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Generalized Mittag-Leffler-type function of arbitrary order and its properties related to integral transforms and fractional calculus

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ABSTRACT: This paper introduces a novel generalization of the Mittag-Leffler function, delving into its fundamental characteristics. The analysis encompasses a thorough exploration of its properties, including the derivation of recurrence relations, differential formulas, and various integral representations such as the Euler, Laplace, Mellin, Whittaker, and Mellin–Barnes transforms. Furthermore, the study establishes connections to other significant special functions, expressing the new generalization in terms of the Fox-Wright function, the generalized hypergeometric function, and the H-function. The paper also defines associated fractional integral and differential operators, highlighting the function's relevance to fractional calculus. Several noteworthy special cases are derived from the main results, demonstrating the breadth and adaptability of this new function. This research provides a comprehensive framework for understanding the properties of this generalized Mittag-Leffler function and suggests its potential for applications in diverse areas, particularly within the realm of fractional analysis and its related fields.

Key Words: Mittag-Leffler function, Fox-Wright function, integral transforms, fractional calculus operators.

Contents

1. Introduction

The function $E_\alpha(z)$ is named after the renowned Swedish mathematician Gösta Magnus Mittag-Leffler, who defined it using a power series [12]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha, z \in \mathbb{C}, \Re(\alpha) > 0. \tag{1.1}$$

First generalizion of the function $E_\alpha(z)$ was introduced by Wiman [27]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.2}$$

The Mittag-Leffler function, denoted as $E_{\alpha,\beta}(z)$, is a fundamental special function in fractional calculus, generalizing the exponential function. Defined by the series (1.2), it plays a key role in solving fractional differential equations (FDEs) due to its ability to model power-law memory effects. In fractional calculus, Mittag-Leffler functions appear naturally in solutions to linear FDEs, often replacing exponentials found

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in classical ODEs. For numerical solutions of FDEs, Mittag-Leffler function based methods, such as the Mittag-Leffler exponential integrators, provide efficient tools for handling non-local fractional operators. These techniques are crucial in modeling anomalous diffusion, viscoelasticity, and biological systems with memory effects, where traditional integer-order models fail. Efficient numerical evaluation of the ML function remains an active research area to improve accuracy in computational fractional calculus, see for instance [1,2,9,10,11,13]. In addition, Prabhakar [16] proposed a further generalization of the function $E_\alpha(z)$ in the form

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0, \tag{1.3}$$

where $(\gamma)_n$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function Γ by (see, e.g., [23])

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}:=\{1,2,...\}).\end{cases}$$

Moreover, the generalization of $E^{\gamma}_{\alpha,\beta}(z)$ was given by Shukla and Prajapati [21], defined as follows:

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.4}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $q\in(0,1)\cup\mathbb{N}$ and $(\gamma)_{qn}=\frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$, denotes the generalized Pochhammer symbol, which reduces to $q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n$ if $q\in\mathbb{N}$.
Recently, several generalizations and extensions for Mittag-Leffler functions have been presented and investigated by many authors (see, e.g., [3,4,5,14,20,25]). Very recently, Pathan and Bin-Saad [15] introduced a new Mittag-Leffler-type function of arbitrary order, which is a generalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$. The arbitrary order Mittag-Leffler-type function is defined as

$$E^{j,k}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma(\beta+\alpha(nj+k))},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,j\geq 1,k\geq 0. \tag{1.5}$$

The present work introduces and investigates a novel generalization of the Mittag-Leffler-type function of arbitrary order, defined as follows:

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}z^{nj+k}, \tag{1.6}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. The increased generality of definition (1.6) permits the direct derivation of the subsequent special cases:

(i) $E^{j,k,1}_{\alpha,\beta,1}(z)$ represents the arbitrary order Mittag–Leffler-type function defined in (1.5).

(ii) $E^{1,0,q}_{\alpha,\beta,\gamma}(z)$ represents the generalized Mittag-Leffler function defined in (1.4).

(iii) $E^{1,0,1}_{\alpha,\beta,\gamma}(z)$ represents the Mittag-Leffler function defined in (1.3).

(iv) $E^{1,0,1}_{\alpha,\beta,1}(z)$ represents the Mittag-Leffler function defined in (1.2).

(v) $E^{1,0,1}_{\alpha,1,1}(z)$ represents the Mittag-Leffler function defined in (1.1).

The following well-known notations, formulas, and functions have been used:
The Beta function is defined as [23]

$$B(\nu,\mu)=\int_0^1 t^{\nu-1}(1-t)^{\mu-1}\,dt,\quad \Re(\nu)>0,\Re(\mu)>0, \tag{1.7}$$

or in terms of gamma function as

$$B(\nu,\mu)=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)},\quad \nu,\mu\in\mathbb{C}\setminus\mathbb{Z}_0^-. \tag{1.8}$$

The Fox-Wright function is defined as [24]

$${}_p\Psi_q\left[\begin{array}{c}(d_1,D_1),...,(d_p,D_p)\\(e_1,E_1),...,(e_q,E_q)\end{array}\middle|\, z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(d_i+D_i n)}{\prod_{j=1}^{q}\Gamma(e_j+E_j n)}\frac{z^n}{n!}, \tag{1.9}$$

where $d_i, D_i, e_j, E_j, z\in\mathbb{C}$, $\Re(d_i)>0,\Re(D_i)>0,\ i=1,...,p,\ \Re(e_i)>0,\Re(E_i)>0,\ j=1,...,q$ and $1+\Re\left(\sum_{j=1}^{q}E_j-\sum_{i=1}^{p}D_i\right)\geq 0$.

The H-function is given as

$$H_{P,Q}^{M,N}\left[z\,\middle|\begin{array}{c}(A_1,\alpha_1),...,(A_P,\alpha_P)\\(B_1,\beta_1),...,(B_Q,\beta_Q)\end{array}\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{M}\Gamma(B_j+\beta_j s)\prod_{i=1}^{N}\Gamma(1-A_i-\alpha_i s)}{\prod_{i=N+1}^{P}\Gamma(A_i+\alpha_i s)\prod_{j=M+1}^{Q}\Gamma(1-B_j-\beta_j s)}z^{-s}ds, \tag{1.10}$$

where M,N,P,Q are integers such that $0\leq M\leq Q$, $0\leq N\leq P$, and the parameters $A_i, B_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}^+(i=1,...,p;j=1,...,q)$ with the contour L suitably chosen, and an empty product, if it occurs, is taken to be unity. For more details about the H-function, one can refer to Kilbas and Saigo [6].

