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# On Optimal Control for Cooperative Systems Governed by Heat Equation with Conjugation Conditions

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ABSTRACT: The optimal control for $2 \times 2$ cooperative Dirichlet and Neumann parabolic systems with conjugation conditions is considered. First the existence and uniqueness of the state are proved; then the necessary and sufficient conditions for the control to be an optimal is obtained by a set of inequalities. Also we generalize these discussions to $n \times n$ cooperative parabolic systems with conjugation conditions. The control in our problems is of distributed type and is allowed to be in a Hilbert space.

Key Words: Cooperative parabolic systems - Conjugation conditions - Dirichlet and Neumann conditions - Lax-Milgram lemma - Distributed control.

## Contents



## 1. Introduction

The paper is sincere to the study of the distributed control for $n \times n$ cooperative Dirichlet and Neumann parabolic systems with conjugation conditions. These problems leads us to the minimization of the quadratic cost functional:

$$J(v) = \sum_{i=1}^{n} \|M_i(v) - z_{id}\|^2_{L^2(Q)} + \sum_{i=1}^{n} (Tv_i, v_i)_{L^2(Q)}. \tag{1.1}$$

subject to the Dirichlet state equation

$$\begin{cases} \frac{\partial}{\partial t} M_i(u) + A(t) M_i(x,t,u) = f_i(x,t) + u_i & \text{in } \; Q, \\ M_i(x,0,u) = M_{i,0}(x), \quad M_{i,0}(x) \in L^2(\Omega) & \text{in } \; \Omega, \\ M_i(x,t,u) = 0 & \text{on } \; \Sigma, \\ \qquad i = 1,2,3,....., n, & \end{cases} \tag{1.2}$$

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and the Neumann state equation

$$\begin{cases} \frac{\partial}{\partial t}M_i(u) + A(t)M_i(u) = f_i(x,t) + u_i & \text{in } \ Q, \\ M_i(x,0,u) = M_{i,0}(x), \quad M_{i,0}(x) \in L^2(\Omega) & \text{in } \ \Omega, \\ \frac{\partial M_i(u)}{\partial \nu_A} = g_i & \text{on } \ \Sigma, \\ \qquad i = 1,2,3,.....,n, \end{cases} \tag{1.3}$$

under conjugation conditions:

$$\begin{cases} [M_i(u)] = 0 & \text{on} \quad \gamma, \\ \left[\frac{\partial M_i(u)}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \beta \frac{\partial M_i(u)}{\partial x_j} \cos(v, x_i)\right] = c_i \frac{\partial M_i(u)}{\partial t} & \text{on} \quad \gamma, \quad i = 1,2,...n. \end{cases} \tag{1.4}$$

The model of our systems $A(t)$ is given by

$$\begin{bmatrix} -\nabla\cdot(\beta\nabla) - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -\nabla\cdot(\beta\nabla) - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -\nabla\cdot(\beta\nabla) - a_{nn} \end{bmatrix},$$

where $M_i(u) = \{M_1(u), M_2(u), M_3(u), ...M_n(u)\}$ are the states of the systems, $f_i \in C(Q)$ are the external sources, $Q = \Omega \times (0,T)$ be a complicated cylinder and $\Sigma = \Gamma \times (0,T)$ be the lateral surface of a cylinder $\Omega_T \cup \gamma_T$,$\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi$, $\gamma_T = \gamma \times (0,T)$, $\partial\Omega_i$ be the boundary of a domain $\Omega_i$, $i = 1,2$, $\beta = \beta(x)$ is a positive function having discontinuity along $\gamma$ and $\frac{\partial M}{\partial \nu_A} = \sum_{i,j=1}^{n} \beta \frac{\partial M}{\partial x_j} \cos(\nu, x_i)$ is directional derivative of $M$, $\nu$ is an ort of a normal to $\gamma$ and such normal is directed into the domain $\Omega_2$.

**Definition 1.1** *System (1.2) or (1.3) is called cooperative if*

$$a_{ij} > 0 \ \forall \ \ i \neq \ j, \tag{1.5}$$

*otherwise is called non-cooperative system.*

An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional. Lions [12] have studied the optimal control systems for both elliptic, hyperbolic and parabolic operators with finite order defined on $R^n$. The discussion was expanded to systems involving different types of operators such as infinite order elliptic operators [5] or operators with an infinite number of variables [6,7]. These results have been extended to cooperative systems [2,3, 4,8,13,14,16,20]. Some applications were introduced in [7,8,9,10,15]. New optimal control problems have been introduced by Sergienko and Deineka [[22]-[24]] for distributed parameter systems with conjugation conditions and by a quadratic cost functional. The considered systems in these problems are in the scalar case (system of one equation). Serag, et al [[17]-[20]] and [1] extended this discussion to $n \times n$ systems. In this paper we discuss the optimal control of distributed type for cooperative parabolic systems under conjugation conditions. Our paper is organized as follows: In section two, we introduce some definitions and notations. In section three we discuss the distributed control for $2 \times 2$ cooperative parabolic systems with Dirichlet and conjugation conditions. Section four deals with distributed control for $n \times n$ cooperative Dirichlet parabolic systems with conjugation conditions. The problem with non-homogeneous Neumann conditions under conjugation conditions is studied for cooperative parabolic systems, in section five.

