Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.75754

On the Relationship between Uniqueness Sets and Interpolation Sets in Functional Quasinormed Spaces

José Alfonso López Nicolás

ABSTRACT: Given a functional quasinormed space we obtain conditions for a uniformly discrete set to be either a uniqueness set or an interpolation set. We apply these results to Paley-Wiener spaces. We also obtain new results on both the refinement of stable sampling sets and the extension of stable interpolation sets in quasinormed spaces.

Key Words: Quasinormed spaces, Paley-Wiener spaces, p-stable sampling set, p-stable interpolation set, uniqueness set.

Contents

1. Introduction

In this contribution we obtain conditions in order to know whether, given a quasinormed space and a uniformly discrete set, such a set is either an interpolation set or a uniqueness set for that space, and we apply these results to Paley-Wiener spaces obtaining certain conclusions when the given uniformly discrete set is an arithmetic progression. Indeed the interaction between certain functional spaces and certain uniformly discrete sets gives as a consequence that we can assure that this set is a uniqueness set if we know that it is not an interpolation set, and viceversa: it is an interpolation set whether it is not a uniqueness set. Of course this phenomenon is not usual in Sampling and Interpolation Theory and we obtain some results on it.

In addition, for general functional quasinormed spaces we obtain new results on the extension of a given interpolation set keeping this property (adding a certain quantity of new points), and on the contraction of sampling sets, maintaining this property when certain elements have been extracted of the given set, and we obtain an expression for the new sampling bound.

The research on both uniqueness sets and interpolation sets for spaces of functions is very extense. We will comment a few examples.

In 1955 J. von Neumann proved that the lattice $\mathbb{Z}^2$ is a uniqueness set for the Bargmann-Fock space (see [13]). In 1992 K. Seip and R. Wallstén proved that a sequence $\Gamma \subseteq \mathbb{R}^n$ which is uniformly close to $\mathbb{Z}^2$ is still a uniqueness set for the Bargmann-Fock space. In 2013 M. Mitkovski and B. D. Wick defined the concept of d-regular sequence, and shew when a d-regular sequence is a uniqueness set for the Bargmann-Fock space, and when it is not (see [12]).

Remind the definition of the Paley-Wiener spaces.

Submitted February 16, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 32A10, 32A20, 46B25, 46E99, 46B20.

Definition 1.1. *Let $S \subseteq \mathbb{R}^n$ be a bounded set and $p \in (0, +\infty]$. We define*

$$E_S^p := \{f \in \mathcal{S}'(\mathbb{R}^n) \,:\, \operatorname{supp}(\hat{f}) \subseteq S \text{ and } \|f\|_p < \infty\},$$

which is a closed vector subspace of $(L^p(\mathbb{R}^n), \| \|_p)$. We call (p, S)-Paley-Wiener space to the complete space $(E_S^p, \| \|_p)$, which is a quasi-Banach space if $p \in (0, 1)$ and it is a Banach space if $p \in [1, +\infty]$. $PW_S := E_S^2$ is a Hilbert space called the classical Paley-Wiener space for S.

We need the following definitions.

Definition 1.2 (Uniformly discrete set)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be infinite countable. We say that Λ is uniformly discrete (briefly u.d.) if*

$$\delta(\Lambda) := \inf_{\lambda,\, \lambda' \in \Lambda,\, \lambda \neq \lambda'} \|\lambda - \lambda'\| > 0.$$

The constant $\delta(\Lambda)$ is called the separation constant of Λ.

Given $\Omega \subseteq \mathbb{R}^n$, we denote by $\mathfrak{F}(\Omega, \mathbb{C})$ the vector space of the complex functions defined in Ω, which is a commutative $\mathbb{C}$-algebra with the usual product of functions.

Definition 1.3 (Uniqueness set)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete. We say that Λ is a uniqueness or complete set (briefly, US) for E if for every $f \in E$ we have that*

$$(\forall \lambda \in \Lambda \; f(\lambda) = 0) \Rightarrow f = 0.$$

For the classical spaces of Paley-Wiener, PW_S, in 1967 A. Beurling and P. Malliavin defined the concept of upper Beurling-Malliavin density of a uniformly discrete set, and used it for obtaining very important results on uniqueness sets in their celebrated paper [4].

Definition 1.4 (Beurling - Malliavin Density ([4]))**.** *Let $\Lambda = (\lambda_n)_{n \in \mathbb{Z}^+} \subseteq \mathbb{R}^n$ be a u.d. set.*

- *The distribution function (or counting function) of Λ is defined by*

$$F(t) := \begin{cases} 0, & \text{if } t = 0, \\ |\{k \in \mathbb{Z}^+ \mid \lambda_k \leq t\}|, & \text{if } t > 0. \end{cases}.$$

- *Let $\mathcal{C}$ be the family of all sequences $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{Z}^+}$ of intervals in $(0, +\infty)$ such that $a_n < b_n \leq a_{n+1}$ for all $n \in \mathbb{Z}^+$ and*

$$\sum_{n=1}^{+\infty} \left(\frac{b_n}{a_n} - 1\right)^2 = +\infty.$$

  Define

$$\overline{\delta}(\mathcal{I}) := \limsup_{n \to +\infty} \frac{b_n}{a_n}, \quad l_{\mathcal{I}} := \liminf_{n \to +\infty} \frac{F(b_n) - F(a_n)}{b_n - a_n}.$$

  Also define

$$\mathcal{R} := \bigcup_{\mathcal{I} \in \mathcal{C}} [0, l_{\mathcal{I}}].$$

  Then the Beurling-Malliavin density of Λ, which is denotated by $b(\Lambda)$, is defined as the supremum of $\mathcal{R}$. This is:

$$b(\Lambda) := \sup \mathcal{R} = \sup \{l_{\mathcal{I}} \,:\, \mathcal{I} \in \mathcal{C}\}.$$

Defining $\mathcal{E}_\Lambda := \left\{e^{\pm i \lambda x} \,:\, \lambda \in \Lambda\right\}$, Beurling and Malliaving proved in [4] that

$$\mathcal{R}(\Lambda) = 2\pi\, b(\Lambda),$$

where

$$\mathcal{R}(\Lambda) := \sup\left\{a \in \mathbb{R}^+ \;:\; \mathcal{E}_\Lambda \text{ is complete in } L^2\,(0,\, a)\right\} =$$
$$= \sup\left\{a \in \mathbb{R}^+ \;:\; \Lambda \text{ is a uniqueness set for } E^2_{(0,\,a)}\right\}$$

is called the *radius of completeness* of Λ.

Now remind the definition of both Bernstein spaces and Landau densities.

Definition 1.5 (Bernstein spaces)**.** *Let $\sigma > 0$ be.*

- *Let $f \in \mathcal{H}(\mathbb{C}^n)$ be a holomorphic function. f is said to be of exponential type at most σ if for all $\varepsilon > 0$ there is a constant $A > 0$ such that*

$$|f(z)| \leq A_\varepsilon\, e^{(\sigma+\varepsilon)\|z\|_1} \text{ for all } z \in \mathbb{C}^n.$$

- *Define $E_\sigma := \{f \in \mathcal{H}(\mathbb{C}^n) \;:\; f \text{ is of exponential type at most } \sigma\}$.*

- *The set $B^p_\sigma(\mathbb{R}^n) := \{f \in E_\sigma \;:\; f|_{\mathbb{R}^n} \in L^p(\mathbb{R}^n)\}$ is a closed vector subespace of $(L^p(\mathbb{R}^n),\, \|\;\|_p)$. We call $(\sigma,\, p)$-Bernstein space to the space $(B^p_\sigma(\mathbb{R}^n),\, \|\;\|_p)$, which is Banach if $p \in [1,\, +\infty]$ and quasi-Banach if $p \in (0,\, 1)$.*

We will denote $B^\infty_\sigma(\mathbb{R}^n)$ by $B_\sigma(\mathbb{R}^n)$; or simply, by B_σ when $n = 1$.

Let $p \in (0,\, +\infty]$. By the Paley-Wiener theorem, $E^p_{[-\sigma,\,\sigma]^n} = B^p_\sigma(\mathbb{R}^n)$ holds.

