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# Numerical Studies of Some Partial Orderings on a Set With Height at Most One

Mohammed Aljohani, Sami Lazaar, Abdelwaheb Mhemdi, Neama Zidani, and Ahmed Aboubakr*

ABSTRACT: Recently, some researchers have introduced several novel classes of partially ordered sets (posets) with a height of at most 1. They studied some of their properties and relationships to Alexandrov topologies. This paper extends their work by enumerating some of these poset classes and illustrating their connections to various established discrete structures through digraph representations.

Key Words: Partial Ordered sets, Alexandroff topology, Krul dimension, submaximal spaces, Whyburn spaces, combinatorial functions.

## Contents



## 1. Introduction

A binary relation defined on a set $X$ is characterized as a quasi-order if it satisfies the properties of reflexivity and transitivity. Furthermore, when the quasi-order exhibits the additional characteristic of anti-symmetry, it is designated as a partial order or simply an order. Formally, an ordered set (poset) denoted as $(X, \leq)$ consists of a non-empty set $X$ equipped with an order relation $\leq$. Consider a set $X$ with $n$-set. The total number of binary relations on $X$ is equal to $2^{n^2}$, which corresponds to the number of subsets of the Cartesian product $X \times X$. The cardinality of reflexive relations is equal to $2^{n^2-n}$, while the cardinality of symmetric relations is equal to $2^{\frac{n(n+1)}{2}}$. A poset $(X, \leq)$ is said to have a chain $C$ if $C$ is a subset of the $n$-set $X$ and for any $x, y \in C$, either $x \leq y$ or $x \geq y$. The total number of ordered sets is equal to the factorial of $n$, denoted as $n!$.

The number of total quasi-order relations is known and can be calculated by the formula $\sum_{k=0}^{k=n} k!S(n,k)$, where $S(n,k)$ is the Stirling numbers of the second kind. Also, the number of equivalence (transitive, symmetric, and transitive) relations is $\sum_{k=0}^{k=n} S(n,k)$. We start by stating some notations and terminologies that we use throughout this paper. They are standard and can be found in [29].

Consider a partially ordered set $(X, \leq)$. The closed filter or closed up-set of an element $x$ in the set $X$ is defined as the set $[x\uparrow)$, which consists of all elements $y$ in $X$ such that $x$ is less than or equal to $y$. Similarly, the closed ideal or closed down-set of $x$ is defined as the set $(\downarrow x]$, which consists of all elements $y$ in $X$ such that $y$ is less than or equal to $x$. The closed neighborhood of $x$ is defined as the set $[\updownarrow x]$, which is equal to the union of the half-open interval $[x\uparrow)$ and the half-open interval $(\downarrow x]$. The open filter of $x$ is defined as the set $(x\uparrow)$, which is obtained by removing the element $x$ from the set $[x\uparrow)$. Similarly, the open ideal of $x$ is defined as the set $(\downarrow x)$, which is obtained by removing the element $x$ from the set $(\downarrow x]$. The open neighborhood of $x$ is defined as the set $(\updownarrow x)$, which is obtained by taking the union of the open filter $(x\uparrow)$ and the open ideal $(\downarrow x)$. In a more general sense, if we have a subset $A$ of $X$, we may represent the closed ideal of $A$ as $(\downarrow A] = \cup\{(\downarrow x] : x \in A\}$ and the closed filter of $A$ as

* Corresponding author

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$[A\uparrow) = \cup\{[x\uparrow) : x \in A\}$.
A point $x \in X$ is considered an isolated point if the set $[x \updownarrow]$ contains only the element $x$ itself. The family of all isolated points in the ordered set $(X, \leq)$ shall be denoted as $Iso(X)$. An element $x$ is referred to as a minimum point if any element $y$ that is less than or equal to $x$ must be equal to $x$. Similarly, $x$ is called a maximal point if any element $y$ that is greater than or equal to $x$ must be equal to $x$.
The length of set $C$ is defined as $l(C)$ and is equal to the cardinality of $C$ minus one. The height of a finite partially ordered set $(X, \leq)$ is defined as the maximum length of any chain in $X$. The height of an ordered set is synonymous with its length, or Krull dimension, represented as $h(X)$ or $dim_K(X)$ [7]. The authors in [4] present novel categories of partial orderings on a set with a maximum height of one in the following manner:

**Definition 1.1** A partially ordered set $(X, \leq)$ is referred to as a $T_{DD}$**-poset** if and only if, for any two different points $x$ and $y$ in $X$, the intersection of their lower sets, denoted as $(\downarrow x) \cap (\downarrow y)$, is empty.