The generalized hypergeometric function is given as [18]

$${}_pF_q\left[\begin{array}{c}a_1,...,a_p\\b_1,...,b_q\end{array};z\right]={}_pF_q(a_1,...,a_p;b_1,...,b_q;z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!}, \tag{1.11}$$

where the infinite series converges for all $z\in\mathbb{C}$ when $p\leq q$.

The Euler (Beta) transform of the function $f(z)$ is defined as [22]

$$B\{f(z);\nu,\mu\}=\int_0^1 z^{\nu-1}(1-z)^{\mu-1}f(z)dz,\quad \Re(\nu)>0,\Re(\mu)>0. \tag{1.12}$$

The Laplace transform of the function $f(z)$ is defined as [22]

$$\mathcal{L}\{f(z);s\}=\int_0^\infty e^{-sz}f(z)dz,\quad \Re(s)>0. \tag{1.13}$$

The Mellin transform of the function $f(z)$ is given as [22]

$$\mathcal{M}\{f(z);s\}=\int_0^\infty z^{s-1}f(z)dz=f^*(s),\quad \Re(s)>0, \tag{1.14}$$

and the inverse Mellin transform is defined as

$$f(z)=\mathcal{M}^{-1}\{f^*(s);z\}=\frac{1}{2\pi i}\int_L f^*(s)z^{-s}ds, \tag{1.15}$$

where L is a contour of integration that begins at $-i\infty$ and ends at $i\infty$.

The Whittaker transform is defined as [26]

$$\int_0^\infty u^{\nu-1}e^{-\frac{u}{2}}W_{\lambda,\mu}(u)du=\frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(1-\lambda+\nu\right)}, \tag{1.16}$$

where $\Re(\mu\pm\nu)>-\frac{1}{2}$ and $W_{\lambda,\mu}(u)$ is the Whittaker confluent hypergeometric function.

The fractional-order integration and differentiation are defined by the left-sided Riemann-Liouville fractional integral operator I^ν_{a+} and the right-sided Riemann-Liouville fractional integral operator I^ν_{b-}, and the corresponding Riemann-Liouville fractional derivative operators D^ν_{a+} and D^ν_{b-}, as [19]

$$\left(I^\nu_{a+}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_a^x(x-t)^{\nu-1}f(t)dt,\quad\left(\Re(\nu)>0,x>a\right), \tag{1.17}$$

$$\left(I^\nu_{b-}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_x^b(t-x)^{\nu-1}f(t)dt,\quad\left(\Re(\nu)>0,x<b\right), \tag{1.18}$$

$$\left(D^\nu_{a+}f\right)(x)=\left(\frac{d}{dx}\right)^m\left(I^{m-\nu}_{a+}f\right)(x),\quad\left(\Re(\nu)>0,m=[\Re(\nu)]+1\right), \tag{1.19}$$

and

$$\left(D^\nu_{b-}f\right)(x)=(-1)^m\left(\frac{d}{dx}\right)^m\left(I^{m-\nu}_{b-}f\right)(x),\quad\left(\Re(\nu)>0,m=[\Re(\nu)]+1\right), \tag{1.20}$$

where $\Re(\nu)$ denotes the real part of the complex number $\nu\in\mathbb{C}$ and $[\Re(\nu)]$ represents the integral part of $\Re(\nu)$. Here, we recall the left and right-sided Riemann-Liouville fractional integrations of a power function are defined in [7] by

$$\left(I^\nu_{0+}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)}x^{\lambda+\nu-1},\quad\left(\Re(\nu)>0,\Re(\lambda)>0\right), \tag{1.21}$$

$$\left(I^\nu_{-}t^{\lambda-1}\right)(x)=\frac{\Gamma(1-\nu-\lambda)}{\Gamma(1-\lambda)}x^{\lambda+\nu-1},\quad\left(0<\Re(\nu)<1-\Re(\lambda)\right), \tag{1.22}$$

respectively. The left and right-sided Riemann-Liouville fractional differentiations of a power function are defined, respectively, by (see [7])

$$\left(D^\nu_{0+}t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\nu)}x^{\lambda-\nu-1},\quad\left(\Re(\nu)>0,\Re(\lambda)>0\right), \tag{1.23}$$

and

$$\left(D^\nu_{-}t^{\lambda-1}\right)(x)=\frac{\Gamma(1+\nu-\lambda)}{\Gamma(1-\lambda)}x^{\lambda-\nu-1},\quad\left(\Re(\nu)>0,\Re(\lambda)<\Re(\nu)-[\Re(\nu)]\right). \tag{1.24}$$

In the end, this paper introduces a new generalized Mittag-Leffler function type. In section 2 the authors investigate basic properties of the introduced new generalized type. Namely, the paper including recurrence relations, differential formulas, integral representations, Euler transform, Laplace transform, Mellin transform, Whittaker transform, and Mellin–Barnes integral representation related to the new generalized Mittag-Leffler function. Additionally, the paper expresses it in terms of Fox-Wright function, generalized hypergeometric function, and H-function found in section 3. Furthermore, we establish fractional integral and differential operators associated with the proposed Mittag-Leffler type function. Several interesting special cases of the main results are derived and addressed in sections 4 and 5.

2. Basic properties

In this section, we have derived some useful basic properties of $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$.

Theorem 2.1 *For $\alpha,\beta,\gamma,z\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $q\in\mathbb{N}$, we get*

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\beta E^{j,k,q}_{\alpha,\beta+1,\gamma}(z)+\alpha z\frac{d}{dz}E^{j,k,q}_{\alpha,\beta+1,\gamma}(z), \tag{2.1}$$

$$E^{j,k,q}_{\alpha,\beta-\alpha j,\gamma}(z)-E^{j,k,q}_{\alpha,\beta-\alpha j,\gamma-1}(z)=qz^{j+k}\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+1)-1}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj}. \tag{2.2}$$

Proof. We have

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^{nj+k}}{n!\,(\beta+\alpha k+\alpha nj)\,\Gamma\left(\beta+\alpha k+\alpha nj\right)}\,(\beta+\alpha k+\alpha nj)$$
$$=\beta\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^{nj+k}}{n!\,(\beta+\alpha k+\alpha nj)\,\Gamma\left(\beta+\alpha k+\alpha nj\right)}+\alpha\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,(nj+k)\,z^{nj+k}}{n!\,(\beta+\alpha k+\alpha nj)\,\Gamma\left(\beta+\alpha k+\alpha nj\right)}$$
$$=\beta E^{j,k,q}_{\alpha,\beta+1,\gamma}(z)+\alpha z\frac{d}{dz}E^{j,k,q}_{\alpha,\beta+1,\gamma}(z),$$

which gives the desired result (2.1).