Now we can define the concepts Landau's lower, upper and uniform densities (see [7]).

Definition 1.6 (Landau Densities)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete.*

- *The lower uniform density of a Λ is defined by*

$$D_-(\Lambda) := D_{-,\,n}(\Lambda) := \liminf_{R\to+\infty} \inf_{x\in\mathbb{R}^n} \frac{|\Lambda \cap B[x;\; R]|}{\lambda_n(B[x;\; R])}.$$

- *The upper uniform density of a Λ is defined by*

$$D_+(\Lambda) := D_{+,\,n}(\Lambda) := \limsup_{R\to+\infty} \sup_{x\in\mathbb{R}^n} \frac{|\Lambda \cap B[x;\; R]|}{\lambda_n(B[x;\; R])}.$$

- *If $D_-(\Lambda) = D_+(\Lambda)$, we say Λ has uniform density, and we define its uniform density by that common value: $D(\Lambda) := D_-(\Lambda) = D_+(\Lambda)$.*

A. Beurling proved that in the context of classical Bernstein spaces $(B_\sigma,\, \|\;\|_\infty)$, $\sigma > 0$, a uniformly discrete set Λ is a stable sampling set for B_σ if and only if each weak limit of translates of Λ is a uniqueness set for B_σ (see [2], p. 343; and [18]) and obtained a necessary and sufficient conditions for a given uniformly discrete set to be a stable sampling set using the concept of *density.* He also used this concept in order to obtain a characterization of interpolation sets and sampling sets for these spaces (see [3]), which was extended by K. M. Flornes to $B^p_\sigma := B^p_\sigma(\mathbb{R})$ with $p \in (0,\, 1]$ (see [5]):

Theorem 1.7 (Beurling-Flornes ([2], [3], [5]))**.** *Let $p \in (0,\, 1] \cup \{+\infty\}$ and $\Lambda \subseteq \mathbb{R}$ u.d. Let $\sigma > 0$.*

1. *$D_-(\Lambda) > \frac{\sigma}{\pi}$ if and only if Λ is SS for B^p_σ.*

2. *$D_+(\Lambda) < \frac{\sigma}{\pi}$ if and only if Λ is IS for B^p_σ.*

On the other hand, A. Olevskii and A. Ulanovskii have also obtained important results for these spaces. In 2008 they proved that the following perturbation of $\mathbb{Z}$,

$$\Lambda := \left\{ n + 2^{-|n|} \; : \; n \in \mathbb{Z} \right\},$$

is a uniqueness set for PW_S for every bounded set $S \subseteq \mathbb{R}$ whose Lebesgue measure is less than 1 (see [14]). In 2011 they also proved that for every set $S \subseteq \mathbb{R}^n$ of finite Lebesgue measure there exists a uniformly discrete set $\Lambda \subseteq \mathbb{R}^n$ with uniform Landau density such that both $D(\Lambda) = \frac{\lambda_n(S)}{(2\,\pi)^n}$ and Λ is a uniqueness set for PW_S, where $D(\Lambda)$ is the uniform Landau density of Λ and $\lambda_n(S)$ denotes de Lebesgue measure of S (see [15]). In 2016 they proved the existence of uniqueness sets for Sobolev spaces whose spectrum belongs to a given set S of infinite Lebesgue measure having *periodic gaps*, and also proved that this periodicity condition is crucial; furthemore, in that contribution they also proved the existence of uniqueness sets for rapidly decreasing functions (see [16]).

In addition, for spaces of trigonometric series (Paul J. Cohen, 1958; Alexander S. Kechris and Alain Louveau, 1987) and for Dirichlet-type spaces (Karim Kellay, 2011) many results on uniqueness sets have been obtained.

On the other hand it is obvious that if we extract one or several elements of an interpolation set (respectively, stable interpolation set), the resultant set is also an interpolation set (respectively, stable interpolation set); and if we add one or several elements to a uniqueness set, the resultant set is also a uniqueness one. Similarly, if we add one or several elements to a stable sampling set, the set that we obtain is a stable sampling set provided that this extended set verifies the Plancherel-Polya condition (which happens if the set of elements what have been added verifies the condition of Plancherel-Polya).

The main question is to know what happens if we do the inverse action. That is, we wish to know what happens if we do a refinement of a stable sampling set and an extension of an interpolation set (respectively, stable interpolation set).

In this sense this contribution is also, in part, a continuation of two previous papers (see [9] and [11]) on this topic, for functional quasinormed spaces.

A motivation for our results is the following result, which is well known in sampling and interpolation in the Hilbert spaces $E_S^2 = PW_S$, the classical Paley-Wiener spaces, being $S \subseteq \mathbb{R}^n$ a bounded and Lebesgue measurable set. See [17], Proposition 2.8, p. 16; and Proposition 4.6, p. 36. Also see [26], chapter 4.

Theorem 1.8 (See Definition 1.13)**.** *Let $S \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set, and $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. For each $\lambda \in \Lambda$ consider the function $\varphi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$x \mapsto \varphi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

We also define the set $E(\Lambda) := \{\varphi_\lambda\}_{\lambda \in \Lambda} \subseteq L^2(S)$. Then:

1. *Λ is a 2-SS for E_S^2 if and only if $E(\Lambda)$ is a frame for $L^2(S)$.*
2. *Λ is a 2-SIS for E_S^2 if and only if $E(\Lambda)$ is a Riesz sequence for $L^2(S)$.*
3. *Λ is a 2-SCIS for E_S^2 if and only if $E(\Lambda)$ is a Riesz basis for $L^2(S)$.*

Hence the results on 2-SCIS (respectively, 2-SS, 2-SIS) for the classical Paley-Wiener space E_S^2 can be translated into results on exponential Riesz bases (respectively, frames, Riesz sequences) in $L^2(S)$, and viceversa.

There are many contributions on both the problems of refinement of stable sampling sets and the extension of stable interpolation sets in the Hilbert spaces $L^2(D)$, with $D \subseteq \mathbb{R}^n$ bounded, Lebesgue measurable and disconnected, with the optimal case of obtaining a stable complete interpolation set, what is translated for the Paley-Wiener space E_D^2 in obtaining a Riesz basis.

Furthermore, the research in frame theory, Riesz sequences and bases theory in L^p spaces has allowed to obtain advances in stable sampling and interpolation theory. For example, in 1964 M. I. Kadec proved his celebrated theorem:

Theorem 1.9 (Kadec-1/4 Theorem). *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers. Suppose that*

$$|\lambda_n - n| \leq L < \frac{1}{4} \text{ for every } n \in \mathbb{Z}.$$

Then the set of exponential functions $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2\left((-\pi,\,\pi)\right)$.

This result says, in terms of sampling and interpolation theory, that $\mathbb{Z}$ is a complete interpolation set (this is, both sampling and interpolation set) for the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$, and that every L-perturbation of $\mathbb{Z}$ also verifies it whenever $L < \frac{1}{4}$.

The bound 1/4 is sharp, and Theorem 1.9 improves a previous very important result by R. Paley and N. Wiener, where the bound is $\frac{1}{\pi^2}$ (see [19], page 113). In 1974 S. A. Avdonin obtained a generalization of Theorem 1.9 using a certain type of means of the values λ_n's (see [1]).

Kadec-1/4 Theorem has been generalized in several ways to L^p spaces and to sequences $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, obtaining $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$, as a result, the property of completeness (see for example [8], [21], [23] and [22]). In addition, the Riesz basis problem in the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$ has been proposed for non-exponential basis. In this sense several important results analogous to Kadec-1/4 Theorem have been obtained for sets of sinc functions by Velluci, P., involving the Lamb-Oseen constant (see [24] and [25]).

As said before, our contribution is aimed at obtaining new results on refinement of stable sampling sets and the extension of stable interpolation sets of a given quasinormed space in order to obtain new ones.

In this paper we consider that $0 \in \mathbb{N}$.

1.1. Notation and Definitions

We establish the notation and definitions that we will use.