**Definition 1.2** A poset $(X, \leq)$ is considered a **submaximal poset** if, for every $x \in X$ and every finite set $F \in X$ such that $x \notin F$, the intersection of the upward closure of $x$ with $F$ is empty or the intersection of $\{x\}$ with the upward closure of $F$ is empty.

**Definition 1.3** A poset $(X, \leq)$ is referred to as a **door-poset** if and only if, for any two disjoint finite subsets $F_1$ and $F_2$ in $X$, the intersection of the upward closure of $F_1$ and $F_2$ is empty, or the intersection of $F_1$ and the upward closure of $F_2$ is empty.

**Definition 1.4** A partially ordered set $(X, \leq)$ is referred to as a $T_Y$**-poset** if and only if, for any two unique points $x$ and $y$ in $X$, the cardinality of the intersection between the lower closure of $x$ and the lower closure of $y$ is less than or equal to 1.

**Definition 1.5** A poset $(X, \leq)$ is defined as a **Whyburn-poset** if and only if, for every $x \in X$, the cardinality of the set of elements below $x$ (denoted as $(\downarrow x]$) is less than or equal to 2.

Let $X$ be a set, and let $f : X \longrightarrow X$ be a mapping. The quasi-ordered set $(X, \leq_f)$, hereafter referred to as the primal quasi-ordered set, is defined as follows, $\forall x, y \in X$

$$y \leq_f x \Longleftrightarrow y = f^n(x),$$

for some integer $n \geq 1$. In particular, if $f$ is idempotent ($f^2 = f$), the corresponding quasi-order is an order with a height of at most one.

An element $x \in X$ is considered periodic if $f(x) \neq x$ and there exists a positive integer $n > 1$ such that $f^n(x) = x$. On the other hand, if $f(x) = x$, then $x$ is referred to as a fixed point. The result stated here is a direct outcome in [11].

**Lemma 1.1 (Proposition 2.5 in [11])** *Let $(X, \leq_f)$ be a primal quasi-ordered set.*

*(i) $(X, \leq_f)$ is an equality poset if and only if $f$ is the identity map.*

*(ii) $(X, \leq_f)$ is a poset if and only if $f$ is without periodic point.*

Given that all binary relations provided in our study are orders, it follows that all functions $f$ under consideration are idempotent, meaning they do not have periodic points. This work employs the Hasse diagram as a visual representation to demonstrate the partially ordered sets (posets). The following scenario demonstrates that $a$ is less than $b$:

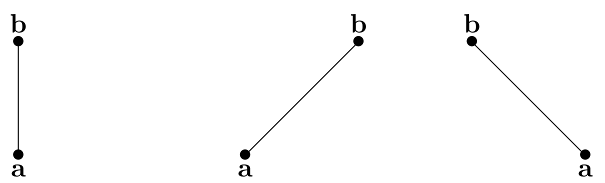



Figure 1

## 2. Numerical and counting formulas in the class of finite posets

There is a widely recognized fact that there is no established formula that provides the number of topologies on a finite set (or equivalently, the number of quasi-partial ordering on a finite set). In this section, we give formulas of submaximal-posets, $T_{DD}$-posets, door-posets, and Whyburn posets on an $n$-sets. Counting $T_Y$-posets is still open.

Remark that by [4] all considered posets are submaximal and thus by combining Theorem 3.1 and Theorem 3.2 in [4], we get immediately that a poset is submaximal if and only if $dim_K(X) \leq 1$. Hence, every point in $(X, \leq)$ is either maximum or minimum. Indeed, suppose the existence of $x$ which is neither maximum nor minimum, which means that $\exists y, z \in X$ satisfying $y < x < z$ which gives $dim_K(X) \geq 2$, a contradiction. As an immediate consequence, all considered posets are composed of two levels, the highest one contains the maximal points not isolated points, and is denoted by level 1 and the lower one contains the minimal points and is denoted by level 0. If the level 1 contains $k$ elements, it will be denoted by $l_{1,k}$ and consequently, the level 0, will contain $n - k$ elements, where $n$ represents the cardinality of the set $X$, and we define it as $l_{0,n-k}$. Since level 1 contains maximal points not isolated points, then $1 \leq k \leq n - 1$, where $k$ is the cardinality of $l_{1,k}$.

