Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–5.
ISSN-0037-8712
doi:10.5269/bspm.75772

Fixed point results using α-admissibility

Amine Jaid[1], Youssef Touail[*,2] and Driss El Moutawakil[3]

ABSTRACT: In this article, using the notion of α-admissibility, we present a new class of condensing mappings. Via this class, we prove some extensions of Darbo's fixed point theorem. Moreover, we give an example to illustrate the usability of the achieved results.

Key Words: Fixed point, α-admissible mapping, condensing mapping, Darbo fixed point theorem.

Contents

1 Introduction **1**

2 Results **2**

3 Conclusion **4**

1. Introduction

In 1912, Brouwer [3] established a foundational result in fixed point theory by proving a fixed point theorem for finite-dimensional spaces. Notably, the theorem is applied to continuous mappings from a closed disk to itself. In 1922, Banach [1] introduced a novel concept known as the "Banach contraction principle" (for short BCP), to prove a fixed point result in the framework of complete metric spaces.
Subsequently, in 1930, Schauder [10] extended Brouwer's theorem to more general setting of Banach spaces, which can be infinite-dimensional. That same year, Kuratowski [6] introduced the concept of measure of noncompactness. Based on this notion, Darbo [4] published a fundamental fixed point theorem in 1955 by utilizing the measure of noncompactness (for short MNC). Precisely, Darbo generalized Schauder's fixed point theorem and modeled with MNC the BCP for the so-called class of condensing maps.
In the same line of research, in 2012, Samet et al. [8] extended the BCP by introducing the concept of α-admissible maps. Their work culminated in a significant fixed point theorem for α-ψ-contractive mappings. For further information on this topic, please consult [5,7,9,11,12,13].
In this paper, we employ the concept of α-admissibility and find a new extension of Darbo's theorem for a new kind of condensing operators. In other words, we extend the Darbo's hypothesis:

$$\varrho(TX) \leq \gamma\varrho(X), \gamma \in [0, 1)$$

to the newly assumption:

$$\alpha(x, Tx)[\varrho(TX) + \theta(\varrho(TX))] \leq \gamma[\varrho(X) + \theta(\varrho(X))], \gamma \in [0, 1),$$

where ϱ is a MNC which we will define further and $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. We point out that our results solve problems raised so far; precisely those related to selfmappings having the Darbo's constant $\gamma \geq 1$. To illustrate this, we will give a concrete example where Darbo fixed point theorem cannot be applied (Example 2.3).
The following definitions and results will be required in the sequel. For this, let E be a Banach space. Denote by $\mathcal{M}_E$ the collection of all bounded subsets of E and by $\mathcal{N}_E$ the collection of all relatively compact subsets of E. For a subset X of E, let $\overline{X}$ and $Co(X)$ represent the closure and closed convex hull of X, respectively.

2010 *Mathematics Subject Classification:* 47H08, 47H10.
Submitted February 17, 2025. Published August 10, 2025

Definition 1.1 ([2]) A map $\varrho : \mathcal{M}_E \to \mathbb{R}_+$ is said to be a measure of non-compactness defined on E if:
(i) $\ker \varrho = \{X \in \mathcal{M}_E : \varrho(X) = 0\} \neq \emptyset$ and $\ker \varrho \subset \mathcal{N}_E$;
(ii) $X \subset Y \Rightarrow \varrho(X) \leq \varrho(Y)$;
(iii) $\varrho(X) = \varrho(\overline{X}) = \varrho(Co(X))$;
(iv) $\varrho(\delta X + (1-\delta)Y) \leq \delta\varrho(X) + (1-\delta)\varrho(Y)$ for all $\delta \in [0,1]$ and $X, Y \in \mathcal{M}_E$;
(v) if $\{X_n\}$ is a decreasing sequence of nonempty closed and bounded subsets of E such that $\lim \varrho(X_n) = 0$, then $X_\infty = \cap_n X_n \neq \emptyset$.