## 2. Definitions and Notations [21]

In this paper, we shall consider $\Omega$ is a domanin that consists of two open, non - intersecting and bounded, continuous, strictly Lipschitz domains $\Omega_1$ and $\Omega_2$ of $R^n$ such that $\Omega = (\Omega_1 \cup \gamma \cup \Omega_2)$,$(\Omega_1 \cap \Omega_2) = \phi$

and $\bar{\Omega} = (\bar{\Omega}_1 \cup \bar{\Omega}_2)$. Furthermore: Let $\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)\backslash\gamma$ be the boundary of the domain $\bar{\Omega}$, In addition,

$$[\Phi] = \varphi^+ - \varphi^-,$$

$$\begin{array}{ll} \varphi^+ = \{\varphi\}^+ = \Phi(x,t) & \text{under } (x,t) \in \gamma_T^+, \\ \varphi^- = \{\varphi\}^- = \Phi(x,t) & \text{under } (x,t) \in \gamma_T^-, \end{array}$$

$\gamma_T^+ = (\partial\Omega_2 \cap \gamma) \times (0,T)$ and $\gamma_T^- = (\partial\Omega_1 \cap \gamma) \times (0,T)$.
Also we assume that the space $L^2(\Omega)$ of real valued functions $u(x)$ which are measurable and is a Hilbert space for the scalar product .

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$$

associated to the norm

$$\|u\|_{L^2(\Omega)} = (\int_\Omega |u(x)|^2 dx)^{\frac{1}{2}} < \infty.$$

Now we define the space

$$V = (H_0^1(\Omega)) = \{M(x,t) = (M_1, M_2) \mid_{\Omega_i} \in (H^1(\Omega_i)),\ i\ = 1\ ,\ 2,\ \ M \mid_\Sigma = 0\}.$$

Analogously, we can define the spaces $L^2(0,T;L^2(\Omega)) = L^2(Q)$, and $L^2(0,T;H^{-1}(\Omega))$, then we have a chain in the form

$$(L^2(0,T;H_0^1(\Omega)))^2 \subseteq (L^2(0,T;L^2(\Omega)))^2 \subseteq (L^2(0,T;H^{-1}(\Omega)))^2.$$

We introduce the Hilbert space

$$W_1(0,T) = \{M : M \in L^2(0,T;H^1(\Omega)), \frac{\partial M}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\},$$

$H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, that being supplied with the norm:

$$\|M(t)\|^2_{W_1(0,T)} = (\int_{]0,T[} \|M(t)\|^2_{H_0^1(\Omega)} dt + \int_{]0,T[} \|\frac{dM}{dt}\|^2_{H^{-1}(\Omega)} dt).$$

We can then introduce the Sobolev space $(W_1(0,T))^n$ by cartesian product:

$$(W_1(0,T))^n = \Pi_{i=1}^n (W_1(0,T))_i,$$

with norm defined by,

$$\|M\|_{(W_1(0,T))^n} = \sum_{i=1}^n \|M_i\|_{W_1(0,T)}.$$

The cartesian product $(W_1(0,T))^n$ may be verified to be a Hilbert space and its dual is denoted by $(W_1(0,T)')^n$. The considered spaces in this paper are assumed to be real.

## 3. Distributed Control for $2 \times 2$ Cooperative Parabolic Systems with Dirichlet and Conjugation Conditions

In this section, we find the necessary and sufficient conditions for the control to be optimal for the following $2 \times 2$ cooperative Dirichlet parabolic system:

$$\left\{ \begin{array}{ll} \frac{\partial M_1}{\partial t} - \nabla \cdot (\beta \nabla M_1) - a_{11} M_1(x,t) - a_{12} M_2(x,t) = f_1(x,t) & \text{in } Q, \\ \frac{\partial M_2}{\partial t} - \nabla \cdot (\beta \nabla M_2) - a_{21} M_1(x,t) - a_{22} M_2(x,t) = f_2(x,t) & \text{in } Q, \\ M_1(x,0) = M_{1,0}(x) & \text{in } \Omega, \\ M_2(x,0) = M_{2,0}(x) & \text{in } \Omega, \\ M_1(x,t) = M_2(x,t) = 0 & \text{in } \Sigma, \end{array} \right. \tag{3.1}$$

under conjugation conditions:

$$\begin{cases} [M_1] = 0 & \text{on } \gamma_T, \\ [M_2] = 0 & \text{on } \gamma_T, \end{cases} \tag{3.2}$$

$$\begin{cases} \left[\frac{\partial M_1}{\partial \nu_A}\right] = c_1 \frac{\partial M_1}{\partial t} & \text{on } \gamma_T, \\ \left[\frac{\partial M_2}{\partial \nu_A}\right] = c_2 \frac{\partial M_2}{\partial t} & \text{on } \gamma_T. \end{cases} \tag{3.3}$$

We assume that

$$c_1,\ c_2 \in L^2(\gamma) \text{ and } c_1,\ c_2 \geq 0 \tag{3.4}$$

**Lemma 3.1** *(Friedrichs inequality). For any bounded Lipschitz domain $\Omega$ there is a constant $m(\Omega) > 0$, which depends only on $\Omega$, such that*

$$\int_\Omega |M|^2 \, dx \leq m(\Omega) \int_\Omega |\nabla M|^2 \quad dx,$$

### 3.1. State equation

To prove the existence and uniqueness of the state for the system (3.1)-(3.3), we define on $(W_1(0,T))^2$ for each $t$, a continuous bilinear form

$$\pi(t, M, \Phi) : (W_1(0,T))^2 \times (W_1(0,T))^2 \to R$$

by

$$\pi(M, \Phi) = \int_\Omega \beta(x)(\nabla M_1 \nabla \varphi_1 + \nabla M_2 \nabla \varphi_2) \, dx - \int_\Omega (a_{11} M_1 \varphi_1 + a_{12} M_2 \varphi_1 + a_{21} M_1 \varphi_2 + a_{22} M_2 \varphi_2) \, dx$$
$$+ \int_\gamma c_1 \frac{\partial M_1}{\partial t} [\varphi_1] \ d\gamma + \int_\gamma c_2 \frac{\partial M_2}{\partial t} [\varphi_2] \ d\gamma. \tag{3.5}$$