Next, we have

$$E^{j,k,q}_{\alpha,\beta-\alpha j,\gamma}(z)-E^{j,k,q}_{\alpha,\beta-\alpha j,\gamma-1}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^{nj+k}}{n!\;\Gamma\left(\beta+\alpha k+\alpha(n-1)j\right)}-\sum_{n=0}^{\infty}\frac{(\gamma-1)_{qn}\,z^{nj+k}}{n!\;\Gamma\left(\beta+\alpha k+\alpha(n-1)j\right)}$$
$$=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{n!\;\Gamma\left(\beta+\alpha k+\alpha(n-1)j\right)}\left[(\gamma)_{qn}-(\gamma-1)_{qn}\right]=qz^{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn-1}\,nz^{nj}}{n!\;\Gamma\left(\beta+\alpha k+\alpha(n-1)j\right)}$$
$$=qz^{k}\sum_{n=1}^{\infty}\frac{(\gamma)_{qn-1}\,z^{nj}}{(n-1)!\;\Gamma\left(\beta+\alpha k+\alpha(n-1)j\right)}=qz^{j+k}\sum_{n=0}^{\infty}\frac{(\gamma)_{q(n+1)-1}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj},$$

which proves (2.2).

If we set $j=1$ and $k=0$ in equation (2.2), we obtain the known result [21].

Theorem 2.2 *For $\alpha,\beta,\gamma,\omega\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $m,q\in\mathbb{N}$, we have*

$$\left(\frac{d}{dz}\right)^{m}E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\frac{(1)_m\,(\gamma)_{\left(\frac{qm-qk}{j}\right)}}{(1)_{\left(\frac{m-k}{j}\right)}}E^{m+1,1,j}_{\alpha j,\beta+\alpha m,j}(z^j), \tag{2.3}$$

$$\left(\frac{d}{dz}\right)^{m}\left[z^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega z^{\alpha})\right]=z^{\beta-m-1}E^{j,k,q}_{\alpha,\beta-m,\gamma}(\omega z^{\alpha}). \tag{2.4}$$

Proof. To prove (2.3), from equation (1.6), we get

$$\left(\frac{d}{dz}\right)^{m}E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\left(\frac{d}{dz}\right)^{m}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj+k}$$
$$=\sum_{n=\frac{m-k}{j}}^{\infty}\frac{(\gamma)_{qn}\,(nj+k)!\;z^{nj+k-m}}{n!\,(nj+k-m)!\;\Gamma\left(\beta+\alpha(nj+k)\right)}$$
$$=\frac{(\gamma)_{q(\frac{m-k}{j})}}{\Gamma(\frac{m-k+j}{j})}\sum_{n=0}^{\infty}\frac{\Gamma\left(nj+m+1\right)}{\Gamma\left(\beta+\alpha(nj+m)\right)\Gamma\left(nj+1\right)}(z^j)^n$$
$$=\frac{(1)_m\,(\gamma)_{\left(\frac{qm-qk}{j}\right)}}{(1)_{\left(\frac{m-k}{j}\right)}}E^{m+1,1,j}_{\alpha j,\beta+\alpha m,j}(z^j).$$

Now, we have

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega z^{\alpha})\right]=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}z^{\beta+\alpha(nj+k)-1}$$
$$=z^{\beta-m-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\ \Gamma\left(\beta-m+\alpha(nj+k)\right)}\left(\omega z^{\alpha}\right)^{nj+k}$$
$$=z^{\beta-m-1}E^{j,k,q}_{\alpha,\beta-m,\gamma}(\omega z^{\alpha}),$$

which is the proof of (2.4).

If we set $\gamma=q=1$ in equation (2.3), we obtain the known result [15].

Theorem 2.3 *For $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 1$, we get*

$$E^{j,k,q}_{\alpha,\beta,\gamma}\left(z\right)=\frac{z^k}{\Gamma(\alpha k)}\int_0^1 t^{\beta-1}\left(1-t\right)^{\alpha k-1}E^{\gamma,q}_{\alpha j,\beta}\left(\left(t^{\alpha}z\right)^j\right)dt. \tag{2.5}$$

Proof. We have

$$\int_0^1 t^{\beta-1}\left(1-t\right)^{\alpha k-1}E^{\gamma,q}_{\alpha j,\beta}\left(\left(t^{\alpha}z\right)^j\right)dt$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ z^{nj}}{n!\ \Gamma\left(\beta+\alpha nj\right)}\int_0^1 t^{\beta+\alpha nj-1}\left(1-t\right)^{\alpha k-1}dt$$
$$=\Gamma(\alpha k)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj}$$
$$=z^{-k}\Gamma(\alpha k)E^{j,k,q}_{\alpha,\beta,\gamma}\left(z\right),$$

which proves (2.5).

Corollary 2.1 *Putting $t=sin^2\theta$ in (2.5), we obtain the following integral representation:*

$$E^{j,k,q}_{\alpha,\beta,\gamma}\left(z\right)=\frac{2z^k}{\Gamma(\alpha k)}\int_0^{\frac{\pi}{2}}\left(sin\theta\right)^{2\beta-1}\left(cos\theta\right)^{2\alpha k-1}E^{\gamma,q}_{\alpha j,\beta}\left(z^j\left(sin\theta\right)^{2\alpha j}\right)d\theta. \tag{2.6}$$

Corollary 2.2 *Putting $t=\frac{u}{1+u}$ in (2.5), we obtain the following integral representation:*

$$E^{j,k,q}_{\alpha,\beta,\gamma}\left(z\right)=\frac{z^k}{\Gamma(\alpha k)}\int_0^{\infty}\frac{u^{\beta-1}}{\left(1+u\right)^{\beta+\alpha k}}E^{\gamma,q}_{\alpha j,\beta}\left(z^j\left(\frac{u}{1+u}\right)^{\alpha j}\right)du. \tag{2.7}$$

3. Representation of $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of other functions

In this section, we establish the representations of $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of Fox-Wright function, generalized hypergeometric function, Mellin-Barnes integral and H-function.