Theorem 1.1 *(Brouwer [3]) All continuous map from a closed disk of a Euclidean space into itself has a fixed point.*

Theorem 1.2 *(Schauder [10]) Suppose that C is a closed convex subset of a Banach space E. Then, if $T : C \to C$ is a compact continuous map on C, then T has a fixed point in C.*

Theorem 1.3 *(Darbo [4]) Let C be a nonempty, bounded, closed and convex subset of a Banach space E. If $T : C \to C$ is a continuous map and there exists $\gamma \in [0,1)$ such that*

$$\varrho(TX) \leq \gamma\varrho(X),$$

where $X \subset C$ and ϱ is a measure of noncompactness. Then T has a fixed point in C.

Definition 1.2 ([8]) Let $T : E \to E$ be a mapping and $\alpha : E \times E \to \mathbb{R}_+$ be a function. T is said to be α-admissible map if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ for all $x, y \in E$.

2. Results

The main result of this paper is:

Theorem 2.1 *Suppose that C is a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \to C$ is a continuous α-admissible mapping. Assume that there exist a closed convex subset X_0 of C and $x_0 \in X_0$ such that $TX_0 \subset X_0$ and $\alpha(x_0, Tx_0) \geq 1$. If we have*

$$a \leq \varrho(X) + \theta(\varrho(X)) \leq b \Rightarrow \alpha(x,Tx)[\varrho(TX) + \theta(\varrho(TX))] \leq \gamma[\varrho(X) + \theta(\varrho(X))], \tag{2.1}$$

for all $0 < a < b < 1$, $X \subset C$ and $x \in X$, where $\gamma \in [0,1)$, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and ϱ is a measure of noncompactness. Then T has a fixed point in C.

Proof: Define sequences of sets $\{X_n\}$ and elements $\{x_n\}$ as follows:

$$X_{n+1} = Co(TX_n) \text{ and } x_{n+1} = Tx_n,$$

for all $n \in \mathbb{N}$. Since $TX_0 \subset X_0$, then

$$X_1 = Co(TX_0) \subset X_0,$$
$$X_2 = Co(TX_1) \subset Co(TX_0) = X_1.$$

Thus, by pursuing this process we obtain

$$X_{n+1} \subset X_n \text{ and } TX_n \subset X_n, \tag{2.2}$$

for all $n \in \mathbb{N}$.
In view of $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ and α-admissibility of T, we obtain

$$\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1,$$

for all $n \in \mathbb{N}$.
If $\varrho(X_N) + \theta(\varrho(X_N)) = 0$ for some $N \in \mathbb{N}$, then $\varrho(X_N) = 0$ and X_N is relatively compact. Moreover, from (2.2) we have $TX_N \subset X_N$, then Schauder's fixed point result implies that T has a fixed point.
Now, suppose that

$$\varrho(X_n) + \theta(\varrho(X_n)) > 0,$$

for all $n \in \mathbb{N}$.
If $\varrho(X_{n_0}) + \theta(\varrho(X_{n_0})) < \varrho(X_{n_0+1}) + \theta(\varrho(X_{n_0+1}))$ for some $n_0 \in \mathbb{N}$, then

$$\begin{aligned} 0 < a := \varrho(X_{n_0}) + \theta(\varrho(X_{n_0})) &\le \varrho(X_{n_0}) + \theta(\varrho(X_{n_0})) \\ &< \varrho(X_{n_0+1}) + \theta(\varrho(X_{n_0+1})) := b. \end{aligned}$$

Using (2.1), so there exists $\gamma \in (0, 1)$ such that

$$\begin{aligned} \varrho(X_{n_0+1}) + \theta(\varrho(X_{n_0+1})) &= \varrho(CoTX_{n_0}) + \theta(\varrho(CoTX_{n_0})) \\ &= \varrho(TX_{n_0}) + \theta(\varrho(TX_{n_0})) \\ &\le \alpha(x_{n_0}, x_{n_0+1})[\varrho(TX_{n_0}) + \theta(\varrho(TX_{n_0}))] \\ &= \alpha(x_{n_0}, Tx_{n_0})[\varrho(TX_{n_0}) + \theta(\varrho(TX_{n_0}))] \\ &\le \gamma[\varrho(X_{n_0}) + \theta(\varrho(X_{n_0}))] \\ &< \gamma[\varrho(X_{n_0+1}) + \theta(\varrho(X_{n_0+1}))], \end{aligned}$$

which leads to $\gamma > 1$, this is a contradiction. Thus

$$\varrho(X_{n+1}) + \theta(\varrho(X_{n+1})) \le \varrho(X_n) + \theta(\varrho(X_n)),$$