Given $A \subseteq \mathbb{R}^n$, we denote the indicator function of A with respect to $\mathbb{R}^n$ by χ_A. If $\|\ \|$ is a quasinorm, we denote by $\tau_{\|\ \|}$ its associated topology. A function $h : E \longrightarrow \mathbb{C}$ defined in a vector space E is even if $h(-x) = h(x)$ for all $x \in E$.

Definition 1.10. *Let $\Omega \subseteq \mathbb{R}^n$ be with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $E \subseteq \mathfrak{F}(\Omega,\,\mathbb{C})$ a vector subspace.*

a) We say that E is invariant (or, also, closed) by translations if for every $x \in \mathbb{R}^n$ the function

$$\tau_x\, f : x + \Omega \to \mathbb{C}$$

defined by

$$\tau_x\, f\,(y) := f\,(y - x) \text{ for all } y \in x + \Omega,$$

belongs to E.

b) Let $\|\ \|$ be a quasinorm on E. We say that the quasinormed space $(E,\,\|\ \|)$ is invariant by translations if both E is invariant by translations and

$$\|\tau_x\, f\| = \|f\| \text{ for each } x \in \mathbb{R}^n.$$

Definition 1.11 (Sequence space $l^p(\Lambda)$). *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\Lambda \subseteq \Omega$ be u.d.*

1. *Let $p \in (0,\,+\infty)$. We define the set*

$$l^p(\Lambda) := \left\{ (a_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^\Lambda \mid \sum_{\lambda\in\Lambda} |a_\lambda|^p < \infty \right\}.$$

The mapping $\|\ \|_p : l^p(\Lambda) \to \mathbb{R}$ given by $\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_p := \left(\sum_{\lambda\in\Lambda} |a_\lambda|^p\right)^{\frac{1}{p}}$, is a quasinorm for $l^p(\Lambda)$, which is a norm if and only if $p \geq 1$. With this quasinorm $l^p(\Lambda)$ is a complete space.

2.

$$l^{\infty}(\Lambda) := \left\{ (a_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^{\Lambda} \mid \sup_{\lambda\in\Lambda} |a_\lambda| < \infty \right\}.$$

The mapping $\| \, \|_\infty : l^\infty(\Lambda) \to \mathbb{R}$ *defined by* $\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_\infty := \sup_{\lambda\in\Lambda} |a_\lambda|$ *is a norm for* $l^\infty(\Lambda)$ *make this space a Banach space.*

Remark 1.12. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a non bounded set with* $\mathrm{Int}\,(\Omega) \neq \emptyset$. *Let* $\Lambda \subseteq \Omega$ *be a u.d. set and let* $p \in (0, +\infty)$. *Define*

$$e_\lambda := (\delta_{\lambda\mu})_{\mu\in\Lambda} \in l^p(\Lambda) \ \textit{for each}\ \lambda \in \Lambda.$$

It is well known that $\mathcal{B} := \{e_\lambda : \lambda \in \Lambda\}$ *is a Schauder basis of* $(l^p(\Lambda), \| \, \|_p)$ *(the so called canonical Schauder basis). In particular,* $\mathcal{B}$ *is a total set for* $(l^p(\Lambda), \| \, \|_p)$*, that is,*

$$\overline{\mathrm{Span}\,\mathcal{B}}^{\| \, \|_p} = l^p(\Lambda).$$

Definition 1.13. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a non bounded set with* $\mathrm{Int}\,(\Omega) \neq \emptyset$. *Let* $\Lambda \subseteq \Omega$ *be a u.d. set. Let* $(E, \| \, \|)$ *be a quasinormed space, verifying* $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. *Let* $p \in (0, +\infty]$ *and* $\Lambda \subseteq \mathbb{R}^n$ *be a uniformly discrete set. Assume that*

$$(f(\lambda))_{\lambda\in\Lambda} \in l^p(\Lambda) \ \textit{for all}\ f \in E.$$

- *The* $\mathbb{C}$*-linear mapping* $S := S_\Lambda : (E, \| \, \|) \to (l^p(\Lambda), \| \, \|_p)$ *given by* $f \to (f(\lambda))_{\lambda\in\Lambda}$ *is called the* p*-sampling operator of* $(E, \| \, \|)$ *with respect to* Λ.
- *We say that* Λ *verifies the p-Plancherel-Polya condition (briefly, p-P.P.C.) for* $(E, \| \, \|)$ *if* S *is continuous, this is, if there exists a constant* $C > 0$ *such that*

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq C \, \|f\| \ \textit{for all}\ f \in E.$$

- Λ *is said to be a p-interpolation set (in short, p-IS) for* E *if* S *is surjective. Given* $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ *and* $f \in E$*, we say that* f *interpolates* c *(over* Λ*) if* $f(\lambda) = c_\lambda$ *for all* $\lambda \in \Lambda$.
- *We say that* Λ *a p-stable interpolation set (briefly, p-SIS) for* $(E, \| \, \|)$ *if* S *is continuous, surjective and has a continuous inverse by right.*
- *We say that* Λ *a p-complete interpolation set (briefly, p-CIS) for* E *if* S *is bijective.*
- Λ *is called a p-stable complete interpolation set (briefly, p-SCIS) for* $(E, \| \, \|)$ *if* S *is a topological isomorphism.*

As consequence of the Banach open mapping theorem, we have the following observation.

Remark 1.14. *Let* $(E, \| \, \|)$ *be a quasi-Banach space, verifying* $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. *Let* $p \in (0, +\infty]$ *and* $\Lambda \subseteq \mathbb{R}^n$ *be a uniformly discrete set. Assume that the sampling operator* $S : (E, \| \, \|) \to (l^p(\Lambda), \| \, \|_p)$ *is continuous. Then*

1. Λ *is a p-IS for* E *if and only if* Λ *is a p-SIS for* $(E, \| \, \|)$.
2. Λ *is a p-CIS for* E *if and only if* Λ *is a p-SCIS for* $(E, \| \, \|)$.

In the rest of this article we will omit the quasinorm of E, except if necessary, and will refer to the quasinormed space $(E, \| \, \|)$ simply as E.

In order to state our main results we need the following definition.

Definition 1.15. *Let* $S \subseteq \mathbb{R}^n$ *be a bounded set and* $p \in (0, +\infty]$. *We define the p-Even Paley-Wiener space with spectrum in* S *by*

$$\begin{aligned} E^p_{e,S} &:= \{f \in E^p_S \,:\, f \text{ is an even function}\} = \\ &= \{f \in E^p_S \,:\, f(-x) = f(x) \ \text{for all}\ x \in \mathbb{R}^n\}, \end{aligned}$$

which is a closed vector subspace of $(E^p_S, \| \, \|_p)$. *Hence,* $\left(E^p_{e,S}, \| \, \|_p\right)$ *is a quasi-Banach space if* $p \in (0, 1)$ *and it is a Banach space if* $p \in [1, +\infty]$.

1.2. Main results

Our main results are the following ones.

Theorem 1.16. *Let $p \in [1, +\infty)$, and $q \in [1, +\infty)$ be its conjugate exponent (this is, $p^{-1} + q^{-1} = 1$). Let $K \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set such that its indicator function χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$. Let $\lambda_0, \lambda \in \mathbb{R}^n$, $\lambda \neq 0$. Define*

$$\lambda_n := \lambda_0 + n\,\lambda \text{ for every } n \in \mathbb{Z} \setminus \{0\}.$$

Also define

$$\Lambda := \{\lambda_n \,:\, n \in \mathbb{N}\} \subseteq \mathbb{R}^n.$$

Suppose that $\Lambda \setminus \{\lambda_0\}$ is not a US for $E^1_{e,K}$. There exists $f \in E^1_{e,K} \setminus \{0\}$ such that $f|_{\Lambda \setminus \{\lambda_0\}} = 0$. Suppose that $f(\lambda_0) \neq 0$. Then

i) Λ is a p-IS for E^p_K (and, thus, it is also an p-SIS for E^p_K).