Then we have

**Theorem 3.1** *For a given $f = (f_1, f_2) \in (W_1(0,T)')^2$, there exists a unique solution $M = (M_1, M_2) \in (W_1(0,T))^2$ for the system (3.1) - (3.3).*

**Proof:** From (3.5), we have

$$\begin{aligned} \pi(t; M, M) =& \frac{1}{2(a_{12}+a_{21})} \int_\Omega \beta(x)(|\nabla M_1|^2 + |\nabla M_1|^2) dx + \frac{1}{2(a_{12}+a_{21})} \int_\Omega \beta(x)(|\nabla M_2|^2 + |\nabla M_2|^2) dx \\ & - \int_\Omega M_1 M_2 dx - \frac{a_{11}}{(a_{12}+a_{21})} \int_\Omega |M_1|^2 \, dx - \frac{a_{22}}{(a_{12}+a_{21})} \int_\Omega |M_2|^2 \, dx \\ & + \int_\gamma c_1 \frac{\partial M_1}{\partial t} [M_1] \, d\gamma + \int_\gamma c_2 \frac{\partial M_2}{\partial t} [M_2] \, d\gamma. \end{aligned}$$

(3.2) implies

$$\begin{aligned} \pi(t; M, M) + & \frac{a_{11}}{(a_{12}+a_{21})} \int_\Omega |M_1(x)|^2 \, dx + \frac{a_{22}}{(a_{12}+a_{21})} \int_\Omega |M_2|^2 \, dx \\ & \geq \frac{1}{2(a_{12}+a_{21})} \int_\Omega \beta(x)(|\nabla M_1|^2 + |\nabla M_1|^2) dx \\ & + \quad \frac{1}{2(a_{12}+a_{21})} \int_\Omega \beta(x)(|\nabla M_2|^2 + |\nabla M_2|^2) dx - \int_\Omega M_1(x) M_2(x) dx. \end{aligned}$$

By Cauchy Schwartz inequality and from Friedrichs inequality, we deduce

$$\begin{aligned}\pi(t;M,M)+&max(\frac{(a_{11},a_{22})}{(a_{12}+a_{21})})[\|M_1\|^2_{L^2(\Omega)}+\|M_2\|^2_{L^2(\Omega)}]\\ &\geq \frac{\beta(x)}{2(a_{12}+a_{21})}[\|M_1\|^2_{H^1_0(\Omega)}+\|M_2\|^2_{H^1_0(\Omega)}]\\ &\quad+\frac{\beta(x)}{2(a_{12}+a_{21})}m(\Omega))^{-1}[\|M_1\|^2+\|M_2\|^2]_{L^2(\Omega)}-\|M_1\|\|M_2\|.\end{aligned}$$

This inequality is equivalent to

$$\begin{aligned}\pi(t;M,M)+&max(\frac{(a_{11},a_{22})}{(a_{12}+a_{21})})[\|M_1\|^2_{L^2(\Omega)}+\|M_2\|^2_{L^2(\Omega)}]\\ &\geq \frac{\beta(x)}{2(a_{12}+a_{21})}[\|M_1\|^2_{H^1_0(\Omega)}+\|M_2\|^2_{H^1_0(\Omega)}]\\ &\quad+\frac{(\beta(x))(m(\Omega))^{-1}}{(a_{12}+a_{21})}[\frac{1}{2}\|M_1\|^2+\|M_2\|^2]_{L^2(\Omega)}-\|M_1\|_{L^2(\Omega)}\|M_2\|_{L^2(\Omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\pi(t;M,M)+&max(\frac{(a_{11},a_{22})}{(a_{12}+a_{21})})[\|M_1\|^2_{L^2(\Omega)}+\|M_2\|^2_{L^2(\Omega)}]\\ &\geq \frac{\beta(x)}{2(a_{12}+a_{21})}[\|M_1\|^2_{H^1_0(\Omega)}+\|M_2\|^2_{H^1_0(\Omega)}]\\ &\quad+\frac{(\beta(x))(m(\Omega))^{-1}}{(a_{12}+a_{21})}(\frac{1}{\sqrt{2}}\|M_1\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|M_2\|_{L^2(\Omega)})^2,\end{aligned}$$

$$\begin{aligned}\pi(t;M,M)+k\|M\|^2_{(L^2(\Omega))^2}\geq&\alpha[\|M_1\|^2_{H^1_0(\Omega)}+\|M_2\|^2_{H^1_0(\Omega)}]\\ \geq&\alpha\|M\|^2_{(W_1(0,T))^2},\end{aligned}\tag{3.6}$$

where

$$k=max(\frac{(a_{11},a_{22})}{(a_{12}+a_{21})}),\alpha=\frac{\beta(x)}{2(\beta_{12}+\beta_{21})},$$

which proves the coerciveness condition. □

Let $\Phi\to L(\Phi)$ be a continuous linear form defined on $W_1(0,T))^2$ by

$$L(\Phi)=\int_Q f_1(x,t)\varphi_1(x)dxdt+\int_Q f_2(x,t)\varphi_2(x)dxdt.$$

Then by Lax-Milgram Lemma, there exists a unique solution $M(x)\in(W_1(0,T))^2$ such that:

$$(\frac{\partial M}{\partial t},\Phi)+\pi(t;M,\Phi)=L(\Phi)\qquad\forall\Phi=(\varphi_1,\varphi_2)\in(H^1_0(\Omega))^2,\tag{3.7}$$

and hence for system (3.1)-(3.3)

