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Fractional Variational Calculus with the Truncated $\mathcal{M}$-series Fractional Derivative

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ABSTRACT: In this paper, we investigate a variational problem defined by a functional involving the truncated $\mathcal{M}$-series fractional derivative of the dependent variable. The necessary optimality conditions are derived in the form of the Euler-Lagrange equation, and several illustrative examples are presented to highlight the results.

Key Words: Fractional calculus, Truncated $\mathcal{M}$-series fractional derivative, Fractional Variational Problems, Fractional Euler-Lagrange equation.

Contents

1. Introduction

Fractional calculus is an extension of classical calculus that deals with operations of integration and differentiation of non-integer order [1,2,3,4]. It was introduced toward the end of the seventeenth century and has since gathered numerous important contributions to mathematics. This is primarily due to its ability to capture the memory effect, nonlocal properties, and its capacity to describe intermediate behaviors. For instance, fractional derivatives are useful in modeling viscoelastic substances and rheology (see e.g. [5]), as well as in economics, where the behavior of economic agents may depend on the history of past economic changes [6]. Today, the field of fractional calculus continues to attract the interest of many researchers, particularly in the solution of fractional differential equations. As a result, various methods have been developed to solve such equations, including the Lie symmetry method, Adomian decomposition method, and others [7,8,9,10,11].

To address certain challenges associated with these fractional derivatives, such as the violation of classical properties, new derivatives have been introduced. These include the conformable fractional derivative, the alternative fractional derivative, the truncated $\mathcal{V}$-fractional derivative, and others [12,13, 14,15].

In mathematical physics, the calculus of variations is an essential topic, underpinning the Lagrangian formulation of mechanics from which numerous dynamical equations can be derived [16,17]. Variational calculus has been developed to address problems where the aim is to identify a function that makes a given quantity optimal, such as the shortest distance between two points on a surface. The objective is to determine the unknown function that optimizes the value of a functional.

In this study, we assume that the derivative in the Lagrangian is the truncated $\mathcal{M}$-series fractional derivative. The formulation we adopt is a generalization that encompasses classical variational calculus results as a special case. Recall that several works have addressed similar topics involving different types of fractional operators [18,19,20].

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The structure of the paper is as follows: Section two provides a brief overview of the fundamental definitions and properties of the truncated $\mathcal{M}$-series fractional derivative. Section three is dedicated to presenting and proving the necessary optimality conditions, followed by the examination of some illustrative examples.

2. Preliminaries

Various definitions of fractional integrals and derivatives exist, but for the purposes of this paper, we address the truncated $\mathcal{M}$-series fractional derivative and the $\mathcal{M}$-series fractional integral as presented in [13].

Definition 2.1 *The truncated M-Series is defined for $\beta > 0$ as*

$$ {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(t) := \sum_{k=0}^{i} \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k} \frac{t^k}{\Gamma(\beta k+\gamma)}, $$

with $\beta, \gamma, t \in \mathbb{R}, p, q \in \mathbb{N}, a_n, c_m \in \mathbb{R}, c_m \neq 0, -1, -2, \dots (n = 1, 2, \dots, p; m = 1, 2, \dots, q)$, where $(\rho)_k$ is a generalization of the Pochhammer symbol, given by :

$$ (\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}. $$

Definition 2.2 *Let $f : [0, \infty) \to \mathbb{R}$. For $\beta > 0, t > 0$ and $0 < \alpha \leq 1$, the truncated $\mathcal{M}$-series fractional derivative of order α of a function f is*

$$ {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(\Gamma(\gamma) t\, {}_i\mathcal{M}^{\beta,\gamma}_{p,\gamma}\left(\varepsilon t^{-\alpha}\right)\right) - f(t)}{\varepsilon}. $$

Note that if f is $\mathcal{M}$-differentiable in some interval $(0, a), a > 0$, and

$$ \lim_{t \to 0^+} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t), $$

exists, then we have

$$ {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(0) = \lim_{t \to 0^+} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t). $$

In the following discussion, we review the main properties of the truncated $\mathcal{M}$-series fractional derivative, highlighting its fundamental properties.