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj+k}=z^k\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+qn\right)}{\Gamma\left(\gamma\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\frac{z^{nj}}{n!}.$$

Using equation (1.9), we can express $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of the Fox-Wright function as

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z)=\frac{z^k}{\Gamma\left(\gamma\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+qn\right)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)}\frac{(z^j)^n}{n!}=\frac{z^k}{\Gamma\left(\gamma\right)}{}_1\Psi_1\left[\begin{array}{c}(\gamma,q)\\(\beta+\alpha k,\alpha j)\end{array}\middle|\,z^j\right]. \tag{3.1}$$

We now establish the following representation formula for $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of the generalized hypergeometric function.

Theorem 3.1 *For $\alpha = m \in \mathbb{N}$ and $q \in \mathbb{N}$, the function (1.6) can be expressed using the generalized hypergeometric function as*

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z) = \frac{z^k}{\Gamma(\beta + mk)}\, {}_qF_{mj}\left[\begin{array}{c}\Delta(q;\gamma)\\ \Delta(mj;\beta+mk)\end{array};\frac{q^q z^j}{(mj)^{mj}}\right], \tag{3.2}$$

where $\Delta(n;\gamma)$ is n-tuple $\left(\frac{\gamma}{n}\right), \left(\frac{\gamma+1}{n}\right), \dots, \left(\frac{\gamma+n-1}{n}\right)$.

Proof. Let $q, \alpha = m \in \mathbb{N}$ and using (1.6), then we have

$$\begin{aligned}
E^{j,k,q}_{\alpha,\beta,\gamma}(z) &= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!\,\Gamma(\beta + \alpha(nj+k))} z^{nj+k}\\
&= \frac{z^k}{\Gamma(\beta+mk)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{(\beta+mk)_{(mj)n}} \frac{z^{nj}}{n!}\\
&= \frac{z^k}{\Gamma(\beta+mk)} \sum_{n=0}^{\infty} \frac{q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n}{(mj)^{mjn} \prod_{s=1}^{mj} \left(\frac{\beta+mk+s-1}{mj}\right)_n} \frac{(z^j)^n}{n!}\\
&= \frac{z^k}{\Gamma(\beta+mk)}\, {}_qF_{mj}\left[\begin{array}{c}\Delta(q;\gamma)\\ \Delta(mj;\beta+mk)\end{array};\frac{q^q z^j}{(mj)^{mj}}\right].
\end{aligned}$$

To express $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of H-function, we start by representing $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ as Mellin-Barnes integral in the following theorem.

Theorem 3.2 *For every $z \in \mathbb{C}$ with $|arg(z)| < \pi$, the function $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ can be represented by the following Mellin-Barnes integral:*

$$E^{j,k,q}_{\alpha,\beta,\gamma}(z) = \frac{z^k}{2\pi i\,\Gamma(\gamma)} \int_L \frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\beta + \alpha k - \alpha js)} (-z^j)^{-s} ds, \tag{3.3}$$

where the contour of integration L joins $-i\infty$ to $+i\infty$, and splitting all the poles at $s = -n$, $(n = 0, 1, 2, \dots)$ to the left and the poles at $s = \frac{\gamma+n}{q}$, $(n = 0, 1, 2, \dots)$ to the right.

Proof. To prove (3.3), we evaluate the contour integral as the sum of residues at the poles $s = -n$, $(n \in \mathbb{N}_0)$. We have

$$\begin{aligned}
&\frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\beta + \alpha k - \alpha js)} (-z^j)^{-s} ds\\
&= \sum_{n=0}^{\infty} \operatorname*{Res}_{s=-n} \left[\frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\beta + \alpha k - \alpha js)} (-z^j)^{-s}\right]\\
&= \sum_{n=0}^{\infty} \lim_{s\to -n} \left[(s+n) \frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\beta + \alpha k - \alpha js)} (-z^j)^{-s}\right]\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + qn)(-z^j)^n}{\Gamma(\beta + \alpha k + \alpha jn)} \lim_{s\to -n} \left[(s+n)\,\Gamma(s)\right]\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + qn)(-z^j)^n}{\Gamma(\beta + \alpha(nj+k))} \lim_{s\to -n} \left[\frac{\Gamma(s+n+1)}{s(s+1)\dots(s+n-1)}\right]\\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + qn)(-z^j)^n}{\Gamma(\beta + \alpha(nj+k))} \frac{(-1)^n}{n!}\\
&= z^{-k}\Gamma(\gamma) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!\,\Gamma(\beta + \alpha(nj+k))} z^{nj+k}\\
&= z^{-k}\Gamma(\gamma) E^{j,k,q}_{\alpha,\beta,\gamma}(z).
\end{aligned}$$

Now by using definition (1.10), then equation (3.3) yields

$$E_{\alpha,\beta,\gamma}^{j,k,q}(z) = \frac{z^k}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[-z^j \,\middle|\, \begin{matrix} (1-\gamma, q) \\ (0,1), (1-\beta-\alpha k, \alpha j) \end{matrix} \right]. \tag{3.4}$$

4. Integral transforms

Here, we present various integral transforms, including the Euler transform, the laplace transform, the Mellin transform and the Whittaker transform of $E_{\alpha,\beta,\gamma}^{j,k,q}(z)$.

