval of $\mathbb{R}$, with $0<a<b<\infty$.
- We denote by $C(\mathcal{T},\mathbb{C})$ the space of all continuous functions with the norm $\|g\|=sup\{|g(t)|:t\in\mathcal{T}\}$.
- We denote by $Car(\mathcal{T}\times\mathbb{C},\mathbb{C})$ the class of functions $\mathcal{H}:\mathcal{T}\times\mathbb{C}\longrightarrow\mathbb{C}$ such that:

  (i) The map $t\longrightarrow\mathcal{H}(t,u)$ is measurable for each $u\in\mathbb{C}$.

  (ii) The map $u\longrightarrow\mathcal{H}(t,u)$ is continuous for each $t\in\mathcal{T}$.

- Define the weighted space $C_{1-\rho,\psi}([a,b])$ by

$$C_{1-\rho,\psi}([a,b])=\left\{x:(a,b]\to\mathbb{R},(\psi(t)-\psi(a))^{1-\rho}x(t)\in C^1[a,b]\right\}, \quad 0<\rho=\Re(\zeta)<1.$$

Clearly, $C_{1-\rho,\psi}([a,b],\mathbb{R})$ is a Banach space endowed with the norm

$$\|x(t)\|_{C_{1-\rho,\psi}([a,b])}=max_{t\in[a,b]}|(\psi(t)-\psi(a))^{1-\rho}x(t)|.$$

- Throughout this paper we consider the function $\psi:[a,b]\to\mathbb{R}$, that is an increasing differentiable function such that $\psi'(t)>0$, for all $t\in[a,b]$.

Lemma 2.1 *Let $t \in \mathcal{T}$, $\vartheta \in \mathbb{C}$ such that $\vartheta = \epsilon + i\varsigma$ with $(\epsilon, \varsigma \in \mathbb{R})$ and let ψ be the function mentioned above, then we have*

$$|(\psi(t) - \psi(a))^{\vartheta}| = (\psi(t) - \psi(a))^{\epsilon}.$$

Proof: We have

$$\begin{aligned} |(\psi(t) - \psi(a))^{\vartheta}| &= |(\psi(t) - \psi(a))^{\epsilon + i\varsigma}| \\ &= \left| e^{\ln((\psi(t)-\psi(a))^{\epsilon+i\varsigma})} \right| \\ &= \left| e^{\epsilon \ln((\psi(t)-\psi(a))} e^{i\varsigma \ln((\psi(t)-\psi(a)))} \right| \\ &= \left| e^{\epsilon \ln(\psi(t)-\psi(a))} \right| |\cos(\varsigma \ln((\psi(t) - \psi(a))) + i \sin(\varsigma \ln((\psi(t) - \psi(a)))| \\ &= (\psi(t) - \psi(a))^{\epsilon}. \end{aligned}$$

□

Definition 2.1 *[13] Let $\beta \in \mathbb{C}$, $(\Re(\beta) > 0)$ and g be an integrable function, we define the left-sided ψ-Riemann-Liouville fractional integral of order β of the function g by*

$$I_{a_+}^{\beta;\psi} g(t) = \frac{1}{\Gamma(\beta)} \int_a^t \psi^{'}(s)(\psi(t) - \psi(s))^{\beta-1} g(s) ds.$$

Definition 2.2 *[12] The Stirling asymptotic formula of the Gamma function for $\tau \in \mathbb{C}$ is defined by*

$$\Gamma(\tau) = (2\pi)^{\frac{1}{2}} \tau^{\frac{\tau-1}{2}} e^{-\tau} \left(1 + O\left(\frac{1}{\tau}\right)\right), \quad (|arg(\tau)| < \pi, |\tau| \to \infty). \tag{2.1}$$

And

$$|\Gamma(\alpha + i\gamma)| = (2\pi)^{\frac{1}{2}} |\gamma|^{\alpha - \frac{1}{2}} e^{-\alpha - \frac{\pi|\gamma|}{2}} \left(1 + O\left(\frac{1}{\tau}\right)\right), \quad (\gamma \to \infty). \tag{2.2}$$

Definition 2.3 *[19] Define the left-sided ψ-Riemann-Liouville fractional derivative of order $\beta \in \mathbb{C}$, $(\Re(\beta) > 0)$ of a function $g \in C^n([a, b], \mathbb{R})$ by*

$$D_{a_+}^{\beta;\psi} g(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a_+}^{n-\beta;\psi} g(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi^{'}(s)(\psi(t) - \psi(s))^{n-\beta-1} g(s) ds,$$

where $n = [\Re(\beta)] + 1$.