Theorem 2.1 *Let $0 < \alpha \leq 1, \beta > 0, a, b \in \mathbb{R}$ and f, g are $\mathcal{M}$-differentiable at the point $t > 0$. Then*

1. ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(af + bg)(t) = a\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t) + b\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} g(t)$.
2. ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f \cdot g)(t) = f(t)\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} g(t) + g(t)\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t)$.
3. ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{f}{g}\right)(t) = \frac{g(t)\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t) - f(t)\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} g(t)}{(g(t))^2}$.
4. ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f(t)) = 0$, *where f is constant .*
5. *If, furthermore, f is differentiable, then* ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t) = \mathcal{K} t^{1-\alpha} \frac{df(t)}{\mathrm{d}t}$, *where* $\mathcal{K} = \frac{a_1 \cdots a_p}{c_1 \cdots c_q} \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$.
6. ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f \circ g)(t) = f'(g(t))\, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} g(t)$, *for f differentiable at $g(t)$.*

Definition 2.3 *Let $a \geq 0, t > a, 0 < \alpha \leq 1$ and f be a function defined in $(a, t]$. Then, the $\mathcal{M}$-series fractional integral of order α the function f is defined by*

$$ \mathcal{I}^{\alpha}_{\mathcal{M}} f(t) := \mathcal{K}^{-1} \int_a^t \frac{f(x)}{x^{1-\alpha}} \mathrm{d}x. $$

The truncated $\mathcal{M}$-series fractional derivative and the $\mathcal{M}$-series fractional integral are connected in the following way :

Theorem 2.2 *Let $a \geq 0, 0 < \alpha \leq 1$ and f be a continuous function such that there exists $\mathcal{I}_{\mathcal{M}}^{\alpha} f$. Then*

1. ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}(\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)) = f(t)$,
2. $\mathcal{I}_{\mathcal{M}}^{\alpha}({}_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(t)) = f(t) - f(a)$.

The formula for integration by parts is expressed as follows :

Theorem 2.3 *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that f, g are differentiables and $0 < \alpha \leq 1$. Then*

$$\int_a^b f(x)_i\mathcal{D}_{\mathcal{M}}^{\alpha} g(x) d_\alpha x = [f(x)g(x)]_a^b - \int_a^b g(x)_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(x) d_\alpha x, \tag{2.1}$$

where $d_\alpha x = \mathcal{K}^{-1} \frac{dx}{x^{1-\alpha}}$.

In what follows, we present, the fundamental lemma for fractional calculus of variations :

Lemma 2.1 *Let f and y be continuous functions on $[a, b]$, if*

$$\int_a^b \eta(x) f(x, y(x), {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y(x)) d_\alpha x = 0,$$

for any $\eta \in C[a, b]$, with $\eta(a) = \eta(b) = 0$, then

$$f(x, y(x), {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y(x)) = 0.$$

3. The truncated $\mathcal{M}$-series fractional variational problem

We examine variational problems in this section that include independent and dependent variables, as well as the truncated $\mathcal{M}$-series fractional derivative. The objective is to find $y \in C^1[a, b]$ that minimizes or maximizes the functional :

$$J[y(x)] = \int_a^b \mathcal{L}\left(x, y(x), {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y(x)\right) d_\alpha x. \tag{3.1}$$

The problem is subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, where $\mathcal{L}$ is the Lagrangian, assumed to be continuous with respect to all of its arguments, and the following conditions are satisfied

$$\mathcal{L} \in C^1([a, b] \times \mathbb{R} \times \mathbb{R}),$$

$$x \to \partial_2 \mathcal{L}\left(x, y(x), {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y(x)\right) \in L^1(a, b),$$
$$x \to \partial_3 \mathcal{L}\left(x, y(x), {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y(x)\right) \in AC[a, b], \forall y \in AC[a, b].$$

The corresponding Euler-Lagrange equation for the variational problem (3.1) is provided by the following result.

Theorem 3.1 *When J defined in (3.1) has a local extremum, the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation is given by :*

$$\frac{\partial \mathcal{L}}{\partial y} - {}_i\mathcal{D}_{\mathcal{M}}^{\alpha}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y\right)}\right) = 0. \tag{3.2}$$

To begin, we consider variation functions of the type $y + \epsilon\eta$, where $\eta : [a, b] \to \mathbb{R}$ is a function of class C^1, with $\eta(a) = \eta(b) = 0$. Then we can write

$$J[y + \epsilon\eta] = \int_a^b \mathcal{L}\left(x, y + \epsilon\eta, {}_i\mathcal{D}_{\mathcal{M}}^{\alpha} y + \epsilon_i\mathcal{D}_{\mathcal{M}}^{\alpha}\eta\right) d_\alpha x.$$