We can illustrate this situation in the following diagram:

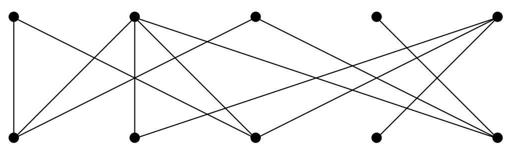



Figure 2: Diagram 1

Note that if the $\leq$ is the trivial order on $X$ ($x \leq y \iff x = y$), all points are both maximal and minimal points but in this case, all points are also isolated points. Hence, we will convenient to consider it as a unique level $l_{0,n}$.

Now we are in a position to give our results.

### 2.1. Finite submaximal-posets and finite submaximal- primal posets

First, consider the following proposition.

**Proposition 2.1** *[4, Proposition 2.3] Consider a set $(X, \leq)$. Then, the statements that follow are equivalent.*

*(i)* *$(X, \leq)$ is a submaximal poset;*

*(ii)* $\dim_K(X, \leq) \leq 1$.

Thus, based on the previous proposition, in a finite set of cardinality $n$.

In this subsection, we use the result given by Lazaar et.al in [20] to enumerate posets with height at most 1 in the general case of posets and in the particular case of primal posets. Then, we give a direct argument to show these results in a simple way.

**Theorem 2.1** *Let $S(n)$ denote the number of submaximal-posets on a finite set of cardinality $n$. Then:*
*If $S(n)$ denotes the number of submaximal-posets on a finite set of cardinality $n$, then*

$$S(n) = \sum_{k=1}^{n} \binom{n}{k} \left(2^k - 1\right)^{n-k}.$$

**Proof:** Let $(X, \leq)$ be a poset. Hence, by the previous proposition:$(X, \leq)$ is a submaximal-poset $\Longleftrightarrow$ $(X, \leq)$ is as in diagram 1. The level 1 has $n$ types $l_{1,k}$ (from $k = 1$ points to $k = n$ points), with $Card(l_{1,k}) = k$.

If $k = n$, there is one possibility ( the trivial order).

If $1 \leq k \leq n-1$, there is $\binom{n}{k}$ choice of $k$ elements from $X$. So that $\binom{n}{k}$ choice of the level $l_{1,k}$.

Now, given a fixed level $l_{1,k}$, the corresponding level $l_{0,n-k}$ contains exactly $n-k$ elements. Thus any point of $l_{1,k}$ has exactly $2^{|l_{0,n-k}|} - 1$ ( $-1$ to say that the empty subset is not considered), so $2^{n-k} - 1$ possibilities. Therefore:

$$S(n) = \sum_{k=1}^{n-1} \binom{n}{k} \left(2^{n-k} - 1\right)^{k} + 1 = \sum_{k=1}^{n} \binom{n}{k} \left(2^{k} - 1\right)^{n-k}.$$

□

**Theorem 2.2** *If $S_f(n)$ be the number of submaximal-primal posets, and $f$ is a map from a finite set of cardinality $n$ to itself, then*

$$S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

**Proof:** Assume that $(X, \leq)$ be a submaximal primal poset. Then for each point from a fixed level $l_{1,k}$, $1 \leq k \leq n-1$, there are exactly $n-k$ choices; that's the cardinality of the family of singletons of $l_{0,n-k}$ (if $k = n$, there is one possibility, the trivial order). Such a situation can be illustrated by the diagram 2.