ii) Define

$$h := \frac{1}{f(\lambda_0)} f \in E^1_{e,K} \setminus \{0\},$$

which verifies that $h(\lambda_0) = 1$ and $h|_{\Lambda \setminus \{\lambda_0\}} = 0$. Let $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$. Then $g_c := \sum_{\lambda \in \Lambda} c_\lambda \cdot \tau_\lambda h \in E^p_K$ interpolates c, that is,

$$c = S_\Lambda(g) \in \operatorname{Im}(S_\Lambda) = S_\Lambda(E^p_K).$$

Theorem 1.17. *Let $p \in [1, +\infty)$, and $q \in [1, +\infty)$ be its conjugate exponent. Let $K \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set such that its indicator function χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$. Let $\lambda_0, \lambda \in \mathbb{R}^n$, $\lambda \neq 0$. Define*

$$\lambda_n := \lambda_0 + n\,\lambda \text{ for each } n \in \mathbb{Z} \setminus \{0\}.$$

Also define

$$\Lambda := \{\lambda_n \,:\, n \in \mathbb{N}\} \subseteq \mathbb{R}^n.$$

Assume both that $\Lambda \setminus \{\lambda_0\}$ is not a US for $E^1_{e,K}$ and Λ is a US for $E^1_{e,K}$. Then

i) Λ is a p-SIS for E^p_K.

ii) Let $h \in E^1_{e,K} \setminus \{0\}$ be such that both $h(\lambda_0) = 1$ and $h|_{\Lambda \setminus \{\lambda_0\}} = 0$. Let $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$. Then $g_c := \sum_{\lambda \in \Lambda} c_\lambda \cdot \tau_\lambda h \in E^p_K$ interpolates c.

The structure of this paper is as follows. Section 1 contains the introduction and the definitions with the main results, Theorem 1.16 and Theorem 1.17. In section 2 we state and prove some results on general spaces of functions which allow us to prove both Theorem 1.16 and Theorem 1.17 when used jointly with Lemma 3.3 of convergence of series in Paley-Wiener spaces . In section 3 we apply the results of section 2 to Paley-Wiener spaces, and prove both Theorem 1.16 and Theorem 1.16. Finally in section 4 we obtain new results on the extension of interpolation sets and the contraction of stable interpolation sets for general quasinormed spaces.

2. General results

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete. Assume that Λ is not a US for E. Then there exist a function $g \in E \setminus \{0\}$ and $x_0 \in \Omega$ verifying both that*

$$\begin{cases} g(x_0) = 1, & \\ g(\lambda) = 0 & \text{for every } \lambda \in \Lambda. \end{cases}$$

Proof. As Λ is not a US for E, then, by definition, there exists a function $f \in E \setminus \{0\}$ such that

$$f(\lambda) = 0 \text{ for all } \lambda \in \Lambda.$$

Since $f \neq 0$, then there exists $x_0 \in \Omega$ verifying that $f(x_0) \neq 0$. Now define

$$g := \frac{1}{f(x_0)} f \in E \setminus \{0\}$$

which verifies the required conditions. □

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete. Let $\lambda_0 \in \Omega \setminus \Lambda$, and define $\Lambda' := \Lambda \cup \{\lambda_0\}$. Suppose both that Λ is not a US for E and Λ' is a US for E. Then there exists $g \in E \setminus \{0\}$ such that*

$$\begin{cases} g(\lambda_0) = 1, \\ g(\lambda) = 0 & \text{for every } \lambda \in \Lambda. \end{cases}$$

Therefore: Let $p \in (0, +\infty]$. Then

$$S_{\Lambda'}(g) = (g(\mu))_{\mu \in \Lambda'} = e_{\lambda_0} \in l^p(\Lambda').$$

Proof. Since Λ is not a US for E, then there exists a function $f \in E \setminus \{0\}$ such that

$$f(\lambda) = 0 \text{ for all } \lambda \in \Lambda.$$

We claim that $f(\lambda_0) \neq 0$. Indeed, suppose that $f(\lambda_0) = 0$. We will obtain a contradiction.

We have that

$$f(\mu) = 0 \text{ for all } \mu \in \Lambda'.$$

As Λ' is a US for E, then $f = 0$, and this is a contradiction.

So that we have proved our claim: $f(\lambda_0) \neq 0$. Define now

$$g := \frac{1}{f(\lambda_0)} f \in E \setminus \{0\},$$

which verifies the conditions of the statement. □

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\Omega, \mathbb{C})$. Let $\Gamma \subseteq \Omega$ be uniformly discrete and $p \in (0, +\infty)$. Consider the sampling mapping*

$$S_\Gamma : (E, \|\,\|) \to (l^p(\Gamma), \|\,\|_p),$$

defined by

$$f \to S_\Gamma(f) := (f(\gamma))_{\gamma \in \Gamma}.$$

Suppose that $\mathcal{B} := \{e_\gamma : \gamma \in \Gamma\} \subseteq S_\Gamma(E)$.

The following statements are equivalent:

i) Γ *is a p-IS for* E.

ii) $\operatorname{Im}(S_\Gamma) = S_\Gamma(E) \subseteq l^p(\Gamma)$ *is a closed set in* $(l^p(\Gamma), \|\,\|_p)$.

Proof. First it is obvious that *i)* implies *ii)*. Indeed, assume that Γ is a p-IS for E, this is, S_Γ is surjective. So

$$\operatorname{Im}(S_\Gamma) = S_\Gamma(E) = l^p(\Gamma)$$

is closed in $(l^p(\Gamma), \|\,\|_p)$.

Let us prove that *ii*) implies *i*). Suppose that

$$\operatorname{Im}(S_\Gamma) = S_\Gamma(E)$$

is closed in $(l^p(\Gamma), \| \|_p)$. We will show that Γ is a p-IS for E.

We know that $\mathcal{B} := \{e_\gamma : \gamma \in \Gamma\}$ is a total set for $l^p(\Gamma)$. In other words

$$\overline{\operatorname{Span}\mathcal{B}}^{\| \|_p} = l^p(\Gamma).$$

Since $\mathcal{B} \subseteq S_\Gamma(E) \subseteq l^p(\Gamma)$, then

$$\operatorname{Span}\mathcal{B} \subseteq S_\Gamma(E) \subseteq l^p(\Gamma),$$

and thus

$$l^p(\Gamma) = \overline{\operatorname{Span}\mathcal{B}}^{\| \|_p} = \overline{S_\Gamma(E)} = S_\Gamma(E) \subseteq l^p(\Gamma).$$

Then

$$S_\Gamma(E) = l^p(\Gamma),$$

that is, Γ is a p-IS for E. □

Remark 2.4. *Notice that under the hypotheses of Lemma 2.3, if Γ is a p-IS for E and $\Lambda \subseteq \Gamma$ is u.d., then Λ is also a p-IS for E.*

Theorem 2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be uniformly discrete, and $p \in (0, +\infty]$. Suppose that*

$$e_\lambda \in S_\Lambda(E) = \operatorname{Im}(S_\Lambda) \text{ for all } \lambda \in \Lambda.$$

For each $\lambda \in \Lambda$ let $f_\lambda \in E$ be such that

$$e_\lambda = S_\Lambda(f_\lambda) = (f_\lambda(\mu))_{\mu\in\Lambda}.$$

Let $a = (a_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ be, and suppose that the series

$$\sum_{\lambda\in\Lambda} a_\lambda f_\lambda$$

converges pointwise in E. Define $f := \sum_{\lambda\in\Lambda} a_\lambda f_\lambda$ as its pointwise limit.

Then $S_\Lambda(f) = a$, and thus $a \in \operatorname{Im}(S_\Lambda)$.

Proof. For each $\mu \in \Lambda$ we have that

$$f(\mu) = \sum_{\lambda\in\Lambda} a_\lambda f_\lambda(\mu) = \sum_{\lambda\in\Lambda} a_\lambda \delta_{\lambda\mu} = a_\mu.$$

Hence

$$a = (a_\mu)_{\mu\in\Lambda} = (f(\mu))_{\mu\in\Lambda} = S_\Lambda(f).$$

□

An immediate consequence of Theorem 2.5 is the following result.