The functional J reaches its extremum if $\frac{dJ[y+\epsilon\eta]}{d\epsilon}=0$, which means that :

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial y}\eta + \frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\eta\right) d_\alpha x = 0,$$

therefore, this leads to :

$$\int_a^b \frac{\partial \mathcal{L}}{\partial y}\eta d_\alpha x + \int_a^b \frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\eta d_\alpha x = 0.$$

Using (2.1), we can conclude that :

$$\int_a^b \frac{\partial \mathcal{L}}{\partial y}\eta d_\alpha x + \left[\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}\eta\right]_a^b - \int_a^b \eta\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}\right) d_\alpha x = 0.$$

Since $\eta(a)=\eta(b)=0$, we conclude that :

$$\int_a^b \left(\frac{\partial \mathcal{L}}{\partial y} - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}\right)\right)\eta d_\alpha x = 0.$$

Based on the lemma and the arbitrary choice of η, we get :

$$\frac{\partial \mathcal{L}}{\partial y} - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}\right) = 0.$$

The following section presents a few examples for illustration.

4. Illustration examples

Example 4.1 Consider the truncated $\mathcal{M}$-series fractional geodesic problem, given by :

$$J[y] = \int_a^b \mathcal{L}\left(x, y, {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right) d_\alpha x = \int_a^b \sqrt{1+\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)^2}d_\alpha x, \tag{4.1}$$

where $y \in C^1[a,b], y(a)=y_a$ and $y(b)=y_b$.

As $\mathcal{L}$ does not explicitly depend on y, using the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation (3.2), we obtain :

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)}\right) = 0, \tag{4.2}$$

therefore, we have

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y}{\sqrt{1+\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)^2}}\right) = 0. \tag{4.3}$$

Using the above equation and the fact that if ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t)=0$, then $f(t)=constant$ we conclude that :

$$\frac{{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y}{\sqrt{1+\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y\right)^2}} = c_0, \tag{4.4}$$

with c_0 is a constant.
By applying simplifications, we obtain :

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}y = c_1, \tag{4.5}$$

where c_1 is a constant given in terms of c_0.
By assuming that y is differentiable and by applying the fifth property from theorem 2.1, we obtain :

$$\mathcal{K}x^{1-\alpha}y'(x) = c_1. \tag{4.6}$$

Therefore, the solution of (4.1) is of the form

$$y(x) = \frac{c_1}{\alpha\mathcal{K}} x^\alpha + c_2, \tag{4.7}$$

where c_2 is an arbitrary constant.

Remark 4.1 In particular, if $\alpha = \beta = \gamma = 1$ and $a_n = c_m = 1 \quad (n = 1, 2, \ldots, p; m = 1, 2, \ldots, q)$, which implies that $\mathcal{K} = 1$, we retrieve the equation of the line, which geometrically is the curve that minimizes the distance between two points, i.e.

$$y(x) = c_1 x + c_2.$$

Example 4.2 Let $\mathcal{L}$ denote the Lagrangian of the system, defined by :

$$\mathcal{L}\left(t, y(t), {}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right) = \frac{1}{2} m \left({}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right)^2 - V(y(t)), \tag{4.8}$$

which characterizes the motion of a particle with the mass m and the truncated $\mathcal{M}$-series fractional derivative velocity ${}_i\mathcal{D}^\alpha_\mathcal{M} y(t)$ under the influence of the potential V in one dimension.
In this situation, the variational problem takes the form of :

$$J[y(t)] = \int_0^1 \mathcal{L}\left(t, y(t), {}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right) d_\alpha t, \tag{4.9}$$

with $y(0) = y_0$ and $y(1) = y_1$.

A solution y to the variational problem exists if it satisfies the corresponding truncated $\mathcal{M}$-series fractional Euler–Lagrange equation (3.2)

$$\frac{\partial \mathcal{L}}{\partial y} - {}_i\mathcal{D}^\alpha_\mathcal{M} \left(\frac{\partial \mathcal{L}}{\partial \left({}_i\mathcal{D}^\alpha_\mathcal{M} y\right)} \right) = 0.$$

Case 1. If $V(y(t)) = 0$ (free particle) the Lagrangian (4.8) takes the form :

$$\mathcal{L}\left(t, y(t), {}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right) = \frac{1}{2} m \left({}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right)^2. \tag{4.10}$$