### 3.2. Formulation of the control problem

The space $(L^2(Q))^2$ being the space of controls. For each control $u = (u_1, u_2) \in (L^2(Q))^2$, let us define the state $M = (M_1(u), M_2(u)) = M(x,t;u)$ of the system as a generalized solution of:

$$\begin{cases} \frac{\partial M_1(u)}{\partial t} - \nabla\cdot(\beta\nabla M_1(u)) - a_{11}M_1(x,t,u) - a_{12}M_2(x,t,u) = f_1(x,t) + u_1 & \text{in } Q, \\ \frac{\partial M_2(u)}{\partial t} - \nabla\cdot(\beta\nabla M_2) - a_{21}M_1(x,t,u) - a_{22}M_2(x,t,u) = f_2(x,t) + u_2 & \text{in } Q, \\ M_1(x,0,u) = M_{1,0}(x) & \text{in } \Omega, \\ M_2(x,0) = M_{2,0}(x) & \text{in } \Omega, \\ M_1(x,t) = M_2(x,t) = 0 & \text{in } \Sigma, \end{cases} \tag{3.8}$$

under conjugation conditions (3.2), (3.3).

**Lemma 3.2** *A unique state, namely, a function $M(x,t;u) \in (W_1(0,T))^2$ corresponds to every control $u \in U$, minimizes the energy functional on $(W_1(0,T))^2$*

$$\Psi(w) = a(w,w) - 2f_m(w) \;\forall\;\; w(x) \in V, \tag{3.9}$$

*and it is the unique solution in $(W_1(0,T))^2$ to the weakly stated problem which stated that: Find an element $M(u) \in (W_1(0,T))^2$ that meets the equations*

$$(\frac{\partial M}{\partial t}, w) + a(M,w) = f_m(u,w), \tag{3.10}$$

$$\int_\Omega M_1(x,0;u)w_1(x)dx + \int_\Omega M_2(x,0;u)w_2(x)dx = \int_\Omega M_{1,0}(x)w_1(x)dx + \int_\Omega M_{2,0}(x)w_2(x)dx, \tag{3.11}$$

*where the bilinear form $a(t;M,w)$ has the form (3.5) and $f_m(w)$ is the linear functional given by*

$$f_m(w) = f_m(u,w) = \int_\Omega f_1(x,t)w_1(x)dx + \int_\Omega f_2(x,t)w_2(x)dx + \int_\Omega u_1w_1dx + \int_\Omega u_2w_2dx \tag{3.12}$$

Specify the observation through the following expression:

$$Z(u) = (z_1(u), z_2(u)) = C_1M(u) = (M_1(u), M_2(u)),$$
$$C_1 \quad \text{is a linear operator} \quad \in \pounds((W_1(0,T))^2; (L^2(Q))^2).$$

For a given $Z_d = (z_{1d}, z_{2d})$ in the space $(L^2(Q))^2$, the energy functional is given by

$$J(u) = \|M_1(u) - z_{1d}\|^2_{L^2(Q)} + \|M_2(u) - z_{2d}\|^2_{L^2(Q)} + (Tu_1, u_1)_{L^2(Q)} + (Tu_2, u_2)_{L^2(Q)}, \tag{3.13}$$

where $T \in\pounds(U;U)$ is a hermitian positive definite operator such that :

$$(Tu,u)_{(L^2(Q))^2} \geq N\|u\|^2_{(L^2(Q))^2}, \qquad N > 0, \quad \forall\; u \in U. \tag{3.14}$$

The optimal control problem then is to find:

$$u = (u_1, u_2) \in U_{ad} \text{ such that: } J(u) = inf\;\, J(v) \qquad \forall v \in U_{ad}, \tag{3.15}$$

where the set of admissible control $U_{ad}$ is a closed convex subset of $(L^2(Q))^2$.

**Definition 3.1** *If an element $u \in U_{ad}$ meets condition (3.15), it is called an optimal control.*

Let us present functional (3.13) as:

$$J(v) = \Pi(v,v) - 2L(v) + \|z_{1d} - M_1(0)\|^2_{L^2(Q)} + \|z_{2d} - M_2(0)\|^2_{L^2(Q)}, \tag{3.16}$$

where the bilinear form $\Pi(u,v)$ and linear functional $L(v)$ are expressed as

$$\begin{aligned}\Pi(u,v) =& (M_1(u) - M_1(0), M_1(v) - M_1(0))_{L^2(Q)} \\ &+ (M_2(u) - M_2(0), M_2(v) - M_2(0))_{L^2(Q)} + (Tu, v)_{(L^2(Q))^2},\end{aligned} \tag{3.17}$$

and

$$L(v) = (z_{1d} - M_1(0), M_1(v) - M_1(0))_{L^2(Q)} + (z_{2d} - M_2(0), M_2(v) - M_2(0))_{L^2(Q)}. \tag{3.18}$$

The form $\Pi(u,v)$ is a continuous bilinear form and from (3.14), it is coercive on $(L^2(Q))^2$, i.e

$$\Pi(v,v) \geq N\|v\|^2_{(L^2(Q))^2}, \tag{3.19}$$

also $L(v)$ is a continuous linear form on $U$. On the basis of the theory of Lions [12], there exists a unique optimal control $u$ of distributed type for problem (3.15); Moreover it is characterized by