Corollary 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a non bounded set with $\operatorname{Int}(\Omega) \neq \emptyset$. Let $(E, \| \|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$ and verifying that the convergence of whatever sequence in $(E, \| \|)$ implies its pointwise convergence. Let $p \in (1, +\infty]$ be, and $q \in [1, +\infty)$ be its conjugate exponent (this is, $p^{-1} + q^{-1} = 1$).*

Let $\Lambda \subseteq \Omega$ be uniformly discrete. Suppose that

$$e_\lambda \in S_\Lambda(E) = \operatorname{Im}(S_\Lambda) \text{ for all } \lambda \in \Lambda,$$

where $S_\Lambda : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$*, given by* $f \to (f(\lambda))_{\lambda\in\Lambda}$*, is the p-sampling operator of* $(E, \| \|)$ *with respect to* Λ*.*

For each $\lambda \in \Lambda$ *let* $f_\lambda \in E$ *be such that*

$$e_\lambda = S_\Lambda (f_\lambda) = (f_\lambda (\mu))_{\mu\in\Lambda} .$$

Assume that

$$w := (w_\lambda := \|f_\lambda\|)_{\lambda\in\Lambda} \in l^p(\Lambda).$$

Then

i) Let $a = (a_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ *be. The series*

$$\sum_{\lambda\in\Lambda} a_\lambda f_\lambda$$

converges in $(E, \| \|)$*, and its limit*

$$f := \sum_{\lambda\in\Lambda} a_\lambda f_\lambda$$

verifies that $S_\Lambda (f) = a$*, and therefore* $a \in \mathrm{Im}\,(S_\Lambda)$*.*

ii) Λ *is a p-IS for* E*.*

Proof. We will prove the first item since the second one is an immediate consequence from it.

First we will show that the series

$$\sum_{\lambda\in\Lambda} a_\lambda f_\lambda$$

converges absolutely in E, which implies that it converges in E because E is a quasi-Banach space. Indeed, by Holder Inequality we have that

$$\sum_{\lambda\in\Lambda} \|a_\lambda f_\lambda\| = \sum_{\lambda\in\Lambda} |a_\lambda| \, \|f_\lambda\| \le \|a\|_p \, \|w\|_q < +\infty.$$

Hence the series $\sum_{\lambda\in\Lambda} a_\lambda f_\lambda$ is convergent in E. Let

$$f := \sum_{\lambda\in\Lambda} a_\lambda f_\lambda \in E$$

its limit. By one of our assumptions we have that $\sum_{\lambda\in\Lambda} a_\lambda f_\lambda$ also pointwise converges to f. Thus

$$f(\mu) = \sum_{\lambda\in\Lambda} a_\lambda f_\lambda(\mu) = \sum_{\lambda\in\Lambda} a_\lambda \delta_{\lambda\,\mu} = a_\mu \text{ for each } \mu \in \Lambda.$$

So we obtain that

$$a = (a_\mu)_{\mu\in\Lambda} = (f(\mu))_{\mu\in\Lambda} = S_\Lambda(f).$$

□

Lemma 2.7. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a non bounded set with* $\mathrm{Int}\,(\Omega) \neq \emptyset$*. Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*, and let* E *be a* $\mathbb{K}$*-vector subspace of* $\mathfrak{F}(\Omega, \mathbb{C})$*. Let* $\Lambda \subseteq \Omega$ *be uniformly discrete, and suppose that* Λ *is not a US for* E*. Let* $g \in E \setminus \{0\}$*,* $x_0 \in \Omega \setminus \Lambda$ *be verifying both that* $g(x_0)$ *and* $g|_\Lambda = 0$ *(both of them exist by Lemma 2.1). Define*

$$\lambda_0 := x_0, \; \Lambda' := \Lambda \cup \{\lambda_0\} \supseteq \Lambda, \; g_{\lambda_0} := g \in E.$$

Observe that $g_{\lambda_0} (\lambda_0) = 1$ *and* $g_{\lambda_0}|_\Lambda = 0$*. Let* $p \in (0, +\infty]$*.*

Consider the sampling mapping

$$S_{\Lambda'} : (E, \| \|) \to (l^p (\Lambda'), \| \|_p),$$

defined by

$$f \to S_{\Lambda'}(f) := (f(\mu))_{\mu \in \Lambda'}.$$

We have that

$$e_{\lambda_0} = (\delta_{\lambda_0\,\mu})_{\mu \in \Lambda'} = S_{\Lambda'}(g_{\lambda_0}).$$

Assume that for each $\lambda \in \Lambda$ there exists a function

$$r_\lambda : \Omega \to \Omega$$

such that

1. $r_\lambda(\Lambda) \subseteq \Lambda'$.
2. $r_\lambda|_\Lambda : \Lambda \to \Lambda'$ *is bijective.*
3. $r_\lambda(\lambda) = \lambda_0$.
4. $g \circ r_\lambda \in E$ *for every* $\lambda \in \Lambda$.

Define

$$g_\lambda := g \circ r_\lambda \in E \text{ for all } \lambda \in \Lambda.$$

Then:

i)

$$e_\lambda = (\delta_{\lambda\,\mu})_{\mu \in \Lambda'} = S_{\Lambda'}(g_\lambda) \text{ for every } \lambda \in \Lambda.$$

ii) $\mathcal{B} := \{e_{\lambda'}\}_{\lambda' \in \Lambda'} \subseteq S_{\Lambda'}(E) = \operatorname{Im}(S_{\Lambda'})$.

Proof. Let $\lambda \in \Lambda$. Then

1. $g_\lambda(\lambda) = (g_{\lambda_0} \circ r_\lambda)(\lambda) = g_{\lambda_0}(r_\lambda(\lambda)) = g_{\lambda_0}(\lambda_0) = 1$.
2. $g_\lambda(\mu) = (g_{\lambda_0} \circ r_\lambda)(\mu) = g_{\lambda_0}(r_\lambda(\mu)) = 0$.
3. $g_\lambda(\lambda_0) = (g_{\lambda_0} \circ r_\lambda)(\lambda_0) = g_{\lambda_0}(r_\lambda(\lambda_0)) = 0$.

□

Hence

$$\begin{cases} e_\lambda = (\delta_{\lambda\,\mu})_{\mu \in \Lambda'} = S_{\Lambda'}(g_\lambda) & \text{for every } \lambda \in \Lambda. \\ e_{\lambda_0} = (\delta_{\lambda_0\,\mu})_{\mu \in \Lambda'} = S_{\Lambda'}(g_{\lambda_0}). \end{cases}$$

So that

$$\mathcal{B} := \{e_{\lambda'}\}_{\lambda' \in \Lambda'} \subseteq S_{\Lambda'}(E) = \operatorname{Im}(S_{\Lambda'}).$$

Remark 2.8. *We can take as $x_0 \in \Omega \setminus \Lambda$ whichever point of Ω such that $g(x_0) \neq 0$ (it is sufficient if we divide g by $g(x_0) \neq 0$, and defining again as g the obtained function).*

Theorem 2.9. *Let E be a vector subspace of $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$ invariant by translations. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete, and $p \in (0, +\infty]$ be. Let $\lambda_0, \lambda \in \mathbb{R}^n$, $\lambda \neq 0$. Define*

$$\lambda_n := \lambda_0 + n\,\lambda \text{ for every } n \in \mathbb{Z} \setminus \{0\}.$$

Also define

$$\Lambda := \{\lambda_n \,:\, n \in \mathbb{N}\} \subseteq \mathbb{R}^n.$$

Suppose that $\Lambda \setminus \{\lambda_0\}$ is not a US for E. There exists $f \in E \setminus \{0\}$ such that $f|_{\Lambda \setminus \{\lambda_0\}} = 0$. Assume that $f(\lambda_0) \neq 0$. Define

$$g := \frac{1}{f(\lambda_0)}\, f \in E \setminus \{0\},$$

which verifies that $g(\lambda_0) = 1$ *and* $g|_{\Lambda\setminus\{\lambda_0\}} = 0$.