Definition 2.4 *[19] The left-sided ψ-Hilfer fractional derivative of a function $g \in C^n([a, b], \mathbb{R})$ of order $\beta \in \mathbb{C}$, $(n - 1 < (\Re(\beta) < n)$ and type $\sigma \in \mathbb{C}$, $(0 \leq (\Re(\sigma) \leq 1)$ is determined as*

$${}^{H}D_{a_+}^{\beta,\sigma,\psi} g(t) = I_{a_+}^{\sigma(n-\beta);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a_+}^{(1-\sigma)(n-\beta);\psi} g(t).$$

In other way

$${}^{H}D_{a_+}^{\beta,\sigma,\psi} g(t) = I_{a_+}^{\sigma(n-\beta);\psi} D_{a_+}^{\zeta,\psi} g(t),$$

where

$$D_{a_+}^{\zeta,\psi} g(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^{(n)} I_{a_+}^{(1-\sigma)(n-\beta);\psi} g(t).$$

with $\zeta = \beta + \sigma(n - \beta)$ and $n = [\Re(\beta)] + 1$.

Theorem 2.1 *[14], (Arzela-Ascoli theorem)*
Let $\mathcal{X}$ be a compact Banach space and $\mathcal{E}$ any space, a subset $\mathcal{M}$ of $C(\mathcal{X}, \mathcal{E})$ is relatively compact if and only if

1) $\mathcal{M}$ is uniformly bounded.

2) $\mathcal{M}$ is equicontinuous.

3) For all $x \in \mathcal{X}$, the space $\mathcal{M}(x) = \{f(x), \forall x \in \mathcal{X}\}$ is relatively compact in $\mathcal{E}$.

3. Existence Results

In this section, we present and study the existence of solution for the given nonlinear quadratically perturbed system (1.3) under the Dhage's fixed point theorem.

Theorem 3.1 *[8] Let $\mathcal{S}$ be a closed, bounded, and convex subset of a Banach algebra $\mathcal{X}$. We consider the two operators $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{P} : \mathcal{S} \to \mathcal{X}$ such that:*

a) $\mathcal{F}$ is Lipschitzian with a Lipschitz constant Θ.

b) $\mathcal{P}$ is completely continuous.

c) $u = FuPv \Rightarrow u \in \mathcal{S}$, for all $v \in \mathcal{S}$.

d) $\Theta\Delta < 1$, where $\Delta = \|\mathcal{P}(\mathcal{S})\|$.
Then, the operator $\Pi u = \mathcal{F}u\mathcal{P}u$ has a fixed point in $\mathcal{S}$.

To give the solution of the problem (3.1), we apply the same reasoning as the authors used in the real case in [23]. Then we have the following definition:

Definition 3.1 *If x is a solution of the problem (1.3), Then x is also a solution of the following integral equation (3.1)*

$$x(t) = \mathcal{W}(t, I_{a^+}^{\beta,\psi} x(t)) \left\{ \frac{\mu(\psi(t)-\psi(a))^{\zeta-1}}{\mathcal{W}(a, I_{a^+}^{\beta,\psi} x(a))} + I_{a^+}^{\beta,\psi} \mathcal{Q}(t, I_{a^+}^{\beta,\psi} x(t)) \right\}. \tag{3.1}$$

Now, we give some hypotheses on which we will rely to prove the existence of the solution to the problem (1.3).