**Theorem 3.2** *Assume that (3.6) holds, the cost functional is given by (3.13). A necessary and sufficient condition for $u = (u_1, u_2) \in (L^2(Q))^2$ to be an optimal control is that the following equations and inequalities are satisfied:*

$$\begin{cases} \frac{-\partial p_1(u)}{\partial t} - \nabla\cdot(\beta\nabla p_1(u)) = a_{11}p_1(u) + a_{21}p_2(u) + M_1(u) - z_{1d} & \text{in } Q, \\ \frac{-\partial p_2(u)}{\partial t} - \nabla\cdot(\beta\nabla p_2(u)) = a_{12}p_1(u) + a_{22}p_2(u) + M_2(u) - z_{2d} & \text{in } Q, \\ p_1(u) = p_2(u) = 0 & \text{in } \Sigma, \\ p_1(x,T,u) = p_2(x,T,u) = 0 & \text{in } \Omega, \end{cases} \tag{3.20}$$

under conjugation conditions:

$$\begin{cases} [p_1(u)] = [p_2(u)] = 0, & \text{on} \quad \gamma_T, \\ \left[\frac{\partial p_1(u)}{\partial v_{A^*}}\right] = -c_1\frac{\partial p_1(u)}{\partial t}, & \text{on} \quad \gamma_T \\ \left[\frac{\partial p_2(u)}{\partial v_{A^*}}\right] = -c_2\frac{\partial p_2(u)}{\partial t}, & \text{on} \quad \gamma_T, \end{cases} \tag{3.21}$$

and

$$\int_Q (p_1(u) + Tu_1)(v_1 - u_1)\, dxdt + \int_Q (p_2(u) + Tu_2)(v_2 - u_2)\, dxdt \geq 0, \forall \qquad v = (v_1, v_2) \in U_{ad}, \tag{3.22}$$

together with (3.8), where $P(u) = (p_1(u), p_2(u))$ is the adjoint state.

**Proof:** The control $u = (u_1, u_2) \in (L^2(Q))^2$ is optimal if and only if [12]:

$$(J'(u), v - u) \geq 0, \forall v \in U_{ad}, \tag{3.23}$$

which is equivalent to:

$$\begin{aligned}&(M_1(u) - z_{1d}, M_1(v) - M_1(u))_{L^2(Q)} + (M_2(u) - z_{2d}, M_2(v) - M_2(u))_{L^2(Q)} \\ &+ (Tu, v - u)_{(L^2(Q))^2} \geq 0 \;\forall\; v \in U_{ad}.\end{aligned}$$

This inequality can be written as:

$$\begin{aligned}&\int_0^T((M_1(u)-z_{1d},M_1(v)-M_1(u))_{L^2(\Omega)}+(M_2(u)-z_{2d},M_2(v)-M_2(u))_{L^2(\Omega)})dt\\&+\int_0^T((Tu,v-u)_{(L^2(\Omega))^2}\geq 0.\end{aligned}\tag{3.24}$$

Now, since

$$(P,AM)=(A^*P,M)=\pi(P,M),P\in V^*,M\in V,$$

then, (3.24) is equivalent to

$$\begin{aligned}&\int_0^T(\frac{-\partial p_1(u)}{\partial t}-\nabla\cdot(\beta\nabla p_1(u))-a_{11}p_1(u)-a_{21}p_2(u),M_1(u))_{L^2(\Omega)}dt\\&+\int_0^T(\frac{-\partial p_2(u)}{\partial t}-\nabla\cdot(\beta\nabla p_2(u))-a_{12}p_1(u)-a_{22}p_2(u),M_2(u))_{L^2(\Omega)}dt+\int_0^T((Tu,v-u)_{(L^2(\Omega))^2}dt\geq 0.\end{aligned}$$

Using Green's formula, we obtain

$$\begin{aligned}&(p_1(u),\frac{\partial}{\partial t}(M_1(v)-M_1(u)))_{L^2(Q)}+(p_2(u),\frac{\partial}{\partial t}(M_2(v)-M_2(u)))_{L^2(Q)}\\&+(p_1(u),-\Delta(M_1(v)-M_1(u)))_{L^2(Q)}+(p_2(u),-\Delta(M_2(v)-M_2(u)))_{L^2(Q)}\\&+(p_1(u),\frac{\partial}{\partial\nu_A}(M_1(v)-M_1(u)))_{L^2(\Sigma)}+(p_2(u),\frac{\partial}{\partial\nu_A}(M_2(v)-M_2(u))_{L^2(\Sigma)}\\&+(p_1(u),-a_{11}(M_1(v)-M_1(u)))_{L^2(Q)}+(p_2(u),-a_{21}(M_1(v)-M_1(u)))_{L^2(Q)}\\&+(p_1(u),-a_{12}(M_2(v)-M_2(u)))_{L^2(Q)}+(p_2(u),-a_{22}(M_2(v)-M_2(u)))_{L^2(Q)}\\&+(T_1u_1,v_1-u_1)_{L^2(Q)}+(T_2u_2,v_2-u_2)_{L^2(Q)}\geq 0.\end{aligned}$$

Using equation (3.8), we obtain

$$\int_Q(p_1(u)+Tu_1)(v_1-u_1)dxdt+\int_Q(p_2(u)+Tu_2)(v_2-u_2)\;dxdt\geq 0.\tag{3.25}$$

Thus, the optimal control $u\in U_{ad}$ can be determined by relations (3.2), (3.3), (3.8), (3.20),(3.21) and (3.25). □

## 4. The $n\times n$ Dirichlet Systems Under Conjugation Conditions

In this section, we generalize the discussion which has been introduced in section three to the following $n\times n$ cooperative parabolic Dirichlet systems

$$\begin{cases}\frac{\partial}{\partial t}M_i+A(t)M_i(x,t)=f_i(x,t) & \text{in } Q,\\ M_i(x,0)=M_{i,0}(x),\quad M_{i,0}(x)\in L^2(\Omega) & \text{in } \Omega,\\ M_i(x,t)=0 & \text{on } \Sigma,\\ \quad i=1,2,3,.....,n.\end{cases}\tag{4.1}$$