For every $n \in \mathbb{Z}$ *we define*

$$g_n := \tau_{n\,\lambda}\, g : \mathbb{R}^n \to \mathbb{C},$$

given by

$$g_n(x) := \tau_{n\,\lambda}\, g = g(x - n\,\lambda) \ \text{for all } x \in \mathbb{R}^n.$$

Then

i) $g_n(\lambda_n) = g(\lambda_0) = 1$ *for every* $n \in \mathbb{N}$.

ii) $g_n(\lambda_m) = g(\lambda_m - n\,\lambda) = g(\lambda_0 + (m-n)\,\lambda) = g(\lambda_{m-n}) = 0$ *for all* $n, m \in \mathbb{Z}$, $n \neq m$.

iii) $e_{\lambda_n} = (\delta_{\lambda_n\,\mu})_{\mu\in\Lambda} = S_\Lambda(g_n) \in \mathrm{Im}(S_\Lambda) = S_\Lambda(E) \subseteq l^p(\Lambda)$ *for every* $n \in \mathbb{Z}$.

iv) $\mathcal{B} := \{e_\lambda\}_{\lambda\in\Lambda} \subseteq S_\Lambda(E) = \mathrm{Im}(S_\Lambda) \subseteq l^p(\Lambda)$.

As consequence of Lemma 2.2 and Theorem 2.9 we obtain the following result.

Corollary 2.10. *Let* E *be a vector subspace of* $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$ *invariant by translations. Let* $\Lambda \subseteq \mathbb{R}^n$ *be uniformly discrete, and* $p \in (0, +\infty]$ *be. Let* $\lambda_0, \lambda \in \mathbb{R}^n$, $\lambda \neq 0$. *Define*

$$\lambda_n := \lambda_0 + n\,\lambda \ \text{for every } n \in \mathbb{Z}\setminus\{0\}.$$

Also define

$$\Lambda := \{\lambda_n \ : \ n \in \mathbb{N}\} \subseteq \mathbb{R}^n.$$

Suppose both that $\Lambda\setminus\{\lambda_0\}$ *is not a US for* E *and* Λ *is a US for* E. *Let* $g \in E\setminus\{0\}$ *be such that*

$$g(\lambda_0) = 1 \ \text{and } g|_{\Lambda\setminus\{\lambda_0\}} = 0.$$

For each $n \in \mathbb{Z}$ *we define the translation*

$$g_n := \tau_{n\,\lambda}\, g : \mathbb{R}^n \to \mathbb{C},$$

given by

$$g_n(x) := \tau_{n\,\lambda}\, g = g(x - n\,\lambda) \ \text{for all } x \in \mathbb{R}^n.$$

Then

i) $g_n(\lambda_n) = g(\lambda_0) = 1$ *for every* $n \in \mathbb{N}$.

ii) $g_n(\lambda_m) = g(\lambda_m - n\,\lambda) = g(\lambda_0 + (m-n)\,\lambda) = g(\lambda_{m-n}) = 0$ *for each* $n, m \in \mathbb{Z}$, $n \neq m$.

iii) $e_{\lambda_n} = (\delta_{\lambda_n\,\mu})_{\mu\in\Lambda} = S_\Lambda(g_n) \in \mathrm{Im}(S_\Lambda) = S_\Lambda(E) \subseteq l^p(\Lambda)$ *for each* $n \in \mathbb{Z}$.

iv) $\mathcal{B} := \{e_\lambda\}_{\lambda\in\Lambda} \subseteq S_\Lambda(E) = \mathrm{Im}(S_\Lambda) \subseteq l^p(\Lambda)$.

3. Application to Paley-Wiener Spaces

In order to apply the previous results we will use a result of convergence of series in Paley-Wiener spaces which allows to make sure the convergence of series under certain conditions. First we need the following auxiliary and well known two results:

Theorem 3.1 (Plancherel-Polya inequality, (see [20])). *Let* $S \subseteq \mathbb{R}^n$ *be bounded set and* $p \in (0, +\infty]$. *Let* $\Lambda \subseteq \mathbb{R}^n$ *be u.d. Then* Λ *verifies the p-P.P.C. for* $(E^p_S, \|\ \|_p)$, *this is, there exists a constant* $C = C(\Lambda, S, p) > 0$ *such that*

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq C\, \|f\|_p \ \text{for each } f \in E^p_S.$$

Besides the constant C *only depends on* p, S *and* $\delta(\Lambda)$.

Lemma 3.2. *Let* $r \in (0, +\infty]$ *and* $S \subseteq \mathbb{R}^n$. *Let* $g \in E^r_S$ *be even. Then:*

1. *The real and imaginary parts of* g *and their Fourier transform,* $Re(g)$, $\widehat{Re(g)}$, $Im\, g$, $\widehat{Img}$, *are even.*

2. $\widehat{Re(g)}(t) \in \mathbb{R}$ *and* $\widehat{Im\, g}(t) \in \mathbb{R}$ *for each* $t \in \mathbb{R}^n$.

3. $Re(g),\ Im\, g \in E_S^r$.

Now we need the following result of convergence of series in Paley-Wiener spaces (see [10]).

Lemma 3.3 (Lemma of Convergence). *Let* $p \in [1, +\infty)$, *and* $K \subseteq \mathbb{R}^n$ *be a bounded and Lebesgue measurable set such that its indicator function* χ_K *is a Fourier multiplier for* $\mathcal{F}L^q(\mathbb{R}^n)$. *Let* $\Lambda \subseteq \mathbb{R}^n$ *be a uniformly discrete set and let* $h \in E_K^1$ *be real valued. For every* $\lambda \in \Lambda$ *we define the function* $h_\lambda := \tau_\lambda h : \mathbb{R}^n \longrightarrow \mathbb{R}$ *by* $h_\lambda(x) := (\tau_\lambda h)(x) = h(x-\lambda)$ *for all* $x \in \mathbb{R}^n$. *Then there exists a constant* $D > 0$ *such that*

$$\left\| \sum_{\lambda \in \Lambda} c_\lambda \cdot h_\lambda \right\|_p \leq D \cdot \|c\|_p \ \text{ for each } c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda).$$

In particular, for each $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$ *we have that* $g_c := \sum_{\lambda \in \Lambda_1} c_\lambda \cdot h_\lambda \in L^p(\mathbb{R}^n)$, *and thus* $g_c \in E_K^p$. *In addition, the constant* D *only depends on p,* $\|h\|_1$, $\delta(\Lambda)$ *and on* K.

Remark 3.4. *Observe that Lemma 3.3 is also true for even* $h \in E_K^1$. *This is an immediate consequence of applying the lemma to the real and imaginary parts of* h *because of Lemma 3.2. In addition, note that* $h_\lambda \in E_K^1 \subseteq E_K^p$ *for all* $\lambda \in \Lambda$.

Proof. If $h = 0$, the result is obvious. Suppose that h is not the function identically zero.

Let $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$. We define $g_c := \sum_{\lambda \in \Lambda} c_\lambda \cdot h_\lambda \in \mathcal{S}'(\mathbb{R}^n)$. We will prove that there exists a constant $D > 0$ independent of c such that

$$|< g_c, \varphi >| \leq D \cdot \|c\|_p \cdot \|\varphi\|_q \ \text{ for each } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

$$|< g_c, \varphi >| = |< \widehat{g_c}, \hat{\varphi} >| = \left| < \sum_{\lambda \in \Lambda} c_\lambda \cdot e^{-it\lambda} \hat{h}(t), \hat{\varphi} > \right| =$$
$$= \left| < \sum_{\lambda \in \Lambda} c_\lambda \cdot e^{-it\lambda} \hat{h}(t) \cdot \chi_K(t), \hat{\varphi} > \right| = \left| \sum_{\lambda \in \Lambda} c_\lambda < e^{-it\lambda} \hat{h}(t) \cdot \chi_K(t), \hat{\varphi} > \right| \leq$$
$$\leq \sum_{\lambda \in \Lambda} |c_\lambda| \left| < e^{-it\lambda} \hat{h}(t) \cdot \chi_K(t), \hat{\varphi} > \right| = \sum_{\lambda \in \Lambda} |c_\lambda| \left| < \hat{\varphi}, e^{-it\lambda} \hat{h}(t) \cdot \chi_K(t) > \right|.$$