for all $n \in \mathbb{N}$.
Hence, the sequence $\{\varrho(X_n) + \theta(\varrho(X_n))\}$ is non-increasing and nonnegative.
Let

$$\lim_{n\to\infty} \varrho(X_n) + \theta(\varrho(X_n)) = c. \tag{2.3}$$

Assume that $c > 0$. Therefore

$$0 < a := c \le \varrho(X_n) + \theta(\varrho(X_n)) \le \varrho(X_0) + \theta(\varrho(X_0)) := b,$$

for all $n \in \mathbb{N}$.
By using (2.1), there exists $\gamma \in [0, 1)$ such that

$$\begin{aligned} \varrho(X_{n+1}) + \theta(\varrho(X_{n+1})) &= \varrho(CoTX_n) + \theta(\varrho(CoTX_n)) \\ &= \varrho(TX_n) + \theta(\varrho(TX_n)) \\ &\le \alpha(x_n, x_{n+1})[\varrho(TX_n) + \theta(\varrho(TX_n))] \\ &= \alpha(x_n, Tx_n)[\varrho(TX_n) + \theta(\varrho(TX_n))] \\ &\le \gamma[\varrho(X_n) + \theta(\varrho(X_n))], \end{aligned}$$

for all $n \in \mathbb{N}$.
Using (2.3) and letting $n \to \infty$, we obtain $c \le \gamma c$, which implies that $\gamma \ge 1$, this is a contradiction. Hence, $c = 0$ and

$$\lim_{n\to\infty} \varrho(X_n) + \theta(\varrho(X_n)) = 0.$$

Thus

$$\lim_{n\to\infty} \varrho(X_n) = 0.$$

Using (2.2), it follows that $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$ and $TX_\infty \subset X_\infty$. Then, from Theorem 1.2, T has a fixed point in X_∞. □

A direct consequence of this result is the following:

Theorem 2.2 *Suppose that C is a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \to C$ is a continuous α-admissible mapping. Assume that there exist a closed convex subset $X_0 \subset C$ and $x_0 \in X_0$ such that $TX_0 \subset X_0$ and $\alpha(x_0, Tx_0) \geq 1$. If we have*

$$\alpha(x, Tx)[\varrho(TX) + \theta(\varrho(TX))] \leq \gamma[\varrho(X) + \theta(\varrho(X))],$$

for all $0 < a < b < 1$, $X \subset C$ and $x \in X$, where $\gamma \in [0, 1)$, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and ϱ is a measure of noncompactness. Then T has a fixed point in C.

Example 2.3 *Let us consider $E = BC(\mathbb{R}_+)$ the space of all continuous functions on $\mathbb{R}_+$ into $\mathbb{R}$. Define the operator $T : E \to E$ by*

$$Tx = \begin{cases} \frac{x}{2}, & \text{if} \quad \|x\| \leq 1, \\ 2x - \frac{3}{2}, & \text{if} \quad \|x\| > 1 \end{cases}$$

and the mapping $\alpha : E \times E \to \mathbb{R}_+$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad \|x\| \leq 1, y \in E, \\ 0, & \text{if} \quad \|x\| > 1, y \in E. \end{cases}$$

If $x, y \in E$ such that $\|x\|, \|y\| > 1$, we have

$$\varrho(TX) = 2\varrho(X),$$

where $X := \{x \in E : \|x\| > 1\}$ and $\varrho(X) := Diam(X)$ is the measure of noncompactness of the diameter (see [2]). Hence, we observe that Darbo's result cannot be applied.
Otherwise, prove that:

$$\alpha(x, Tx)[\varrho(TX) + \theta(\varrho(TX))] \leq \frac{1}{2}[\varrho(X) + \theta(\varrho(X))], \tag{2.4}$$

for all $X \subset E$ and $x \in X$, where $\theta(t) = t$ $(t \in \mathbb{R}_+)$. We have the following two cases:
- *If $X \subset E$ and $x \in X$ with $\|x\| \leq 1$, we have $Tx = \frac{x}{2}$, $\alpha(x, Tx) = 1$ and $\alpha(x, Tx)[\rho(TX) + \theta(\rho(TX))] \leq \frac{1}{2}[\rho(X) + \theta(\rho(X))]$.*
- *If $X \subset E$ and $x \in X$ with $\|x\| > 1$, we have $Tx = 2x - \frac{3}{2}$ and $\alpha(x, Tx) = 0$. Therefore, the identity (2.4) is clearly satisfied.*

Thus, we conclude that for all $X \subset E$ and $x \in X$ the condition (2.4) holds and $T0 = 0$.