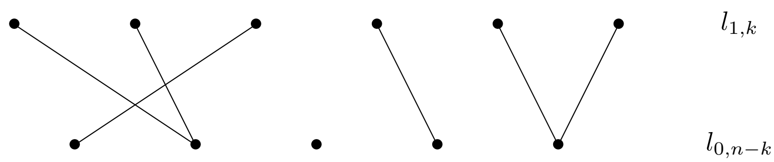



Figure 3: Diagram 2

Hence, we get:

$$\begin{aligned} S_f(n) &= \sum_{k=1}^{n-1} \binom{n}{k} (n-k)^k + 1 \\ &= \sum_{j=1}^{n-1} \binom{n}{n-j} j^{n-j} + 1 \\ &= \sum_{j=1}^{n-1} \binom{n}{j} j^{n-j} + 1 \\ &= \sum_{j=1}^{n} \binom{n}{j} j^{n-j} \end{aligned}$$

□

**Remark 2.1** Consider the $C^{\infty}$ function defined on $\mathbb{R}$ by $x \longmapsto e^{xe^x}$. Using the Taylor series of $f$, It is observable that $S_f(n) = (e^{xe^x})^{(n)}(0)$.

Indeed,

$$\begin{aligned} e^{xe^x} &= \sum_{k=0}^{\infty} \frac{(xe^x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(x)^k}{k!}.e^{xk} \\ &= \sum_{k=0}^{\infty} \frac{(x)^k}{k!}.\sum_{k=0}^{\infty} \frac{(xk)^j}{j!} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^{k+j}}{j!k!}.k^j \end{aligned}$$

Now, set $n = k + j$, we get

$$\begin{aligned} e^{xe^x} &= \sum_{n=0}^{\infty}(\sum_{k=1}^{n} \frac{k^{n-k}}{k!(n-k)!}).x^n \\ &= \sum_{n=0}^{\infty}(\sum_{k=1}^{n} \frac{n!k^{n-k}}{k!(n-k)!}).\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}(\sum_{k=1}^{n} \binom{n}{k} k^{n-k})\frac{x^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} S_f(n)\frac{x^n}{n!} \end{aligned}$$

Finally, We come to the conclusion that:

$$S_f(n) = (e^{xe^x})^{(n)}(0), \forall n \in \mathbb{N}$$

.

**Remark 2.2** $S_f(n)$ gives the sequence *A000248* in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [31] as the number of forests with $n$ vertices and height at most 1.

### 2.2. Finite door-posets

**Theorem 2.3** *let $D(n)$ denote the number of door-posets on a finite set of order $n$. Then:*

$$D(n) = n2^n - (n^2 + n - 1).$$

**Proof:** By [4, Proposition 3.4], $(X, \leq)$ is a door-poset if and only if $(X, \tau(\leq))$ is a door space. Now, using [20, Thorem 3.2], we get

$$D(n) = 1 + 2\binom{n}{2} + 2\sum_{k=3}^{n} k\binom{n}{k} = n2^n - (n^2 + n - 1).$$

□

**Theorem 2.4** *Let $D_f(n)$ denote the number of door-posets in the particular case when $\leq$ is $\leq_f$, for a given map $f$ from a finite set of cardinality $n$ to itself. Then*

$$D_f(n) = 1 + n(2^{n-1} - 1).$$

**Proof:** It is clear that $(X, \leq_f)$ is a door poset $\Longleftrightarrow$ $(X, \tau(\leq_f))$ is a primal door space. Now, using [20, Thorem 3.2], we get

$$D_f(n) = 1 + n(2^{n-1} - 1).$$

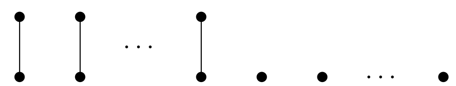
□

### 2.3. Finite $T_{DD}$-posets

**Theorem 2.5** *Let $(X, \leq)$ be a primal poset, for a given map $f$ from a finite set of cardinality $n$ to itself. Let $T_{DD,f}(n)$ denote the number of $T_{DD}$-posets $X$. Then:*

$$T_{DD,f}(n) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(n-2k)!k!}.$$

**Proof:** By [4, Proposition 3.4], all components of a $T_{DD}$-poset are singleton points or a pair of elements $(b < a)$. Such a situation can be illustrated as follows:



Let $k$ be the number of pairs and $P(k)$ the number of orders with exactly $k$ pairs. Then $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $P(k) = \frac{n!}{(n-2k)!k!}$.

Indeed, firstly we choose $n - 2k$ fixed points from $n$ points, so that $\binom{n}{n-2k}$ possibilities. For the $2k$ other points, we divide it into two identical levels $l_0$ and $l_1$ with exactly $k$ elements and thus there is $\binom{2k}{k}$. Now the first point from $l_{1,k}$ has $k$, choice, the second $k-1$ choice,...., the latest one has one choice. Consequently, $P(k) = \binom{n}{n-2k}.\binom{2k}{k}.k.(k-1)....2.1$.