$(H_{\mathcal{W}})$: The function $\mathcal{W} \in C(\mathcal{T} \times \mathbb{C}, \mathbb{C} \setminus \{0\})$ is bounded and there exist constants $\delta > 0$ and $\mathcal{L} > 0$ such that for all $u, v \in \mathbb{C}$, and $t \in \mathcal{T}$ we have

$$|\mathcal{W}(t, u) - \mathcal{W}(t, v)| \leq \delta|u - v|, \quad |\mathcal{W}(t, u)| \leq \mathcal{L}.$$

$(H_{\mathcal{Q}})$: Let $\mathcal{Q} \in Car(\mathcal{T} \times \mathbb{C}, \mathbb{C})$, we consider the function $\mathcal{K} \in C_{(1-\rho,\psi)}(\mathcal{T}, \mathbb{C})$ such that:

$$|\mathcal{Q}(t, u)| \leq (\psi(t) - \psi(a))^{1-\rho} |\mathcal{K}(s)| \quad t \in \mathcal{T}, u \in \mathbb{C}.$$

- We consider the Banach space $\mathcal{X} = \left(C_{(1-\rho,\psi)}(\mathcal{T}, \mathbb{C}), \| \cdot \|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\right)$.
- We define the closed, bounded and convex subset $\mathcal{S}$ of $\mathcal{X}$ by

$$\mathcal{S} = \left\{ u \in \mathcal{X}, \| u \|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} \leq \Omega \right\},$$

where

$$\Omega = \mathcal{L}\left(\left| \frac{\mu}{\mathcal{W}(a, I_{a^+}^{\beta,\psi} x(a))} \right| + \frac{(\psi(b) - \psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|} \|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} \right).$$

- Define the operators $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{P} : \mathcal{S} \to \mathcal{X}$ by

$$\mathcal{F}x(t) = \mathcal{W}(t, x(t)).$$

$$\mathcal{P}x(t) = \frac{\mu(\psi(t)-\psi(a))^{\zeta-1}}{\mathcal{W}(a, I^{\beta,\psi}_{a^+}x(a))} + I^{\beta,\psi}_{a^+}Q(t, I^{\beta,\psi}_{a^+}x(t)).$$

- We consider the operator $\Pi : \mathcal{S} \to \mathcal{X}$ defined by

$$\Pi x(t) = \mathcal{F}x(t)\mathcal{P}x(t).$$

Theorem 3.2 *Suppose that the hypotheses $(H_{\mathcal{W}}) - (H_{\mathcal{Q}})$ are satisfied. Then, the problem (1.3) has a solution $u \in \mathcal{X}$ provided that*

$$\frac{\delta(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\beta)|}\left(\left|\frac{\mu}{\mathcal{W}(a, I^{\beta,\psi}_{a^+}x(a))}\right| + \frac{(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\right) < 1. \tag{3.2}$$

Proof: It suffices to show that the two operators $\mathcal{F}$ and $\mathcal{P}$ satisfy the four conditions of Dhage's fixed point Theorem.
The proof is given in the following four claims.
Claim 1: $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a Lipschitz operator:
Using the hypothesis $(H_{\mathcal{W}})$ and Lemma 2.1, we get

$$\begin{aligned}
& |(\psi(t)-\psi(a))^{1-\rho}(\mathcal{F}x(t)-\mathcal{F}y(t))| \\
= \; & |(\psi(t)-\psi(a))^{1-\rho}(\mathcal{W}(t, I^{\beta,\psi}_{a^+}x(t)) - \mathcal{W}(t, I^{\beta,\psi}_{a^+}y(t)))| \\
\leq \; & \delta|(\psi(t)-\psi(a))^{1-\rho}(I^{\beta,\psi}_{a^+}x(t) - I^{\beta,\psi}_{a^+}y(t))| \\
\leq \; & \delta\frac{(\psi(t)-\psi(a))^{1-\rho}}{|\Gamma(\beta)|}\int_a^t \psi^{'}(s)\,|(\psi(t)-\psi(s))|^{\beta-1}\,|(x(s)-y(s))|ds \\
\leq \; & \delta\frac{(\psi(b)-\psi(a))^{1-\rho}}{|\Gamma(\beta)|}\parallel x-y \parallel_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}ds \\
\leq \; & \frac{\delta(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\parallel x-y \parallel_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} \\
\leq \; & \Theta \parallel x-y \parallel_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})},
\end{aligned}$$

where $\Theta = \frac{\delta(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}$.
Therefore, the operator $\mathcal{F}$ is Lipschitzian, with a Lipschitz constant Θ.
Claim 2: The operator $\mathcal{P} : \mathcal{S} \to \mathcal{X}$ is completely continuous:

(i) The operator $\mathcal{P} : \mathcal{S} \to \mathcal{X}$ is continuous:
We consider a sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$ such that $x_n \to x$ as $n \to \infty$ in $\mathcal{S}$. Then, we have

$$\begin{aligned}
& |(\psi(t)-\psi(a))^{1-\rho}(\mathcal{P}x_n(t)-\mathcal{P}x(t))| \\
= \; & |(\psi(t)-\psi(a))^{1-\rho}(I^{\beta,\psi}_{a^+}\mathcal{Q}(t, I^{\beta,\psi}_{a^+}x_n(t)) - I^{\beta,\psi}_{a^+}\mathcal{Q}(t, I^{\beta,\psi}_{a^+}x(t)))| \\
\leq \; & (\psi(t)-\psi(a))^{1-\rho}I^{\beta,\psi}_{a^+}|\mathcal{Q}(t, I^{\beta,\psi}_{a^+}x_n(t)) - \mathcal{Q}(t, I^{\beta,\psi}_{a^+}x(t))|.
\end{aligned}$$

Using the fact that $\mathcal{Q} \in Car(T \times \mathbb{C}, \mathbb{C})$ and Lebesgue dominated convergence theorem, we get $\|\mathcal{P}x_n - \mathcal{P}x\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} \to 0$ as $n \to \infty$.
This proves that $\mathcal{P} : \mathcal{S} \to \mathcal{X}$ is continuous.

(ii) $\mathcal{P}(\mathcal{S}) = \{\mathcal{P}x : x \in \mathcal{S}\}$ is uniformly bounded.

Using hypotheses $(H_{\mathcal{W}})$, $(H_{\mathcal{Q}})$ and Lemma 2.1, for any $x\in\mathcal{S}$ and $t\in\mathcal{T}$, we get

$$\begin{aligned}
&|(\psi(t)-\psi(a))^{1-\rho}\mathcal{P}x(t)|\\
=&\left|(\psi(t)-\psi(a))^{1-\rho}\left(\frac{(\psi(t)-\psi(a))^{\zeta-1}}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\mu+I_{a^+}^{\beta,\psi}\mathcal{Q}(t,I_{a^+}^{\beta,\psi}x(t))\right)\right|\\
\leq&\frac{|(\psi(b)-\psi(a))^{i\eta}||\mu|}{|\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))|}+(\psi(b)-\psi(a))^{1-\rho}\frac{1}{|\Gamma(\beta)|}\int_a^t\psi^{'}(s)|(\psi(t)-\psi(s))^{\beta-1}||\mathcal{Q}(s,I_{a^+}^{\beta,\psi}x(s))|ds\\
\leq&\frac{|\mu|}{|\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))|}+(\psi(b)-\psi(a))^{1-\rho}\frac{1}{|\Gamma(\beta)|}\int_a^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|(\psi(s)-\psi(a))^{1-\rho}\mathcal{K}(s)|ds\\
\leq&\frac{|\mu|}{|\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))|}+\frac{(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})},
\end{aligned}$$

therefore

$$\|\mathcal{P}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\leq\frac{|\mu|}{|\mathcal{W}(a,I_{0^+}^{\beta,\psi}x(a))|}+\frac{(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}. \tag{3.3}$$

Then the operator $\mathcal{P}$ is uniformly bounded.