Theorem 4.1 *(Euler transform): For $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\nu) > 0$, $\Re(\mu) > 0$, $\Re(\sigma) > 0$ and $j \geq 1$, $k \geq 0$, $q \in \mathbb{N}$, we have*

$$\int_0^1 z^{\nu-1}(1-z)^{\mu-1} E_{\alpha,\beta,\gamma}^{j,k,q}(xz^\sigma)\, dz = \frac{x^k \Gamma(\mu)}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q), (\nu+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (\nu+\mu+\sigma k, \sigma j) \end{matrix} \,\middle|\, x^j \right]. \tag{4.1}$$

Proof. Applying (1.6) and (1.12), then

$$\begin{aligned}
&\int_0^1 z^{\nu-1}(1-z)^{\mu-1} E_{\alpha,\beta,\gamma}^{j,k,q}(xz^\sigma)\, dz \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\ x^{nj+k}}{n!\ \Gamma(\beta+\alpha(nj+k))} \int_0^1 z^{\nu+\sigma nj+\sigma k-1}(1-z)^{\mu-1}\, dz \\
&= \frac{x^k \Gamma(\mu)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\,\Gamma(\nu+\sigma k+\sigma nj)}{\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(\nu+\mu+\sigma k+\sigma nj)} \frac{(x^j)^n}{n!} \\
&= \frac{x^k \Gamma(\mu)}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q), (\nu+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (\nu+\mu+\sigma k, \sigma j) \end{matrix} \,\middle|\, x^j \right],
\end{aligned}$$

which proves the required result (4.1).

Corollary 4.1 *Putting $\gamma = q = 1$ in (4.1), we have*

$$\int_0^1 z^{\nu-1}(1-z)^{\mu-1} E_{\alpha,\beta}^{j,k}(xz^\sigma)\, dz = x^k \Gamma(\mu)\, {}_2\Psi_2 \left[\begin{matrix} (1,1), (\nu+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j), (\nu+\mu+\sigma k, \sigma j) \end{matrix} \,\middle|\, x^j \right]. \tag{4.2}$$

If we set $j = 1$ and $k = 0$ in equation (4.1), we obtain the known result [21].

Theorem 4.2 *(Laplace transform): For $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(p) > 0$, $\Re(\sigma) > 0$, $\Re(s) > 0$ and $j \geq 1$, $k \geq 0$, $q \in \mathbb{N}$, we have*

$$\int_0^\infty z^{p-1} e^{-sz} E_{\alpha,\beta,\gamma}^{j,k,q}(xz^\sigma)\, dz = \frac{s^{-(p+\sigma k)} x^k}{\Gamma(\gamma)} {}_2\Psi_1 \left[\begin{matrix} (\gamma, q), (p+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j) \end{matrix} \,\middle|\, \left(\frac{x}{s^\sigma}\right)^j \right]. \tag{4.3}$$

Proof. Using (1.6) and (1.13), we obtain

$$\begin{aligned}
&\int_0^\infty z^{p-1} e^{-sz} E_{\alpha,\beta,\gamma}^{j,k,q}(xz^\sigma)\, dz \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\ x^{nj+k}}{n!\ \Gamma(\beta+\alpha(nj+k))} \int_0^\infty z^{p+\sigma nj+\sigma k-1} e^{-sz} dz \\
&= \frac{s^{-(p+\sigma k)} x^k}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\,\Gamma(p+\sigma k+\sigma nj)}{\Gamma(\beta+\alpha k+\alpha nj)} \frac{(s^{-\sigma j} x^j)^n}{n!} \\
&= \frac{s^{-(p+\sigma k)} x^k}{\Gamma(\gamma)} {}_2\Psi_1 \left[\begin{matrix} (\gamma, q), (p+\sigma k, \sigma j) \\ (\beta+\alpha k, \alpha j) \end{matrix} \,\middle|\, \left(\frac{x}{s^\sigma}\right)^j \right],
\end{aligned}$$

which proves the required result (4.3).

If we set $j = 1$ and $k = 0$ in equation (4.3), we obtain the known result [21].

Theorem 4.3 *(Mellin transform): For $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(s)>0$ and $j\geq 1$, $k\geq 0$, $q\in\mathbb{N}$, we have*

$$\mathcal{M}\left[(-\omega z)^{-\frac{k}{j}}E^{j,k,q}_{\alpha,\beta,\gamma}\left((-\omega z)^{\frac{1}{j}}\right);s\right]=\frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\gamma)\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s}. \tag{4.4}$$

Proof. According to Theorem 3.2, the function (1.6) can be expressed as

$$(-\omega z)^{-\frac{k}{j}}E^{j,k,q}_{\alpha,\beta,\gamma}\left((-\omega z)^{\frac{1}{j}}\right)=\frac{1}{2\pi i}\int_L f^*(s)(z)^{-s}ds, \tag{4.5}$$

where

$$f^*(s)=\frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\gamma)\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s}.$$

Applying the definition of the Mellin transform in (4.5), we get

$$\mathcal{M}\left[(-\omega z)^{-\frac{k}{j}}E^{j,k,q}_{\alpha,\beta,\gamma}\left((-\omega z)^{\frac{1}{j}}\right);s\right]=\frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\gamma)\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s},$$

which proves the required result (4.4).

Corollary 4.2 *Putting $\gamma=q=1$ in (4.4), we have*

$$\mathcal{M}\left[(-\omega z)^{-\frac{k}{j}}E^{j,k}_{\alpha,\beta}\left((-\omega z)^{\frac{1}{j}}\right);s\right]=\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s}. \tag{4.6}$$

If we set $j=1$ and $k=0$ in equation (4.4), we obtain the known result [21].

Theorem 4.4 *(Whittaker transform): For $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\sigma)>0$, $\Re(\nu)>0$ and $j\geq 1$, $k\geq 0$, $q\in\mathbb{N}$, we have*

$$\begin{aligned}&\int_0^\infty u^{\nu-1}e^{-\frac{1}{2}tu}W_{\lambda,\mu}(tu)E^{j,k,q}_{\alpha,\beta,\gamma}(\omega u^\sigma)\,du\\&=\frac{\omega^k t^{-(\nu+\sigma k)}}{\Gamma(\gamma)}{}_3\Psi_2\left[\begin{array}{c}(\gamma,q),(\frac{1}{2}+\mu+\nu+\sigma k,\sigma j),(\frac{1}{2}-\mu+\nu+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(1-\lambda+\nu+\sigma k,\sigma j)\end{array}\middle|\left(\frac{\omega}{t^\sigma}\right)^j\right].\end{aligned} \tag{4.7}$$