under conjugation conditions (1.4). For this we define a bilinear form

$$\pi:(W_1(0,T))^n\times(W_1(0,T))^n\to R$$

by

$$\pi(t;M,\Phi)=\sum_{i=1}^n\int_\Omega\beta(x)\nabla M_i\nabla\varphi_i dx-\sum_{i\neq j}^n\int_\Omega a_{ij}M_j\varphi_i dx-\sum_{i=j=1}^n\int_\Omega a_{ij}M_i\varphi_i dx+\sum_{i=1}^n\int_\gamma c_i\frac{\partial M_i}{\partial t}\,[\varphi_i]\;d\gamma,\tag{4.2}$$

$$\pi(M,M)+k\|M\|^2_{(L^2(\Omega))^n}\geq\alpha\|M\|^2_{(W_1(0,T))^n}\qquad\forall\ M=(M_i)_{i=1}^n\in(W_1(0,T))^n, \tag{4.3}$$

as in theorem (3.1), there exists a unique solution $M=(M_i)_{i=1}^n\in(W_1(0,T))^n$ of the initial boundary value problem (4.1) and (1.4) for $f=(f_i)_{i=1}^n\in(W_1(0,T)')^2$ and $M_{i,0}(x)\in L^2(\Omega)$. For each a control $u=(u_i)_{i=1}^n\in(L^2(Q))^n$, let us define a state $M=(M_i(u))_{i=1}^n$ as a generalized solution of the initial boundary value problem (1.2) under conditions (1.4). The generalized problem corresponds to initial boundary value problem (1.2), (4.1), (1.4) is to define the function $M(x,t;u)=(M_i(u))_{i=1}^n\in(W_1(0,T))^n$, satisfying the following equations

$$(\frac{\partial M}{\partial t},w)+\pi(M,w)=f_N(u,w), \tag{4.4}$$

$$\int_\Omega M_i(x,0;u)w_i dx=\int_\Omega M_{i,0}(x)w_i dx, \tag{4.5}$$

where $\pi(M,w)$ has the form (4.2) and a linear functional is given by

$$f_N(w)=\sum_{i=1}^n\int_\Omega f_i(x)w_i(x)dx. \tag{4.6}$$

We will specify the observation by expression

$$Z(u)=(z_i(u))_{i=1}^n=M(u)=(M_i(u))_{i=1}^n.$$

For a given $Z_d=(z_{id})_{i=1}^n\in(L^2(Q))^n$, the cost functional is given by (1.1). The control problem then is to find

$$\begin{cases} u=(u_1,u_2,...u_n)\in U_{ad},\ \text{closed convex subset of}\ (L^2(Q))^n\quad\text{such that:}\\ J(u)=\inf_{v\in U_{ad}}J(v). \end{cases} \tag{4.7}$$

The cost functional (1.1) can be written as

$$J(v)=\Pi(v,v)-2L(v)+\sum_{i=1}^n\|z_{id}-M_i(0))\|^2_{L^2(Q)},$$

where

$$\Pi(u,v)=\sum_{i=1}^n((M_i(u)-M_i(0),M_i(v)-M_i(0))_{L^2(Q)})+\sum_{i=1}^n(Tu_i,v_i)_{L^2(Q)}, \tag{4.8}$$

is a continuous bilinear form and it is coercive, that is:

$$\Pi(v,v)\geq\eta\|v\|^2_{(L^2(Q))^n},\quad\eta>0,$$

and

$$L(v)=\sum_{i=1}^n(z_{id}-M_i(0),M_i(v)-M_i(0))_{L^2(Q)}, \tag{4.9}$$

is a continuous linear form on $(L^2(Q))^n$. Then the necessary and sufficient conditions for $u=(u_i)_{i=1}^n\in(L^2(Q))^n$ to be an optimal control is that the following equations and inequalities are satisfied:

$$\begin{cases} -\frac{\partial}{\partial t}p_i(u)+A^*(t)p_j(u)=M_i(u)-z_{id} & \text{in}\ \ Q,\\ p_i(u)(x,T,0)=0 & \text{in}\ \ \Omega,\\ p_i(u)=0 & \text{on}\ \ \Sigma,\\ \qquad i=1,2,3,\cdots,n, & \end{cases} \tag{4.10}$$

under conjugation conditions:

$$[p_i(u)] = 0 \quad \text{on} \quad \gamma_T, \tag{4.11}$$

$$\left[\frac{\partial p_i(u)}{\partial \nu_A *}\right] = -c_i \frac{\partial p_i(u)}{\partial t} \quad \text{on} \quad \gamma_T, \tag{4.12}$$

where$A^*(t)$ is the transpose of $A(t)$ such that

$$A^*(t) = \begin{bmatrix} -\nabla\cdot(\beta\nabla) - a_{11} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & -\nabla\cdot(\beta\nabla) - a_{22} & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & -\nabla\cdot(\beta\nabla) - a_{nn}, \end{bmatrix} \tag{4.13}$$

and

$$\sum_{i=1}^{n} \int_Q (p_i(u) + Tu_i)(v_i - u_i)\ dxdt\ \geq 0,\ \forall \qquad v = (v_i)_{i=1}^n \in U_{ad}, \tag{4.14}$$

together with (1.2) and (1.4) where $P(u) = (p_i(u))_{i=1}^n$ is the adjoint state.