(iii) $\mathcal{P}(\mathcal{S})$ is equicontinuous.
Let $t_1,t_2\in\mathcal{T}$ such that $t_1<t_2$ and Let $x\in\mathcal{S}$. Then using hypotheses $(H_{\mathcal{W}})$, $(H_{\mathcal{Q}})$, and Lemma (2.1), we get

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(a))^{1-\rho}\mathcal{P}u(t_2)-(\psi(t_1)-\psi(a))^{1-\rho}\mathcal{P}u(t_1)\right|\\
=&\left|(\psi(t_2)-\psi(a))^{1-\rho}\left(\frac{(\psi(t_2)-\psi(a))^{\zeta-1}}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\mu+I_{a^+}^{\beta,\psi}Q(t_2,I_{a^+}^{\beta,\psi}x(t_2))\right)\right.\\
&\left.-(\psi(t_1)-\psi(a))^{1-\rho}\left(\frac{(\psi(t_1)-\psi(a))^{\zeta-1}}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\mu+I_{a^+}^{\beta,\psi}\mathcal{Q}(t_1,I_{a^+}^{\beta,\psi}x(t_1))\right)\right|\\
\leq&\left|\frac{\mu}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\right|\left||(\psi(t_2)-\psi(a))^{i\eta}|-|(\psi(t_1)-\psi(a))^{i\eta}|\right|\\
&+\left|(\psi(t_2)-\psi(a))^{1-\rho}I_{a^+}^{\beta,\psi}|\mathcal{Q}(t_2,I_{a^+}^{\beta,\psi}x(t_2))|-(\psi(t_1)-\psi(a))^{1-\rho}I_{a^+}^{\beta,\psi}|\mathcal{Q}(t_1,I_{a^+}^{\beta,\psi}x(t_1))|\right|\\
\leq&\left|(\psi(t_2)-\psi(a))^{1-\rho}-(\psi(t_1)-\psi(a))^{1-\rho}\right|\frac{(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}.
\end{aligned}$$

Using the continuity of the function ψ, we get $\|\mathcal{P}u(t_2)-\mathcal{P}u(t_1)\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\to 0$ as $|t_1-t_2|\to 0$ which implies that the operator $\mathcal{P}$ is equicontinuous.
From (ii), (iii) and by applying the Arzela-Ascoli theorem, we deduce that $\mathcal{P}(\mathcal{S})$ is relatively compact. According to steps 1 and 2, it follows that the operator $\mathcal{P}(\mathcal{S})$ is continuous and compact, hence it is completely continuous.
Claim 3:
Let $x(t)=\mathcal{F}x(t)\mathcal{P}y(t)$ for any $x\in\mathcal{X}$ and $y\in\mathcal{S}$, then by using our hypotheses and Lemma (2.1) for all $t\in\mathcal{T}$, we have

$$\begin{aligned}
&|(\psi(t)-\psi(a))^{1-\rho}x(t)|\\
=&|(\psi(t)-\psi(a))^{1-\rho}\mathcal{F}x(t)\mathcal{P}y(t)|\\
\leq&|\mathcal{W}(t,I_{a^+}^{\beta,\psi}x(t))|\left(\left|\frac{(\psi(t)-\psi(a))^{i\eta}\mu}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\right|+|(\psi(t)-\psi(a))^{1-\rho}I_{a^+}^{\beta,\psi}\mathcal{Q}(t,I_{a^+}^{\beta,\psi}xy(t))|\right)\\
\leq&\mathcal{L}\left(\left|\frac{\mu}{\mathcal{W}(a,I_{a^+}^{\beta,\psi}x(a))}\right|+\frac{\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\right)\\
\leq&\Omega,
\end{aligned}$$

this implies that $x \in \mathcal{S}$.
Claim 4:
Let $\Delta = \|\mathcal{P}(\mathcal{S})\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} = sup\{\|\mathcal{P}x\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})} : x \in \mathcal{S}\}$.
From inequality (3.2) and (3.3), we have

$$\begin{aligned} \Theta\Delta &\leq \Theta\left(\left|\frac{\mu}{\mathcal{W}(a, I_{a^+}^{\beta,\psi}x(a))}\right| + \frac{(\psi(b)-\psi(a))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\|\mathcal{K}\|_{C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})}\right) \\ &< 1. \end{aligned}$$