Observe that

$$\overline{\hat{h}(t)} = \overline{\int_{\mathbb{R}^n} h(x) \cdot e^{-itx} dx} = \int_{\mathbb{R}^n} \overline{h(x)} \cdot e^{itx} dx =$$
$$= \int_{\mathbb{R}^n} h(x) \cdot e^{itx} dx = \hat{h}(-t) \text{ for each } t \in \mathbb{R}^n.$$

We define $f_h := h \circ (-1_{\mathbb{R}^n}) \in L^1(\mathbb{R}^n)$, and we have that $\|f_h\|_1 = \|h\|_1$, and $\widehat{f_h}(t) = \hat{h}(-t)$ for all $t \in \mathbb{R}^n$. Let $\lambda \in \Lambda$.

$$< \hat{\varphi}, e^{-it\lambda} \hat{h}(t) \cdot \chi_K(t) >= \int_{\mathbb{R}^n} \hat{\varphi}(t)\, \overline{\hat{h}(t)} \cdot \chi_K(t) e^{it\lambda} dt =$$
$$= \int_{\mathbb{R}^n} \hat{\varphi}(t) \cdot \widehat{f_h}(t) \cdot \chi_K(t) e^{it\lambda} dt = \mathcal{F}^{-1}\left(\hat{\varphi} \cdot \widehat{f_h} \cdot \chi_K \right)(\lambda) =$$
$$= \mathcal{F}^{-1}\left(\widehat{(\varphi * f_h)} \cdot \chi_K \right)(\lambda).$$

Then:

$$|< g_c, \varphi >| \leq \sum_{\lambda \in \Lambda} |c_\lambda| \left| \mathcal{F}^{-1} \left((\widehat{\varphi * f_h}) \cdot \chi_K \right) (\lambda) \right| \leq \|c\|_p \cdot \|a\|_q ,$$

where $a_\lambda := \mathcal{F}^{-1} \left((\widehat{\varphi * f_h}) \cdot \chi_K \right) (\lambda)$ for each $\lambda \in \Lambda$, and $a := (a_\lambda)_{\lambda \in \Lambda}$.
Since χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$, then

$$\left\| \mathcal{F}^{-1} \left((\widehat{\varphi * f_h}) \cdot \chi_K \right) \right\|_q \leq C_q \|\varphi * f_h\|_q \leq C'_q \|\varphi\|_q \cdot \|f_h\|_1 =$$
$$= C'_q \|\varphi\|_q \cdot \|h\|_1 = C'_{q,h} \|\varphi\|_q ,$$

where $C'_{q,h} := C'_q \cdot \|h\|_1 > 0$. In addition, using the Plancherel-Polya inequality, Theorem 3.1, we have:

$$\|a\|_q = \left\| (a_\lambda)_{\lambda \in \Lambda} \right\|_q = \left\| \left(\mathcal{F}^{-1} \left((\widehat{\varphi * f_h}) \cdot \chi_K \right) (\lambda) \right)_{\lambda \in \Lambda} \right\|_q \leq$$
$$\leq C''_{q,K} \cdot \left\| \mathcal{F}^{-1} \left((\widehat{\varphi * f_h}) \cdot \chi_K \right) \right\|_q \leq C''_{q,K} \cdot C'_{q,h} \cdot \|\varphi\|_q = D \cdot \|\varphi\|_q .$$

So that $|< g_c, \varphi >| \leq D \cdot \|c\|_p \cdot \|\varphi\|_q$, and this is true for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence $\|g_c\|_p \leq D \cdot \|c\|_p$. □

Now we can prove the main results.

3.1. Proof of Theorem 1.16

Proof. Theorem 1.16 is an immediate consequence of both Theorem 2.9 and Lemma 3.3. □

3.2. Proof of Theorem 1.17

Proof. We obtain Theorem 1.17 immediately from Lemma 2.2, Corollary 2.10 and Lemma 3.3. □

In the following two sections we will state and prove two new results on extension of interpolation sets and contraction of sampling sets, respectively, in general quasinormed spaces. This is a continuation of a previous contribution (see [11]), obtaining complementary results.

4. Extension of IS

Theorem 4.1. *Let* $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, *and let* $(X, \| \|)$ *be a quasinormed space with* $X \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. *Let* $E, F \subseteq X$ *be vector subspaces. Let* $E_1, E_2 \subseteq E$ *be vector subspaces. Let* $p \in (0, +\infty]$, *and* $\Lambda_1, \Lambda_2 \subseteq \Omega$ *be uniformly discrete sets and disjoint. Define* $\Lambda := \Lambda_1 \cup \Lambda_2$, *and assume that*

i) Λ *is not a US for* F.

ii) *The space* F *contains the constant function equals to 1.*

iii) *There exists* $h \in F$ *such that*

$$h(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \Lambda_1 \\ 0, & \text{if } \lambda \in \Lambda_2, \end{cases} , \text{ for each } \lambda \in \Lambda.$$

(We say that h *is a function that distinghishes between* Λ_1 *and* Λ_2*.).*

iv) $E_j \cdot F \subseteq E$ *for each* $j \in \{1, 2\}$.

v) Λ_j *is a* p*-IS for* E_j *for every* $j \in \{1, 2\}$.

Then Λ *is a* p*-IS for* E.

Proof of Theorem 4.1. It is obvious that Λ is u.d. We will prove that Λ is a p-IS for E.
First observe that as Λ is not a US for F, then there exists $r \in F \setminus \{0\}$ such that

$$r(\lambda) = 0 \text{ for each } \lambda \in \Lambda.$$

Define $s := r + 1 \in F \setminus \{0\}$, which verifies that

$$s(\lambda) = 1 \text{ for all } \lambda \in \Lambda.$$

Also define $g := s - h \in F$, that verifies that

$$g(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \Lambda_1 \\ 1, & \text{if } \lambda \in \Lambda_2, \end{cases}, \text{ for every } \lambda \in \Lambda.$$

Let $a := (a_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$. Let us see that there exists $f_a \in E$ such that $f_a(\lambda) = a_\lambda$ for every $\lambda \in \Lambda$. Define

$$a^j := (a_\lambda)_{\lambda \in \Lambda_j} \in l^p(\Lambda_j) \text{ for every } j \in \{1,\, 2\}.$$

Let $j \in \{1,\, 2\}$. By the assumption v) we have that there exists $f_{a^j} \in E_j$ such that

$$f_{a^j}(\lambda) = a_\lambda \text{ for each } \lambda \in \Lambda_j.$$

Consider now that the function

$$f_a := f_{a^1} \cdot h + f_{a^2} \cdot g.$$

By the assumption iv) and since E is closed by sums, then $f_a \in E$.
Let $\lambda \in \Lambda$. Then we obtain that

$$f_a(\lambda) = f_{a^1}(\lambda) \cdot h(\lambda) + f_{a^2}(\lambda) \cdot g(\lambda) = a_\lambda \in \mathbb{C}.$$

That is, f_a interpolates $a = (a_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$.
Conclusion: Λ is a p-IS for E. □

Remark 4.2. *Under the assumptions of Theorem 4.1 observe that:*

1. *Suppose that*

   A) Neither Λ_1 nor Λ_2 are US for F.

   B) $f \cdot g \in F$ for all $f,\, g \in F$.

   C) For every $f,\, g \in F$ we have that

   $$f \cdot g = 0 \in F \Rightarrow (f = 0 \vee g = 0).$$

   Then $\Lambda = \Lambda_1 \cup \Lambda_2$ is not a US for F. (Notice that the second and third conditions on F exactly mean that F is an integral domain with the usual product of functions.).

2. *The quasinorm is not used at all, whereby we need no quasinorm in Theorem 4.1.*

5. Contraction of SS

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(E,\, \|\,\|)$ be a quasinormed space with $E \subseteq \mathfrak{F}(\Omega,\, \mathbb{C})$. Let $\Lambda_1,\, \Lambda_1 \subseteq \Omega$ be u.d. and disjoint sets. Define $\Lambda := \Lambda_1 \cup \Lambda_2 \subseteq \Omega$. Let $p \in (0,\, +\infty]$. Suppose that*

1. *Λ is a p-SS for E, and let $C > 0$ be such that*

   $$\|f\| \leq C \, \|(f(\lambda))_{\lambda \in \Lambda}\|_p \textit{ for every } f \in E.$$

2. *$C \cdot \|S_{\Lambda_2}\| < 1$, where $\|S_{\Lambda_2}\|$ is the norm of the sampling operator S_{Λ_2}.*

Then Λ_1 *is a p-SS for* E.