## 5. The $2 \times 2$ Neumann Parabolic Systems Under Conjugation Conditions

In this section we discuss the distributed control for the following $2 \times 2$ cooperative parabolic system with non homogenous Neumann conditions

$$\begin{cases} \frac{\partial M_1}{\partial t} - \nabla\cdot(\beta\nabla M_1) - a_{11}M_1(x,t) - a_{12}M_2(x,t) = f_1(x,t) & \text{in } Q, \\ \frac{\partial M_2}{\partial t} - \nabla\cdot(\beta\nabla M_2) - a_{21}M_1(x,t) - a_{22}M_2(x,t) = f_2(x,t) & \text{in } Q, \\ M_1(x,0) = M_{1,0}(x) & \text{in } \Omega, \\ M_2(x,0) = M_{2,0}(x) & \text{in } \Omega, \\ \frac{\partial M_1}{\partial \nu_A} = g_1 & \text{on } \Sigma, \\ \frac{\partial M_2}{\partial \nu_A} = g_2 & \text{on } \Sigma, \end{cases} \tag{5.1}$$

with conjugation conditions (3.2), (3.3), where $(g_1, g_2) \in (L^2(\Sigma))^2$ are given functions. Let us define

$$(W_2(0,T))^2 = \{M : M \in (L^2(0,T;H^1(\Omega)))^2, \frac{\partial M}{\partial t} \in (L^2(0,T;H^{-1}(\Omega)))^2\},$$

where the Sobolev space $H^1(\Omega)$ forms a Hilbert space endowed with the scalar product defined by

$$(u,v)_{H^1(\Omega)} = \sum_{|\alpha|\leq 1} (D^\alpha u, D^\alpha v)dx,$$

and $D^\alpha$ is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{|\alpha|}x_1 ... \partial^{|\alpha|}x_2, \cdots}, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i,$$

where the differentiation in the sense of distribution function and after the completion, we obtain the Sobolev space $H^1(\Omega)$. We introduce again the bilinear form (3.5) which is coercive on $(H^1(\Omega))^2$, since

$$(H_0^1(\Omega))^2 \subseteq (H^1(\Omega))^2.$$

Then by Lax-Milgram lemma, there exists a unique solution $M \in (W_2(0,T))^2$ for system (5.1) according the equation (3.7), where

$$L_g(\Phi) = \int_\Omega f_1(x)\varphi_1(x)dx + \int_\Omega f_2(x)\varphi_2(x)dx + \int_\Gamma g_1(x)\varphi_1(x)d\Gamma + \int_\Gamma g_2(x)\varphi_2(x)d\Gamma,$$

is a continuous linear form defined on $(W_2(0,T))^2$.
For each control $u = (u_1, u_2) \in (L^2(Q))^2$, let us define the state $M(x,t;u) = (M_1(u), M_2(u))$ as a generalized solution of the initial-boundary problem given by

$$\begin{cases} \frac{\partial M_1(u)}{\partial t} - \nabla\cdot(\beta\nabla M_1(u)) - a_{11}M_1(x,t) - a_{12}M_2(x,t) = f_1(x,t) + u_1 & \text{in } Q, \\ \frac{\partial M_2(u)}{\partial t} - \nabla\cdot(\beta\nabla M_2(u)) - a_{21}M_1(x,t) - a_{22}M_2(x,t) = f_2(x,t) + u_2 & \text{in } Q, \\ M_1(x,0,u) = M_{1,0}(x) & \text{in } \Omega, \\ M_2(x,0,u) = M_{2,0}(x) & \text{in } \Omega, \\ \frac{\partial M_1(u)}{\partial \nu_A} = g_1 & \text{on } \Sigma, \\ \frac{\partial M_2(u)}{\partial \nu_A} = g_2 & \text{on } \Sigma, \end{cases} \tag{5.2}$$

with conjugation conditions (3.2), (3.3). The generalized problem corresponding to the initial boundary value problems (5.2), (3.2), (3.3) is to define the function $M(x,t;u) = (M_1(u), M_2(u)) \in (W_2(0,T))^2$, satisfying $\forall w(x) \in (H^1(\Omega))^2$

$$\int_\Omega \frac{\partial M}{\partial t} w dx + \pi(M,w) = f_M(u,w), \tag{5.3}$$

$$\int_\Omega M_1(x,0;u)w_1(x)dx + \int_\Omega M_2(x,0;u)w_2(x)dx = \int_\Omega M_{1,0}(x)w_1(x)dx + \int_\Omega M_{2,0}(x)w_2(x)dx, \tag{5.4}$$

where the bilinear form $\pi(t;M,w)$ has the form of expression (3.5) and the linear functional is

$$\begin{aligned} f_M(w) = f_M(u,w) = &\int_\Omega f_1(x,t)w_1(x)dx + \int_\Omega f_2(x,t)w_2(x)dx + \int_\Gamma g_1 w_1 d\Gamma \\ &+ \int_\Gamma g_2 w_2 d\Gamma + \int_\Omega u_1 w_1 d\Gamma + \int_\Omega u_2 w_2 d\Gamma. \end{aligned} \tag{5.5}$$

So, the initial boundary value problem (5.2), (3.2) and (3.3) has a unique generalized solution $M(x,t;u) \in (W_2(0,T))^2 \ \forall \ u \in U$. For a given $Z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$, the cost functional is given a gain by (3.13). Then there exists a unique optimal control $u \in U_{ad}$ according the equation (3.15), moreover it is characterized by the following equations and inequalities

$$\begin{cases} \frac{-\partial p_1(u)}{\partial t} - \nabla\cdot(\beta\nabla p_1(u)) - a_{11}p_1(u) - a_{21}p_2(u) = M_1(u) - z_{1d} & \text{in } Q, \\ \frac{-\partial p_2(u)}{\partial t} - \nabla\cdot(\beta\nabla p_2(u) - a_{12}p_1(u) - a_{22}p_2(u) = M_2(u) - z_{2d} & \text{in } Q, \\ p_1(x,T,u) = p_2(x,T,u) = 0 & \text{in } \Omega, \\ \frac{\partial p_1(u)}{\partial \nu_{A^*}} = \frac{\partial p_2(u)}{\partial \nu_{A^*}} = 0 & \text{on } \Sigma, \end{cases} \tag{5.6}$$