Finally,

$$T_{DD,f}(n) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(n-2k)!k!}.$$

□

**Theorem 2.6** *If $T_{DD}(n)$ be the number of $T_{DD}$-posets on a finite set of order $n$, then*

$$T_{DD}(n) = S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

**Proof:** Let $(X, \leq)$ be a poset. By [4, Proposition 3.4], $(X, \leq)$ is a $T_{DD}$-poset $\Longleftrightarrow$ $|[x \uparrow)| \leq 2, \forall x \in X$. Now, since the correspondence $(X, \leq) \longmapsto (X, \geq)$ is a bijection, it remains to say that $T_{DD}(n)$ is the cardinality of posets satisfying $|(\downarrow x]| \leq 2, \forall x \in X$. That is the number of submaximal-primal posets, for a given map $f$ from $X$ to itself. we get,

$$T_{DD}(n) = S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

. □

### 2.4. Finite Whyburn-posets

**Theorem 2.7** *If $W(n)$ be the number of Whyburn posets on a finite set of order $n$, then*

$$W(n) = S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

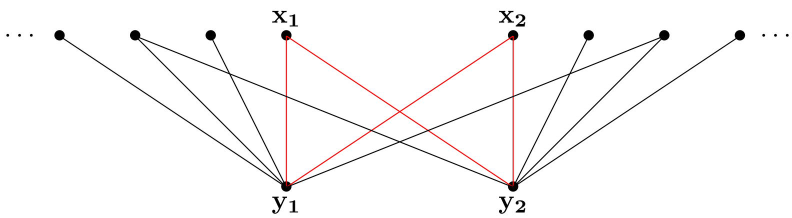
**Proof:** Immediate consequence of Definition 1.5. □

**Remark 2.3** Let $(X, \leq)$ be a primal poset, for a given map $f$ from a finite set of cardinality $n$ to itself. By [20, Corollary 4.8] $(X, \leq)$ is $T_Y$-poset $\Longleftrightarrow$ $(X, \leq)$ is submaximal poset and consequently if we denote by $T_{Y,f}(n)$ the number of $T_Y$-posets in the particular case when $(X, \leq)$ is a submaximal-primal poset. Then:

$$T_{Y,f}(n) = S_f(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

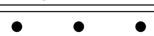
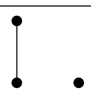
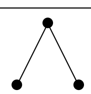
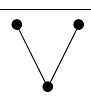
To close this paper we give some comments on the $T_Y$-posets for finite sets.

Let $NT_Y$ denote a submaximal poset $(X, \leq)$ which is not $T_Y$-poset. This is equivalent to saying, by [4, Proposition 2.4], that $(X, \leq)$ is a submaximal poset with the existence of two distinct points $x_1$ and $x_2$ such that $|(\downarrow x_1) \cap (\downarrow x_2)| \geq 2$. Such a situation can be illustrated as follows:

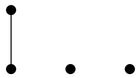
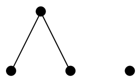
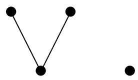
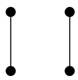
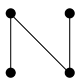
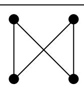
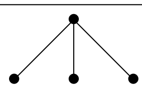
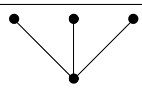



**Remark 2.4** It is clear that for $1 \leq n \leq 3$, $NT_Y(n) = 0$ and thus $T_Y(n) = S(n)$.

**Example 2.1** (1) For $n = 3$, if $X = \{a, b, c\}$. Then the non-isomorphic submaximal posets are:

| Diagram form | number of submaximal spaces | $S_f$ | $D$ | $D_f$ | $T_{DD}$ | $T_{DD,f}$ | $T_Y$ | $T_{Y,f}$ | $W$ |
|---|---|---|---|---|---|---|---|---|---|
|  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|  | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
|  | 3 | 0 | 3 | 3 | 0 | 0 | 3 | 0 | 0 |
|  | 3 | 3 | 3 | 0 | 0 | 3 | 3 | 3 | 3 |
| Total | 13 | 10 | 13 | 10 | 7 | 10 | 13 | 10 | 10 |