Hence deduce that all the conditions of Dhage's fixed point theorem (3.1) are satisfied, then the operator Π has a fixed point in $\mathcal{S}$.
This implies that the problem (1.3) has a solution in $C_{(1-\rho,\psi)}(\mathcal{T},\mathbb{C})$. □

4. Example

In this section, we give an example to illustrate our results. We Consider the following quadratically perturbed system involving the Hadamard fractional derivative.

$$\begin{cases} D_{1^+}^{\beta,\sigma,\psi}\left(\frac{x(t)}{\frac{1}{95}(tI_{1^+}^{\beta,\psi}x(t)+95)}\right) = \frac{tI_{1^+}^{\beta,\psi}x(t)}{tI_{1^+}^{\beta,\psi}x(t)+2}, \quad t \in [1,e] \\ (\psi(t)-\psi(1))^{1-\zeta}x(t)|_{t=1} = \frac{1}{4} + i\frac{1}{4}, \end{cases} \tag{4.1}$$

where $\psi(t) = ln(t)$, $\beta = \frac{2}{3} + i\frac{1}{2}$, $\sigma = \frac{1}{2}$, and $\mu = \frac{1}{4} + i\frac{1}{4}$.
Comparing the system (4.1) with the problem (1.3). Then we have

$$\mu = \frac{1}{4} + i\frac{1}{4} \quad and \quad \mathcal{T} = [1,e].$$

$$\mathcal{W}(t, I_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)) = \frac{1}{95}(tI_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t) + 95).$$

$$\mathcal{Q}(t, I_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)) = \frac{tI_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)}{tI_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)+2}.$$

Let's check the hypotheses $H_{\mathcal{W}}$ and $H_{\mathcal{Q}}$:
We have

$$\begin{aligned} |\mathcal{W}(t,u) - \mathcal{W}(t,v)| &= \frac{1}{95}|tu + 95 - tv - 95| \\ &\leq \frac{e}{95}|u-v|. \end{aligned}$$

Hence the function $\mathcal{W}$ is Lipschitzian, with a Lipschitz constant $\frac{e}{95}$.
It is clear that

$$\begin{aligned} \left|\mathcal{Q}(t, I_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t))\right| &= \frac{tI_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)}{tI_{1^+}^{\frac{2}{3}+i\frac{1}{2},ln(t)}x(t)+2} \\ &\leq 1. \end{aligned}$$

We notice that hypotheses $H_{\mathcal{W}}$ and $H_{\mathcal{Q}}$ are satisfied with $\delta = \frac{e}{95}$ and $\mathcal{K}(t) = 1$.
Now it only remains to verify the condition (3.2). Using definition (2.2), we find $|\Gamma(\beta)| \simeq 0,522$, then we have

$$\begin{aligned} &\frac{\delta(\psi(e)-\psi(1))^{\alpha}}{\alpha|\Gamma(\beta)|}\left(\left|\frac{\mu}{W(1, I_{1^+}^{\beta,\psi}x(1))}\right| + \frac{\psi(e)-\psi(1))^{1-\rho+\alpha}}{\alpha|\Gamma(\beta)|}\right) \\ &= \frac{\delta}{\alpha|\Gamma(\beta)|}\left\{|\mu| + \frac{1}{\alpha|\Gamma(\beta)|}\right\} \\ &\approx 0,265 < 1, \end{aligned}$$

where $\alpha = \Re(\beta) = \frac{2}{3}$.
We notice that all the conditions are satisfied then, problem (4.1) has a solution in $C_{(1-\rho,\psi)}(\mathcal{T}, \mathbb{C})$.

5. Conclusion

In this paper, we developed the theory of quadratically perturbed fractional systems with derivatives of complex order. We investigated the existence of solutions for a nonlinear hybrid fractional differential equation involving the ψ-Hilfer fractional derivative of complex order $\beta \in \mathbb{C}$ by utilizing Dhage's fixed-point theorem. Additionally, we presented an example to illustrate our main findings.

As a possible avenue for future research, we plan to extend these results by exploring the existence of solutions and Ulam-Hyers stability for a novel class of Langevin equations incorporating the ψ-Hilfer generalized proportional derivative of complex order.

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