Proof. Setting $tu=r$ in the left-hand side of (4.7), then we get

$$\begin{aligned}&\frac{1}{t}\int_0^\infty\left(\frac{r}{t}\right)^{\nu-1}e^{-\frac{r}{2}}W_{\lambda,\mu}(r)E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega\left(\frac{r}{t}\right)^\sigma\right)dr\\&=\omega^k t^{-(\alpha+\sigma k)}\sum_{n=0}^\infty\frac{(\gamma)_{qn}}{n!\,\Gamma(\beta+\alpha(nj+k))}\left(\frac{\omega}{t^\sigma}\right)^{nj}\int_0^\infty r^{\nu+\sigma nj+\sigma k-1}e^{-\frac{r}{2}}W_{\lambda,\mu}(r)dr\\&=\frac{\omega^k t^{-(\alpha+\sigma k)}}{\Gamma(\gamma)}\sum_{n=0}^\infty\frac{\Gamma(\gamma+qn)\,\Gamma\left(\frac{1}{2}+\mu+\nu+\sigma k+\sigma nj\right)\Gamma\left(\frac{1}{2}-\mu+\nu+\sigma k+\sigma nj\right)}{n!\,\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(1-\lambda+\nu+\sigma k+\sigma nj)}\left(\frac{\omega^j}{t^{\sigma j}}\right)^n\\&=\frac{\omega^k t^{-(\nu+\sigma k)}}{\Gamma(\gamma)}{}_3\Psi_2\left[\begin{array}{c}(\gamma,q),(\frac{1}{2}+\mu+\nu+\sigma k,\sigma j),(\frac{1}{2}-\mu+\nu+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(1-\lambda+\nu+\sigma k,\sigma j)\end{array}\middle|\left(\frac{\omega}{t^\sigma}\right)^j\right],\end{aligned}$$

which proves the required result (4.7).

If we set $j=1$ and $k=0$ in equation (4.7), we obtain the known result [21].

5. Fractional calculus operators

In this section, we consider the Riemann-Liouville fractional integrals and fractional derivatives involving the generalized arbitrary order Mittag-Leffler-type function $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$.

Theorem 5.1 *Let $\nu, \alpha, \beta, \gamma, \omega \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$, $x > a$. Let I^{ν}_{a+} be the left-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I^{\nu}_{a+}\left[(t-a)^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x) = (x-a)^{\beta+\nu-1}\,E^{j,k,q}_{\alpha,\beta+\nu,\gamma}\left(\omega(x-a)^{\alpha}\right). \tag{5.1}$$

Proof.

$$\begin{aligned}
&\left(I^{\nu}_{a+}\left[(t-a)^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x)\\
&= \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}(t-a)^{\beta-1}\,E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega(t-a)^{\alpha}\right)dt\\
&= \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\;\omega^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}\int_a^x (x-t)^{\nu-1}(t-a)^{\beta+\alpha(nj+k)-1}\,dt\\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\;\omega^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}\left(I^{\nu}_{a+}(t-a)^{\beta+\alpha(nj+k)-1}\right)(x).
\end{aligned}$$

Using the relation (1.17), we get

$$\begin{aligned}
&\left(I^{\nu}_{a+}\left[(t-a)^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x)\\
&= (x-a)^{\beta+\nu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!}\frac{\left(\omega(x-a)^{\alpha}\right)^{nj+k}}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)}\\
&= (x-a)^{\beta+\nu-1}E^{j,k,q}_{\alpha,\beta+\nu,\gamma}\left(\omega(x-a)^{\alpha}\right).
\end{aligned}$$

This completes the desired proof.

Corollary 5.1 *Let $\nu, \alpha, \beta, \gamma, \omega \in \mathbb{C}$ such that $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Then the following left fractional integral formula holds true:*

$$\left(I^{\nu}_{0+}\left[t^{\beta-1}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega t^{\alpha}\right)\right]\right)(x) = x^{\beta+\nu-1}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega x^{\alpha}\right). \tag{5.2}$$

Corollary 5.2 *If we set $\gamma = q = 1$ in (5.1), we obtain the following new and interesting result:*

$$\left(I^{\nu}_{0+}\left[t^{\beta-1}E^{j,k}_{\alpha,\beta}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x) = x^{\beta+\nu-1}E^{j,k}_{\alpha,\beta+\nu}\left(\omega(t-a)^{\alpha}\right). \tag{5.3}$$

Further, setting $j = 1$ and $k = 0$ in (5.1), we arrive at the result [17] given by Prajapati and Shukla.

Theorem 5.2 *Let $\nu, \alpha, \beta, \gamma, \omega \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $j \geq 1$, $k \geq 0$, $q \in (0,1) \cup \mathbb{N}$. Let I^{ν}_{-} be the right-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I^{\nu}_{-}\left[t^{-\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{-\alpha})\right]\right)(x) = x^{-\beta}E^{j,k,q}_{\alpha,\beta+\nu,\gamma}(\omega x^{-\alpha}). \tag{5.4}$$

Proof. For convenience, let Λ be the left-hand side of (5.4), then using the definition (1.6), we get

$$\begin{aligned}
\Lambda &= \left(I^{\nu}_{-}\left[t^{-\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{-\alpha})\right]\right)(x)\\
&= \left(I^{\nu}_{-}\;t^{-\nu-\beta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\;\omega^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}t^{-\alpha(nj+k)}\right)(x),
\end{aligned}$$

this can be written as,

$$\Lambda = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)} \left(I_{-}^{\nu}\ t^{-(\alpha(nj+k)+\nu+\beta)}\right)(x).$$

Applying (1.22), we obtain

$$\Lambda = x^{-\beta} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!} \frac{\left(\omega x^{-\alpha}\right)^{nj+k}}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)},$$

which gives us the desired result.

Corollary 5.3 *Let $\nu,\alpha,\beta,\gamma,\omega \in \mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j \geq 1$, $k \geq 0$, $q \in (0,1)\cup\mathbb{N}$. Then the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu}\left[t^{-\nu-\beta} E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{-\alpha})\right]\right)(x) \;=\; x^{-\beta}\left((\beta+\nu)E_{\alpha,\beta+\nu+1,\gamma}^{j,k,q}(\omega x^{-\alpha}) - x\frac{d}{dx}E_{\alpha,\beta+\nu+1,\gamma}^{j,k,q}(\omega x^{-\alpha})\right). \quad (5.5)$$

The proof can be easily obtained.

Now, we establish the following results of the Riemann-Liouville fractional integrals involving the Mittag-Leffler function of arbitrary order $E_{\alpha,\beta,\gamma}^{j,k,q}(z)$ in terms of the Fox-Wright function.