(1) For $n = 4$, if $X = \{a, b, c, d\}$. Then the non-isomorphic submaximal posets are:

| Diagram form | number of submaximal spaces | $S_f$ | $D$ | $D_f$ | $T_{DD}$ | $T_{DD,f}$ | $T_Y$ | $T_{Y,f}$ | $W$ |
|---|---|---|---|---|---|---|---|---|---|
|  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|  | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |
|  | 12 | 0 | 12 | 0 | 12 | 0 | 12 | 0 | 0 |
|  | 12 | 12 | 12 | 12 | 0 | 0 | 12 | 12 | 12 |
|  | 12 | 12 | 0 | 0 | 12 | 12 | 12 | 12 | 12 |
|  | 24 | 0 | 0 | 0 | 0 | 0 | 24 | 0 | 0 |
|  | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|  | 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 0 |
|  | 4 | 4 | 4 | 4 | 0 | 0 | 4 | 4 | 4 |
| Total | 87 | 41 | 45 | 29 | 41 | 25 | 81 | 41 | 41 |

(3) For $n = 5$, if $X = \{a, b, c, d, e\}$. Then, the different non isomorphic $NT_Y$-poset are represented by the following diagrams:

The number of different possible diagrams of each form in the previous table is explained below:

1. **Form 1:** To obtain a diagram of this form, we have to choose 3 elements of $X$ for the level $l_{0,3}$ ($\binom{5}{3}$ possibilities) and for each possibility we have to choose 2 elements from $l_{0,3}$ to be linked to the 2 elements of $l_{1,2}$ ($\binom{3}{2}$ possibilities). Totally, we have $\binom{5}{3}\binom{3}{2} = 30$ different diagrams of the form 1.
2. **Form 2:** To obtain a diagram of this form, we have to choose 3 elements of $X$ for the level $l_{1,3}$ ($\binom{5}{3}$ possibilities) and for each possibility we have to choose 2 elements from $l_{1,3}$ to be linked to the 2 elements of $l_{0,2}$ ($\binom{3}{2}$ possibilities) and for each case of these last possibilities we have 2 different possibilities to link the third point of level $l_{1,3}$. Totally, we have $\binom{5}{3}\binom{3}{2}2 = 60$ different diagrams of the form 2.
3. **Form 3:** To obtain a diagram of this form, we have to choose 3 elements of $X$ for the level $l_{1,3}$. Totally, we have $\binom{5}{3} = 10$ different diagrams of the form 1.
4. **Form 4:** The dual of form 2.
5. **Form 5:** The dual of form 3.

Therefore:

$$T_Y(5) = S(5) - NT_Y(5) = 841 - 170 = 671.$$

This naturally leads to the following question.

**Question 2.1** *Determine* $T_Y(n)$, $\forall n \in \mathbf{N}$.

The following table gives explicitly the number of our separation axioms on a set $X$ with cardinality $n \leq 15$.

| $n$ | $S(n)$ | $D(n)$ | $D_f(n)$ | $T_{DD,f}(n)$ | $S_f(n)$ | $T_Y(n)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 13 | 13 | 10 | 7 | 10 | 13 |
| 4 | 87 | 45 | 29 | 25 | 41 | 81 |
| 5 | 841 | 131 | 76 | 81 | 196 | 671 |
| 6 | 11643 | 343 | 187 | 331 | 1057 | ? |
| 7 | 227893 | 841 | 442 | 1303 | 6322 | ? |
| 8 | 6285807 | 1977 | 1017 | 5937 | 41393 | ? |
| 9 | 243593041 | 4519 | 2296 | 26785 | 293608 | ? |
| 10 | 13262556723 | 10131 | 5111 | 133651 | 2237921 | ? |
| 11 | 1014466283293 | 22397 | 11254 | 669351 | 18210094 | ? |
| 12 | 109128015915207 | 48997 | 24565 | 3609673 | 157329097 | ? |
| 13 | 16521353903210521 | 106315 | 53236 | 19674097 | 1436630092 | ? |
| 14 | 3524056001906654763 | 229167 | 114675 | 113525595 | 13810863809 | ? |
| 15 | 1059868947134489801413 | 491281 | 245746 | 664400311 | 139305550066 | ? |

**Remark 2.5** Some of the sequences from this table appear in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [31] as follow:

- $S(n)$ gives the sequence $A001831$ which is the number of Labeled digraphs where every vertex has either indegree 0 or outdegree 0.
- $D_f(n)$ gives the sequence $A048493$ which is the number of connected induced subgraphs in the $n$-sunlet graph.
- $T_{DD,f}(n)$ gives the sequence $A047974$ The count of partial permutation matrices denoted as $P$ in $\mathrm{GL}_n$ where $P^2 = 0$.
- $S_f(n)$ gives the sequence $A000248$ the number of forests consisting of $n$ vertices where the height does not exceed 1. Alternatively, the number of idempotent mappings $f$ from an $n$-set to itself.

The exact formula for counting $T_Y(n)$ appears to be challenging to obtain. However, can one show that $\frac{T_Y(n)}{S(n)} \to c$ as $n \to \infty$? Is it true that $c = 0$?

## 3. Conclusion

In this manuscript, we have list of novel partial orderings on a set with a maximum height of 1. These posets were introduced and characterized by Lazaar et .al in [20].

Given the widespread application of set theory in addressing real-world problems across several disciplines, we anticipate that the findings presented here will facilitate future study in both theoretical and practical domains.

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### Ethical Approval

Not applicable

### Funding

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### Availability of data and materials

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### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

### Conflict of interest

The authors declare that they have no competing interests.

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*Mohammed Aljohani,*
*Department of Mathematics,*
*College of Science, Taibah University, Madinah 41477,*
*Kingdom of Saudi Arabia.*
*E-mail address:* mhjohany@taibahu.edu.sa

*and*

*Sami Lazaar,*
*Department of Mathematics,*
*College of Science, Taibah University, Madinah 41477,*
*Kingdom of Saudi Arabia.*
*E-mail address:* slazar@taibahu.edu.sa

*and*

*Abdelwaheb Mhemdi,*
*Department of Mathematics,*
*Prince Sattam Bin Abdulaziz University, College of Sciences and Humanities in Aflaj,*
*Kingdom of Saudi Arabia.*
*E-mail address:* a.mhemdi@psau.edu.sa

*and*

*Neama Zidani,*
*Department of Mathematics,*
*Tunis El Manar University, Tunis,*
*Tunisia.*
*E-mail address:* zidani.neama@gmail.com

*and*

*Ahmed Aboubakr,*
*Department of Mathematics,*
*College of Science, Taibah University, Madinah 41477,*

*Kingdom of Saudi Arabia.*
*Faculty of Science, Fayoum University,*
*63514, Fayoum, Egypt.*
*E-mail address:* aaboubakr@taibahu.edu.sa , afs00@fayoum.edu.eg

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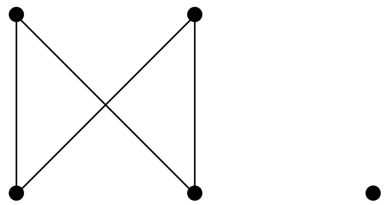
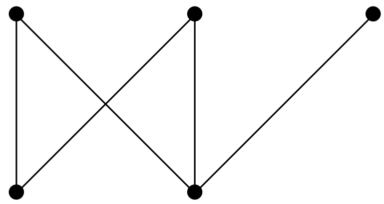
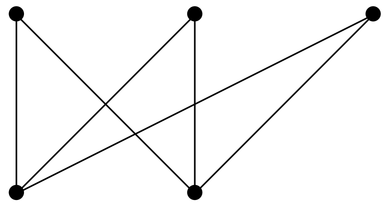
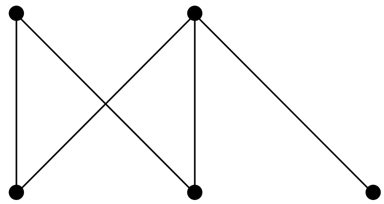
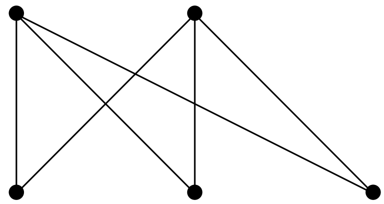
| 1 |  | 30 |
| 2 |  | 60 |
| 3 |  | 10 |
| 4 |  | 60 |
| 5 |  | 10 |
| | Total | 170 |