Remark 2.1 *The above example shows that Darbo fixed point theorem cannot ensure the existence of fixed point for such mappings. Indeed, the condition of Darbo's theorem is not satisfied. Then, our result is a real extension of it; thanks to our result, we prove the existence of fixed point for a large class of mappings satisfying $\varrho(TX) \leq \gamma\varrho(X)$ where $\gamma \geq 1$.*
If we take $\alpha(x, y) = 1$ for all $x, y \in E$ and $\theta(t) = 0$ for all $t \in \mathbb{R}_+$ in Theorem 2.2, we obtain Darbo's fixed point result.

3. Conclusion

This paper offers a recent extension of Darbo's fixed point theorem for a new class of mappings by employing the concept of α-admissibility. To emphasize the utility of our results, we have presented an example demonstrating the insufficiency of Darbo's original theorem.

References

1. Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund.Math., 3(1), 133-181 (1922).
2. Banaś, J. and Goebel, K., *Measures of Non-compactness in Banach Spaces*, Marcel Dekker, New York (1980).
3. Brouwer, L. E. J., *Über abbildung von mannigfaltigkeiten*, Math. Ann., 71(1), 97-115 (1911).
4. Darbo, G., *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova., 24, 84-92 (1955).

5. Imdad, M., Khan, A. R., Saleh, H. N., Alfaqih, W. M., *Some φ-fixed point results for $(F,\varphi,\alpha\text{-}\psi)$-contractive type mappings with applications.* Mathematics, 7(2), 122 (2019). https://doi.org/10.3390/math7020122

6. Kuratowski, C., *Sur les espaces complets*, Fund. Math., 1(15), 301-309 (1930).

7. Raji, M., *Generalized α-ψ-contractive type mappings and related coincidence fixed point theorems with applications.* The Journal of Analysis, 31(2), 1241-1256 (2023). https://doi.org/10.1007/s41478-022-00498-8

8. Samet, B., Vetro, C., Vetro, P., *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal., 75(4), 2154-2165 (2012). https://doi.org/10.1016/j.na.2011.10.014

9. Samet, B., *Fixed points for α-ψ-contractive mappings with an application to quadratic integral equations.* Electron. J. Differ. Equ, 152 (2014).

10. Schauder, J., *Der Fixpunktsatz in Funktional-raumen*, Studia Math., 2(1), 171-180 (1930).

11. Touail, Y., Jaid, A., El Moutawakil, D., *Fixed Point Theorem via Measure of Non-Compactness for a New Kind of Contractions.* Vestnik St. Petersburg University, Mathematics, 56(2), 198-202 (2023). https://doi.org/10.1134/S1063454123020164

12. Touail, Y., Jaid, A., El Moutawakil, D., *A note on common fixed point theorems in a bounded metric space.* Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 27(2), 241-249 (2023). https://doi.org/10.14498/vsgtu1940

13. Touail, Y., *Set-valued mappings: fixed point results with β-function and some applications.* Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta, 63, 61-79 (2024). https://doi.org/10.35634/2226-3594-2024-63-05

Amine Jaid,
Equipe de Recherche en Mathématiques Appliquées,
Technologies de l'Information et de la Communication,
Faculté Polydisciplinaire de Khouribga,
Université Sultan Moulay Slimane de Béni-Mellal,
Morocco.
E-mail address: aminejaid1990@gmail.com

and

Youssef Touail,
Department of Mathematics,
Faculty of Sciences Dhar El Mahraz,
Sidi Mohamed Ben Abdellah University, Fez,
Morocco.
E-mail address: youssef9touail@gmail.com

and

Driss El Moutawakil,
Team of Sciences and Technologies, Higher School of Education and Training El Jadida,
University Chouaïb Doukkali Ucd, Morocco.
E-mail address: d.elmoutawakil@gmail.com