under conjugation conditions (3.21), and

$$\int_Q (p_1(u) + T_1u_1)(v_1 - u_1)\ dxdt + \int_Q (p_2(u) + T_2u_2)(v_2 - u_2)\ dxdt\ \geq 0, \quad \forall \qquad v = (v_1, v_2) \in U_{ad}, \tag{5.7}$$

together with (5.2) where

$$P(u) = (p_1(u), p_2(u)) \text{ is the adjoint state.}$$

**Remark 5.1** *To discuss the distributed control for the following $n \times n$ cooperative parabolic systems with Neumann condition*

$$\begin{cases} \frac{\partial}{\partial t} M_i + A(t) M_i(x,t) = f_i(x,t) & in \;\; Q, \\ M_i(x,0) = M_{i,0}(x), \quad M_{i,0}(x) \in L^2(\Omega) & in \;\; \Omega, \\ \frac{\partial M_i}{\partial \nu_A} = g_i & on \;\; \Sigma, \\ \quad i = 1, 2, 3, ....., n. \end{cases} \tag{5.8}$$

*with conjugation conditions (1.4), we introduce again the bilinear form (4.2) which is coercive on $(W_2(0,T))^n$, since*

$$(H_0^1(\Omega))^n \subseteq (H^1(\Omega))^n \;,$$

*then based on (4.3) and Lax- Milgram lemma there exists a unique solution $M = (M_i)_{i=1}^n \in (W_2(0,T))^n$ for system (5.8) such that*

$$(\frac{\partial M}{\partial t} \Phi) + \pi(t; M, \Phi) = f_g(\Phi), \quad \forall \Phi \in (H^1(\Omega))^2, \tag{5.9}$$

*where*

$$f_g(\Phi) = \sum_{i=1}^n \int_\Omega f_i(x) \varphi_i(x) dx + \sum_{i=1}^n \int_\Gamma g_i(x) \varphi_i(x) d\Gamma, \tag{5.10}$$

*is a continuous linear form defined on $(W_2(0,T))^n$, then from (4.2), (5.10), $M$ is a solution of*

$$\frac{\partial M_i}{\partial t} - \nabla \cdot (\beta \nabla M_i) - \sum_{j=1}^n a_{ij} M_j(u) = f_i(x,t) in \; Q, \quad \forall 1 \;\leq i \;\leq n$$

*this equation satisfies the Neumann condition. Multiplying both sides by $\Phi \in (H^1(\Omega))^n$ and integrating over $Q$, we get*

$$\int_Q \frac{\partial M_i}{\partial t} \varphi_i(x) dx dt + \int_Q (-\nabla \cdot (\beta \nabla M_i) \varphi_i(x) dx dt - \sum_{j=1}^n \int_Q a_{ij} M_j \varphi_i dx dt = \sum_{i=1}^n \int_Q f_i(x,t) \varphi_i dx dt,$$

*using Green's formula we obtain*

$$\pi(t; M, \Phi) - \sum_{i=1}^n \int_\Gamma \frac{\partial M_i}{\partial \nu} \varphi_i d\Gamma = \sum_{i=1}^n \int_Q f_i(x,t) \varphi_i dx dt,$$

*from (5.9), we get*

$$\sum_{i=1}^n \int_\Omega f_i(x) \varphi_i(x) dx + \sum_{i=1}^n \int_\Gamma g_i(x) \varphi_i(x) d\Gamma,$$

*hence we obtain the Neumann conditions*

$$\beta \frac{\partial M_i}{\partial \nu_A} = g_i \quad on \; \Sigma,$$

*so we can formulate corresponding the control problem:*
*The space $(L^2(Q))^n$ is the space of controls. The state $M(u) \in (W_2(0,T))^n$ of the system is given*

*by the solution of* (1.3) *under conjugation conditions* (1.4). *For a given* $z_d \in (L^2(Q))^n$, *the cost functional is again given by* (1.1), *then there exists a unique optimal control* $u = (u_1, u_2, ..., u_n) \in U_{ad}$ *such that:*

$$J(u) = \inf_{v \in U_{ad}} J(v), \quad \forall v = (v_1, v_2, ..., v_n) \in U_{ad}.$$

*Moreover it is characterized by the following equations and inequalities*

$$\begin{cases} -\frac{\partial p_i(u)}{\partial t} - \nabla \cdot (\beta \nabla p_i(u)) - \sum_{j=1}^{n} a_{ji} p_j(u) = M_i(u) - z_{id} & in \quad Q, \\ p_i(x; T, u) = 0 & in \quad \Omega, \\ \beta \frac{\partial p_i(u)}{\partial \nu_{A^*}} = 0 & on \quad \Sigma, \\ [p_i(u)] = 0 & on \quad \gamma, \\ \left[ \frac{\partial p_i(u)}{\partial \upsilon_{A^*}} \right] = -c_i \frac{\partial p_i(u)}{\partial t} & on \quad \gamma, \end{cases}$$

*and*

$$\sum_{i=1}^{n} \int_Q (p_i(u) + T_i u_i)(v_i - u_i) \; dxdt \; \geq 0, \quad \forall \qquad v = (v_i)_{i=1}^n \in U_{ad}, \tag{5.11}$$

*together with* (1.3), *where* $P(u) = (p_i(u))_{i=1}^n$ *is the adjoint state.*

## 6. Conclusions

In this paper, we focused on optimal control problems for cooperative systems governed by heat equation in the presence of concentrated heat capacity under conjugation conditions. Under some conditions on the coefficients, we proved the existence and uniqueness of the state for$2 \times 2$ Dirichlet cooperative parabolic systems under conjugation conditions. Then we demonstrated the existence and uniqueness of the optimal control of distributed type for these systems. We gave the set of equations and inequalities that characterizes this control. Also, we studied the problem with Neumann condition. Finally, we generalized the discussion to $n \times n$ cooperative parabolic systems under conjugation conditions.

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