Because $\mathcal{L}$, as defined in (4.10), is independent of y and taking into account that

$$\frac{\partial \mathcal{L}}{\partial \left({}_i\mathcal{D}^\alpha_\mathcal{M} y\right)} = m {}_i\mathcal{D}^\alpha_\mathcal{M} y(t),$$

then (3.2) becomes

$${}_i\mathcal{D}^\alpha_\mathcal{M} \left(m {}_i\mathcal{D}^\alpha_\mathcal{M} y(t)\right) = 0,$$

By performing some algebraic calculations, we derive the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation of motion for a free particle, which takes the form of a second-order differential equation :

$$y''(t) + \frac{(1-\alpha)}{t} y'(t) = 0. \tag{4.11}$$

The general solution to this equation is expressed as :

$$y(t) = \frac{c_1}{\alpha} t^\alpha + c_2. \tag{4.12}$$

where c_1 and c_2 are arbitrary constants.

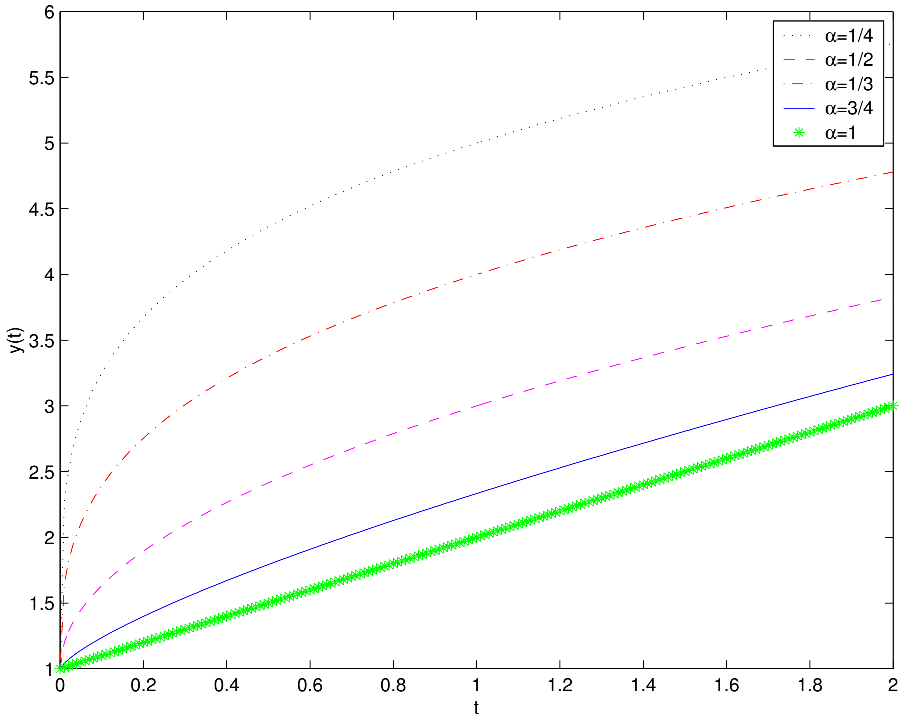

Figure 1: Solution of (4.12) with $c_1 = c_2 = 1$ and different values of α.

In this figure, we see the impact of the fractional order α on the solution. In fact as α gets closer to 1, the solution increasingly becomes a linear equation

Case 2. The equation of motion of a charged particle in gravity $V(y) = mgy$ where g is the acceleration of gravity.

The Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L}\left(t, y(t); {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right) = \frac{1}{2} m \left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right)^2 - mgy(t). \tag{4.13}$$

Based on (3.2), the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation becomes

$$-mg - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(m_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right) = 0.$$

From this, we deduce that :

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y\right) + g = 0.$$

We obtain :

$$\mathcal{K}^2 t^{2-2\alpha} y'' + (1-\alpha)\mathcal{K}^2 t^{1-2\alpha} y' + g = 0. \tag{4.14}$$

The truncated $\mathcal{M}$-series fractional Euler-Lagrange equation of motion of a charged particle in gravity is simplified to be of the form

$$y'' + \frac{(1-\alpha)}{t} y' + \frac{g}{\mathcal{K}^2 t^{2-2\alpha}} = 0. \tag{4.15}$$

The general solution of this equation is given by

$$y(t) = \frac{-g}{2\alpha^2 \mathcal{K}^2} t^{2\alpha} + a t^{\alpha} + b. \tag{4.16}$$

where a and b are the arbitrariness constants.