Theorem 5.3 *Let $\nu,\lambda,\alpha,\beta,\gamma,\omega,\sigma \in \mathbb{C}$ with $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\sigma)>0$ and $j \geq 1$, $k \geq 0$, $q \in (0,1)\cup\mathbb{N}$. Let I_{0+}^{ν} be the left-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I_{0+}^{\nu}\left[t^{\lambda-1} E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{\sigma})\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(\lambda+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(\nu+\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{\sigma})^j\right]. \quad (5.6)$$

Proof. By using (1.6) and (1.17), we have

$$\begin{aligned}
&\left(I_{0+}^{\nu}\left[t^{\lambda-1} E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{\sigma})\right]\right)(x)\\
&= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\, t^{\lambda-1} E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{\sigma})\, dt\\
&= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\, t^{\lambda-1}\left(\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)} t^{\sigma(nj+k)}\right) dt\\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\, t^{\lambda+\sigma(nj+k)-1}\ dt\right)\\
&= \frac{\omega^k x^{\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\,\Gamma(\lambda+\sigma k+\sigma nj)}{\Gamma(\beta+\alpha k+\alpha nj)\,\Gamma(\nu+\lambda+\sigma k+\sigma nj)}\frac{(\omega^j x^{\sigma j})^n}{n!},
\end{aligned}$$

which in accordance with the definition (1.9), completes the proof.

Theorem 5.4 *Let $\nu,\lambda,\alpha,\beta,\gamma,\omega,\sigma \in \mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $0<\Re(\nu)<1-\Re(\lambda-\sigma k)$ and $j \geq 1$, $k \geq 0$, $q \in (0,1)\cup\mathbb{N}$. Let I_{-}^{ν} be the right-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I_{-}^{\nu}\left[t^{\lambda-1} E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{-\sigma})\right]\right)(x) = \frac{\omega^k x^{\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\, {}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(1-\nu-\lambda+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(1-\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right]. \quad (5.7)$$

Proof. By using (1.6) and (1.18), we have

$$\begin{aligned}
&\left(I_-^\nu\left[t^{\lambda-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)\\
&=\frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}\,t^{\lambda-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)dt\\
&=\frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}\,t^{\lambda-1}\left(\sum_{n=0}^\infty \frac{(\gamma)_{qn}\;\omega^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}t^{-\sigma(nj+k)}\right)dt\\
&=\sum_{n=0}^\infty \frac{(\gamma)_{qn}\;\omega^{nj+k}}{n!\;\Gamma\left(\beta+\alpha(nj+k)\right)}\left(\frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}\,t^{\lambda-\sigma(nj+k)-1}\;dt\right)\\
&=\frac{\omega^k x^{\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\Gamma\left(\gamma+qn\right)\Gamma\left(1-\nu-\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)\Gamma\left(1-\lambda+\sigma k+\sigma nj\right)}\frac{(\omega^j x^{-\sigma j})^n}{n!},
\end{aligned}$$

which in accordance with the definition (1.9), completes the proof.

Theorem 5.5 *Let $\nu,\alpha,\beta,\gamma,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$, $x>a$. Let D_{a+}^ν be the left-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right]\right)(x)=(x-a)^{\beta-\nu-1}\,E_{\alpha,\beta-\nu,\gamma}^{j,k,q}\left(\omega(x-a)^\alpha\right). \tag{5.8}$$

Proof. To prove the theorem, we have

$$\begin{aligned}
&\left(D_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right]\right)(x)\\
&=\left(\frac{d}{dx}\right)^m\left[I_{a+}^{m-\nu}(t-a)^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right](x),
\end{aligned}$$

which on using (5.1) takes the following form:

$$\begin{aligned}
&\left(D_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right]\right)(x)\\
&=\left(\frac{d}{dx}\right)^m\left[(t-a)^{\beta-\nu+m-1}E_{\alpha,\beta-\nu+m,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right](x).
\end{aligned}$$

By applying (2.4), we get

$$\begin{aligned}
&\left(D_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega(t-a)^\alpha\right)\right]\right)(x)\\
&=(x-a)^{\beta-\nu-1}\,E_{\alpha,\beta-\nu,\gamma}^{j,k,q}\left(\omega(x-a)^\alpha\right),
\end{aligned}$$

which completes the desired proof.

Corollary 5.4 *Let $\nu,\alpha,\beta,\gamma,\omega\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following left fractional derivative formula holds true:*

$$\left(D_{0+}^\nu\left[t^{\beta-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^\alpha\right)\right]\right)(x)=x^{\beta-\nu-1}E_{\alpha,\beta-\nu,\gamma}^{j,k,q}\left(\omega x^\alpha\right). \tag{5.9}$$

Further, taking $j=1$ and $k=0$ in (5.8), we get known result due to Prajapati and Shukla [17].

Theorem 5.6 *Let $\nu,\alpha,\beta,\gamma,\omega\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\beta)>[\Re(\nu)]+1$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Let D_-^ν be the right-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_-^\nu\left[t^{\nu-\beta}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\alpha}\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta-\nu,\gamma}^{j,k,q}\left(\omega t^{-\alpha}\right). \tag{5.10}$$

Proof. Denote L.H.S. of (5.10) by Λ_1, then

$$\Lambda_1 = \left(D^{\nu}_{-}\left[t^{\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega t^{-\alpha}\right)\right]\right)(x).$$

Using definitions (1.20) and (1.6), and further simplification gives

$$\begin{aligned}\Lambda_1 &= (-1)^m\left(\frac{d}{dx}\right)^m\left(I^{m-\nu}t^{\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega t^{-\alpha}\right)\right)(x)\\ &= (-1)^m\left(\frac{d}{dx}\right)^m\frac{1}{\Gamma(m-\nu)}\int_x^\infty (t-x)^{m-\nu-1}\,t^{\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}\left(\omega t^{-\alpha}\right)dt\\ &= x^{-\beta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!}\frac{(\omega x^{-\alpha})^{nj+k}}{\Gamma\left(\beta-\nu+\alpha(nj+k)\right)}\\ &= x^{-\beta}E^{j,k,q}_{\alpha,\beta-\nu,\gamma}\left(\omega x^{-\alpha}\right).\end{aligned}$$

Corollary 5.5 *Let $\nu,\alpha,\beta,\gamma,\omega\in\mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\beta)>[\Re(\nu)]+1$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Then the following right fractional derivative formula holds true:*

$$\left(D^{\nu}_{-}\left[t^{\nu-\beta}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{-\alpha})\right]\right)(x) = x^{-\beta}\left((\beta-\nu)E^{j,k,q}_{\alpha,\beta-\nu+1,\gamma}(\omega x^{-\alpha}) - x\frac{d}{dx}E^{j,k,q}_{\alpha,\beta-\nu+1,\gamma}(\omega x^{-\alpha})\right). \quad (5.11)$$

The proof can be easily obtained.