Proof of Theorem 5.1. Let c, $C > 0$, $c \le C$, be constants such that

$$c\,\|(f(\lambda))_{\lambda\in\Lambda}\|_p \le \|f\| \le C\,\|(f(\lambda))_{\lambda\in\Lambda}\|_p$$

for every $f \in E$.

Let $f \in E$. Obviously,

$$\|(f(\lambda))_{\lambda\in\Lambda_1}\|_p \le \|(f(\lambda))_{\lambda\in\Lambda}\|_p \le \frac{1}{c}\,\|f\|.$$

The same is true for Λ_2 (that is, Λ_2 verifies the p-P.P.C. for E, whereby its sampling operator S_{Λ_2} is continuous).

Now suppose that $p \in (0, +\infty)$. Then we have:

$$\begin{aligned}
\|f\|^p \le C^p\,\|(f(\lambda))_{\lambda\in\Lambda}\|_p^p = C^p \sum_{\lambda\in\Lambda} |f(\lambda)|^p = \\
= C^p \left(\sum_{\lambda\in\Lambda_1} |f(\lambda)|^p + \sum_{\lambda\in\Lambda_2} |f(\lambda)|^p\right) = \\
= C^p \sum_{\lambda\in\Lambda_1} |f(\lambda)|^p + C^p \sum_{\lambda\in\Lambda_2} |f(\lambda)|^p \le \\
\le C^p \sum_{\lambda\in\Lambda_1} |f(\lambda)|^p + C^p\,\|S_{\Lambda_2}\|^p\,\|f\|^p.
\end{aligned}$$

Hence

$$\|f\|^p - C^p\,\|S_{\Lambda_2}\|^p\,\|f\|^p \le C^p \sum_{\lambda\in\Lambda_1} |f(\lambda)|^p.$$

So that

$$\left(1 - (C\cdot\|S_{\Lambda_2}\|)^p\right)\,\|f\|^p \le C^p \sum_{\lambda\in\Lambda_1} |f(\lambda)|^p.$$

Our third assumption consisting that $C\cdot\|S_{\Lambda_2}\| < 1$ give us

$$\|f\|^p \le \frac{C^p}{1 - C^p\cdot\|S_{\Lambda_2}\|^p} \sum_{\lambda\in\Lambda_1} |f(\lambda)|^p,$$

and thus we obtain that

$$\|f\| \le \frac{C}{\left(1 - C^p\cdot\|S_{\Lambda_2}\|^p\right)^{\frac{1}{p}}} \left(\sum_{\lambda\in\Lambda_1} |f(\lambda)|^p\right)^{\frac{1}{p}}.$$

For the case $p = +\infty$ the proof is completely analogous with the standard modifications, and in this case we obtain:

$$\|f\| \le \frac{C}{1 - C\cdot\|S_{\Lambda_2}\|}\,\|(f(\lambda))_{\lambda\in\Lambda_1}\|_\infty.$$

□

References

1. Avdonin, S. A., *On the question of Riesz bases of exponential functions in* L^2, Vestnik Leningrad Univ. **13** (1974), 5–12 (Russian). English translation in Vestnik Leningrad Univ. Math. **7** (1979), 203–211.
2. Beurling, A., *Balayage of Fourier-Stieltjes Transforms. In: The collected Works of Arne Beurling, vol. 2, Harmonic Analysis.*, Birkhäuser, Boston, 1989.
3. Beurling, A., *Interpolation for an interval in* $\mathbb{R}$. In *The collected Works of Arne Beurling 2, Harmonic Analysis*, Birkhäuser, Boston (1989), 351–365.
4. Beurling, A., Malliavin, P. *On the closure of characters and the zeros of entire functions.*, Acta Math., **118** (1967), 79–93.

5. Flornes, Kristin M. *Sampling and Interpolation in the Paley-Wiener Spaces* L^p_π, $0 < p \leq 1$. Publicacions Matemàtiques, Vol **42**, (1998), 103–118.
6. Kadec, M. I., *The exact value of the Paley-Wiener constant*, Sov. Math. Dokl. **5** (1964), 559–561.
7. Landau, H. J., *Necessary density conditions for sampling and interpolation of certain entire functions.*, Acta Math. **117**, (1967), 37–52.
8. Levinson, N., *Gap and Density Theorems*, Colloquium Publications **26**, First Edition, American Mathematical Soc., 1940.
9. López Nicolás, J. A., *Irregular Stable Sampling and Interpolation in Functional Normed Spaces*, Bol. Soc. Paran. Mat. **40** (2019), 1–13, doi 10.5269/bspm.45497.
10. López Nicolás, J. A., *On Stable Sampling and Interpolation in Bernstein Spaces*, Revista Colombiana de Matemáticas **56** (2) (2022), 1–13, doi 10.15446/recolma.v56n2.108383.
11. López Nicolás, J. A., *Recombination of Stable Sampling Sets and Stable Interpolation Sets in Functional Quasinormed Spaces*, Bol. Soc. Paran. Mat. **42** (2024), 1–14, doi 10.5269/bspm.62925.
12. Mitkovski, M., Wick, B. D., *On the Uniqueness Sets in the Fock Space* arXiv-org. preprint, arXiv:1306.0318 [math.CV], 2013, doi 10.48550/ARXIV.1306.0318.
13. Neumann, J. von, *The Mathematical Foundations of Quantum Mechanics* Princeton University Press, NJ, 1955.
14. Olevskii, A., Ulanovskii, A., *Universal Sampling and Interpolation of Band-limited Signals* Geometric and Functional Analysis **18** (3) (2008), 1029–1052, doi 10.1007/s00039-008-0674-7.
15. Olevskii, A., Ulanovskii, A., *Uniqueness sets for unbounded spectra* Comptes Rendus Mathematique **349** (11) (2011), 679–681, doi 10.1016/j.crma.2011.05.010.
16. Olevskii, A., Ulanovskii, A., *Discrete Uniqueness Sets for Functions with Spectral Gaps* Russian Academy of Sciences Sbornik Mathematics **208** (6) (2016), 43–53, doi 10.1070/SM8837.
17. Olevskii, A., Ulanovskii, A., *Functions with Disconnected Spectrum*, University Lecture Series, First Edition, American Mathematical Society, 2016.
18. Olevskii, A., Ulanovskii, A., *On the duality between sampling and interpolation* Analysis Mathematica **42** (2016), 43–53, doi 10.1007/s10476-016-0104-2.
19. Paley, R., Wiener, N. *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications **19** (1934).
20. Plancherel, M., Pólya, G., *Fonctions entiéres et intégrales de Fourier multiples (seconde partie)* Comment. Math. Helv. **10** (1) (1937), 110–163, doi 10.1007/bf01214286.
21. Sedleckii, A. M., *Nonharmonic Fourier Series*, Sib. Math. J. **12** (1971a), 793–802.
22. Sedletskii, A. M., *Nonharmonic Analysis*, J. Math. Sc. **116** (5) (2009), 3551–3619.
23. Ullrich, D., *Divided differences and systems of nonharmonic Fourier series*, Proc. Amer. Math. Soc. **80** (1980), 47–57.
24. Velluci, P., *A simple pointview for Kadec-1/4 theorem in the complex case*, Ricerche di Matematica **64** (2015), 87–92.
25. Velluci, P., *Explicit Bounds of Complex Exponential Frames on Complex Field*, Acta Mathematica Hungarica **159** (2) (2019), 503–510.
26. Young, R. M., *An introduction to Nonharmonic Fourier Series*, Academic Press, Second Edition, 2001.

Consejería de Educación, Cultura y Universidades de la CARM, Avenida de la Fama, nº 15,

C.P. 30003. Murcia, Spain.

E-mail address: jaln4@um.es, josealfonso.lopez@murciaeduca.es