Remark 4.2 When $\alpha = \mathcal{K} = 1$, the equation becomes $y''(t) + g = 0$, which is the standard equation of motion for a charged particle under the influence of gravity, and its general solution is

$$y(t) = -\frac{gt^2}{2} + c_1 t + c_2. \tag{4.17}$$

where c_1 and c_2 are constants.

Case 3. The equation of motion of an harmonic oscillator $V(y) = \frac{1}{2} m\omega^2 y^2$, where w is the natural frequency.

The fractional Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L}\left(t, y(t); {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right) = \frac{1}{2} m \left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right)^2 - \frac{1}{2} m\omega^2 y^2. \tag{4.18}$$

The corresponding truncated $\mathcal{M}$-series fractional Euler-Lagrange equation (3.2) gives

$$-m\omega^2 y - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} \left(m {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y(t)\right) = 0.$$

It yields,

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}} \left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}} y\right) + \omega^2 y = 0.$$

So, we have

$$\mathcal{K}^2 t^{2-2\alpha} y'' + (1-\alpha)\mathcal{K}^2 t^{1-2\alpha} y' + \omega^2 y = 0. \tag{4.19}$$

Consequently, the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation of motion of an harmonic oscillator becomes

$$y'' + \frac{(1-\alpha)}{t} y' + \left(\frac{\omega}{\mathcal{K}}\right)^2 \frac{y}{t^{2-2\alpha}} = 0 \tag{4.20}$$

The general solution of this equation is given by

$$y(t) = a_0 \cos\left(\frac{\omega}{\alpha\mathcal{K}} t^{\alpha}\right) + a_1 \sin\left(\frac{\omega}{\alpha\mathcal{K}} t^{\alpha}\right), \tag{4.21}$$

where a_0 and a_1 are the constants.

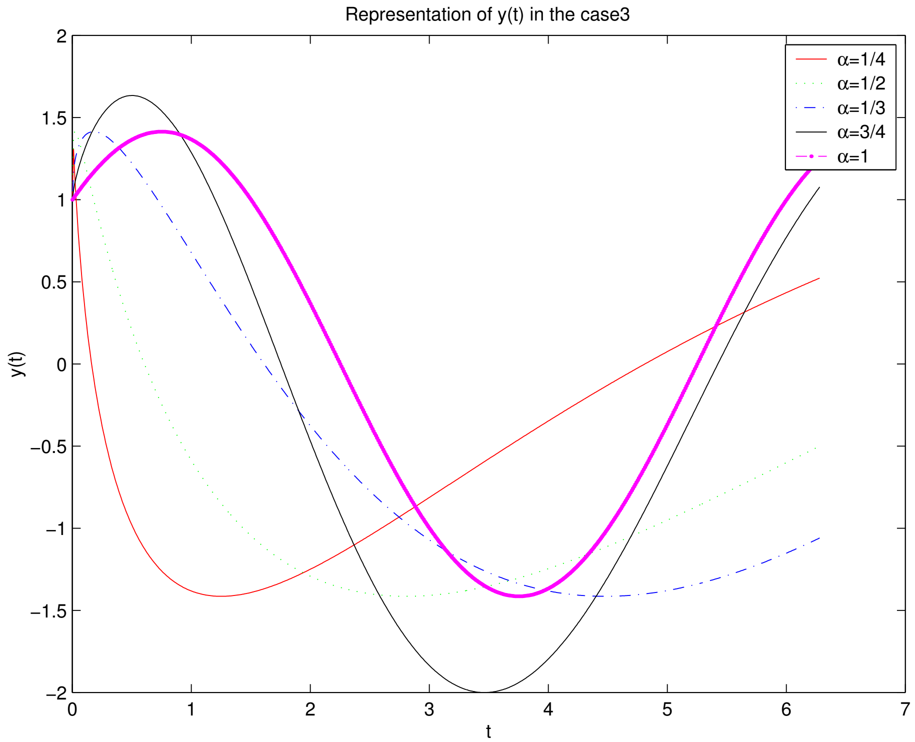

Figure 2: Solution of (4.21) with $a_0 = a_1 = 1, \mathcal{K} = 1$ and different values of α.

The figure demonstrates how the fractional order α affects the solution. It clearly shows that the introduction of α breaks the periodic behavior of the solution. Nevertheless, as α gets closer to 1, this periodicity reemerges.