We now establish the following results of the Riemann-Liouville fractional derivatives related to the $E^{j,k,q}_{\alpha,\beta,\gamma}(z)$ in terms of the Fox-Wright function.

Theorem 5.7 *Let $\nu,\lambda,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\sigma)>0$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Let D^{ν}_{0+} be the left-sided operator of Riemann-Liouville fractional derivative. Then*

$$\begin{aligned}\left(D^{\nu}_{0+}\left[t^{\lambda-1}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{\sigma})\right]\right)(x) &= \frac{\omega^k x^{-\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}\\ &\times {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\lambda+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(-\nu+\lambda+\sigma k,\sigma j)\end{matrix}\,\middle|\,(\omega x^{\sigma})^j\right]. \quad (5.12)\end{aligned}$$

Proof. Using definitions (1.6) and (1.19), and changing the orders of integration and summation, we get

$$\begin{aligned}&\left(D^{\nu}_{0+}\left[t^{\lambda-1}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{\sigma})\right]\right)(x)\\ &= \left(D^{\nu}_{0+}\left[\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}t^{\lambda+\sigma(nj+k)-1}\right]\right)(x)\\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(D^{\nu}_{0+}t^{\lambda+\sigma(nj+k)-1}\right)(x).\end{aligned}$$

By applying (1.23), we have

$$\begin{aligned}&\left(D^{\nu}_{0+}\left[t^{\lambda-1}E^{j,k,q}_{\alpha,\beta,\gamma}(\omega t^{\sigma})\right]\right)(x)\\ &= \omega^k x^{-\nu+\lambda+\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma\left(\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(-\nu+\lambda+\sigma k+\sigma nj\right)}x^{\sigma nj}\\ &= \frac{\omega^k x^{-\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+qn\right)\Gamma\left(\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)\Gamma\left(-\nu+\lambda+\sigma k+\sigma nj\right)}\frac{(\omega^j x^{\sigma j})^n}{n!}\\ &= \frac{\omega^k x^{-\nu+\lambda+\sigma k-1}}{\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\lambda+\sigma k,\sigma j)\\ (\beta+\alpha k,\alpha j),(-\nu+\lambda+\sigma k,\sigma j)\end{matrix}\,\middle|\,(\omega x^{\sigma})^j\right],\end{aligned}$$

which is the required result.

Theorem 5.8 *Let $\nu,\lambda,\alpha,\beta,\gamma,\omega,\sigma\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\nu)>0$, $\Re(\lambda-\sigma k)<\Re(\nu)-[\Re(\nu)]$ and $j\geq 1$, $k\geq 0$, $q\in(0,1)\cup\mathbb{N}$. Let D_-^ν be the right-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_-^\nu\left[t^{\lambda-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)=\frac{\omega^k x^{-\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}$$
$$\times\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(1+\nu-\lambda+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(1-\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right].\tag{5.13}$$

Proof. Using definitions (1.6) and (1.20), and changing the orders of integration and summation, we get

$$\begin{aligned}&\left(D_-^\nu\left[t^{\lambda-1}E_{\alpha,\beta,\gamma}^{j,k,q}\left(\omega t^{-\sigma}\right)\right]\right)(x)\\&=\left(D_-^\nu\left[\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}t^{\lambda-\sigma(nj+k)-1}\right]\right)(x)\\&=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj+k}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(D_-^\nu t^{\lambda-\sigma(nj+k)-1}\right)(x).\end{aligned}$$

By applying (1.24), we have

$$\begin{aligned}&\left(D_-^\nu\left[t^{\lambda-1}E_{\alpha,\beta,\gamma}^{j,k,q}(\omega t^{-\sigma})\right]\right)(x)\\&=\omega^k x^{-\nu+\lambda-\sigma k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\ \omega^{nj}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma\left(1+\nu-\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(1-\lambda+\sigma k+\sigma nj\right)}x^{-\sigma nj}\\&=\frac{\omega^k x^{-\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+qn\right)\Gamma\left(1+\nu-\lambda+\sigma k+\sigma nj\right)}{\Gamma\left(\beta+\alpha k+\alpha nj\right)\Gamma\left(1-\lambda+\sigma k+\sigma nj\right)}\frac{(\omega^j x^{-\sigma j})^n}{n!}\\&=\frac{\omega^k x^{-\nu+\lambda-\sigma k-1}}{\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{array}{c}(\gamma,q),(1+\nu-\lambda+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j),(1-\lambda+\sigma k,\sigma j)\end{array}\middle|(\omega x^{-\sigma})^j\right],\end{aligned}$$

which is the required result.

6. Conclusion

In conclusion, this manuscript successfully introduced a novel generalized Mittag-Leffler function type and rigorously investigated its fundamental properties. The derived recurrence relations, differential formulas, integral representations (Euler, Laplace, Mellin, Whittaker, and Mellin–Barnes), and its expressions in terms of other special functions (Fox-Wright, generalized hypergeometric, and H-functions) provide a comprehensive understanding of the newly defined function. Furthermore, the establishment of associated fractional integral and differential operators underscores its potential utility within fractional calculus. The derivation of several interesting special cases highlights the versatility and broad applicability of this generalization. This work lays a solid foundation for future research exploring the applications of this new generalized Mittag-Leffler function in diverse fields, particularly in the analytical and numerical treatment of fractional differential and integral equations.

Generalizations of the Mittag-Leffler function play a crucial role in advancing fractional calculus, leading to broader classes of fractional differential equations that incorporate these extended functions. This expansion, in turn, drives the growing need for numerical methods and computational techniques to solve such equations. The applications discussed in this work represent only a fraction of the potential impact in both theoretical and applied mathematics.

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