Example 4.3 Let $\mathcal{L}$ represent the Lagrangian of the LC circuit system, defined as

$$\mathcal{L}\left(t, Q(t), {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}Q\right) = \frac{1}{2}L\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}Q\right)^2 - \frac{Q^2}{2C}. \tag{4.22}$$

The LC-circuit consists of an inductor, L, and a capacitor, C, connected in series. It is a simplified model of the RLC-circuit, where resistance is neglected, meaning there is no energy loss.

The variational problem is defined by :

$$J[Q(t)] = \int_0^1 \mathcal{L}\left(t, Q(t), {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}Q(t)\right) d_{\alpha}t, \tag{4.23}$$

with $Q(0) = Q_0$ and $Q(1) = Q_1$.

The variational problem admits a solution Q provided it satisfies the truncated $\mathcal{M}$-series fractional Euler–Lagrange equation (3.2).

$$\frac{\partial \mathcal{L}}{\partial Q} - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{\partial \mathcal{L}}{\partial\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}Q\right)}\right) = 0. \tag{4.24}$$

$$-\frac{Q}{C} - {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(L{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}Q(t)\right) = 0.$$

Thus,

$$L{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\right) + \frac{1}{C}Q = 0,$$

So, we have

$$\mathcal{K}^2\left(\frac{(1-\alpha)}{t^\alpha}t^{1-\alpha}\dot{Q}+t^{2-2\alpha}\ddot{Q}\right)+\frac{1}{LC}Q=0. \tag{4.25}$$

Consequently, the truncated $\mathcal{M}$-series fractional Euler-Lagrange equation of the LC-circuit becomes

$$\ddot{Q}+\frac{(1-\alpha)}{t}\dot{Q}+\frac{1}{LC\mathcal{K}^2}\frac{Q}{t^{2-2\alpha}}=0. \tag{4.26}$$

The general solution of this equation is given by

$$Q(t)=a\cos\left(\frac{1}{\alpha\mathcal{K}\sqrt{LC}}t^\alpha\right)+b\sin\left(\frac{1}{\alpha\mathcal{K}\sqrt{LC}}t^\alpha\right), \tag{4.27}$$

where a and b are the constants.

Example 4.4 Let $\mathcal{L}$ denote the Lagrangian of the system defined by

$$\mathcal{L}\left(t,\theta(t),{}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta(t)\right)=\frac{1}{2}mR^2\left({}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta(t)\right)^2-mgR\cos(\theta(t)), \tag{4.28}$$

which describes a particle sliding on a hemisphere of fixed radius R without friction with the mass m and the truncated $\mathcal{M}$-series fractional derivative (α)-velocity ${}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta(t)$.
In this case the variational problems is given by,

$$J[\theta(t)]=\int_0^1\mathcal{L}\left(t,\theta(t),{}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta(t)\right)d_\alpha t, \tag{4.29}$$

with $\theta(0)=\theta_0$ and $\theta(1)=\theta_1$.

The variational problem admits a solution θ if it satisfies the corresponding truncated $\mathcal{M}$-series fractional Euler–Lagrange equation (3.2)

$$\frac{\partial\mathcal{L}}{\partial\theta}-{}_i\mathcal{D}^\alpha_{\mathcal{M}}\left(\frac{\partial\mathcal{L}}{\partial\left({}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta\right)}\right)=0.$$

Thus,

$$mgR\sin(\theta)-{}_i\mathcal{D}^\alpha_{\mathcal{M}}\left(mR^2{}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta\right)=0.$$

Therefore, we have

$${}_i\mathcal{D}^\alpha_{\mathcal{M}}\left({}_i\mathcal{D}^\alpha_{\mathcal{M}}\theta\right)-\frac{g}{R}\sin(\theta)=0.$$

So, we get the solution of the equation of motion,

$$\ddot{\theta}+\frac{(1-\alpha)}{t}\dot{\theta}-\frac{g(\Gamma(\beta+1))^2}{Rt^{2-2\alpha}}\sin(\theta)=0. \tag{4.30}$$

5. Conclusion

In this study, we have presented a variational problem involving the truncated $\mathcal{M}$-series-fractional derivative. By utilizing the principle of local extremum, we derive the corresponding truncated $\mathcal{M}$-series fractional Euler-Lagrange equation. Additionally, we explored how the α degree of the derivatives influences the behavior of the solution through several illustrative examples.

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Conflict of interest The authors declare that they have no conflict of interest.

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