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Existence and Uniqueness of Solutions for Nonlinear Fractional Boundary Value Problems with ψ-Caputo Derivatives

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ABSTRACT: This work aims to investigate the existence and uniqueness of solutions for a boundary value problem using a nonlinear fractional differential equation that involves the ψ-Caputo fractional derivative. Our findings are demonstrated using Krasnoselkii's fixed point theorem and the Banach contraction principle.

Key Words: Fractional Differential Equations, ψ-Caputo Derivative, Boundary Value Problems, Fixed Point Theorems

Contents

1. Introduction

In recent years, fractional differential equations (FDEs) and fractional calculus (FC) have become the most significant and well-known fields of interdisciplinary study. Although FC has been around for more than 300 years, its versatility in several fields has only lately become apparent. The field has had exponential growth in the previous three decades, and many academics worldwide are currently researching on this subject see [7, 8, 9]. By taking into consideration the Caputo fractional derivative of a function with respect to another function ψ, Almeida [1] expanded the concept of the Caputo fractional derivative and examined several helpful aspects of the fractional calculus. The benefit of this new definition of the fractional derivative is that by selecting an appropriate function ψ, the model's accuracy could be increased. Some sufficient conditions for the existence of solutions to the linear fractional boundary value issue were recently provided by Benlabess, Benbachir, and Lakrib in [5]:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), t \in J := [0, 1], 2 < \alpha \leq 3 \\ D_{0^+}^{\alpha-1} u(1) = 0, u(0) = 0, u'(0) = 0 \end{cases}$$

Where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional differential operator of order α and the non linear function $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ is continuous. Inspired by the previously stated works, this paper deals with the existence of solutions for the following nonlinear fractional boundary value problem, generalizing the conclusions found in [5] using ψ-Caputo type fractional derivative of order $3 < \alpha \leq 4$.

$$\begin{cases} {}^{C}D_{0^+}^{\alpha;\psi} u(t) = f(t, u(t)), t \in J := [0, 1] \\ u(0) = u'(0) = 0 \text{ and } {}^{C}D_{0^+}^{\alpha-1;\psi} u(1) = 0, {}^{C}D_{0^+}^{\alpha-2;\psi} u(1) = 0 \end{cases} \tag{1.1}$$

Where ${}^{C}D_{0^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $3 < \alpha \leq 4$ and $f : J \times [0, \infty) \to \mathbb{R}$ is a given continuous function. The paper is organized as follows. In section 2, we introduce notations, definitions and preliminary facts which are used .In section 3, we introduce the basic assumptions and the state the main result on the existence and uniqueness of nonlinear fractional boundary value problem.

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2. Preliminaries.

In order to lay the groundwork for future improvements, we begin this part by providing some essential definitions and fundamental findings. The Banach space of all continuous functions from $J = [0,1]$ into $\mathbb{R}$ with the norm was designated by $\|u\|_\infty = \sup_{t\in J}|u(t)|$

Definition 2.1 *(ψ-Riemann-Liouville fractional integral [4])*
It is assumed that $\alpha > 0, f$ is an integrable function defined on $[a,b]$ and that $\psi : [a,b] \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in [a,b]$. The definition of a function f's ψ-Riemann-Liouville fractional integral operator of order α is

$$I_a^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s)ds$$

Definition 2.2 *(ψ-Riemann-Liouville fractional derivative [4])*
Assume that $n \in \mathbb{N}, f, \psi \in \mathcal{C}^n([a,b])$ are two functions such that, for all $t \in [a,b]$, ψ is growing with $\psi'(t) > 0$. The ψ-Riemann-Liouville The definition of the fractional derivative of order α of a function f is

$$\begin{aligned} D_a^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \left(I_a^{n-\alpha;\psi} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds \end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 *(ψ-Caputo fractional derivative [4])*
Assume that $n \in \mathbb{N}, f, \psi \in \mathcal{C}^n([a,b])$ are two functions such that, for all $t \in [a,b]$, ψ is growing with $\psi'(t) eq 0$. A function f's ψ-Caputo fractional derivative of order α is defined by

$$\begin{aligned} {}^C D_a^{\alpha;\psi} f(t) &= \left(I_a^{n-\alpha;\psi} f_\psi^{[n]}\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f_\psi^{[n]}(s)ds \end{aligned}$$

where $n = [\alpha]+1$, for $\alpha \notin \mathbb{N}$, and $f_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$ on $[a,b]$.

According to the definition, when $\alpha = n \in \mathbb{N}$, we have

$${}^C D_a^{\alpha;\psi} f(t) = f_\psi^{[n]}(t)$$

We observe that if $f \in \mathcal{C}^n([a,b])$. The ψ-Caputo fractional derivative of order α of f can be found as

$${}^C D_a^{\alpha;\psi} f(t) = D_a^{\alpha;\psi}\left(f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!}(\psi(t)-\psi(a))^k\right)$$

Theorem 2.1 (4) *Let $f \in \mathcal{C}^n([a,b])$ and $\alpha > 0$. Then we have*

$$I_a^{\alpha;\psi C} D_a^{\alpha;\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!}(\psi(t)-\psi(a))^k$$

In particular, given $\alpha \in (0,1)$ we have:

$$I_a^{\alpha;\psi^C} D_a^{\alpha;\psi} f(t) = f(t) - f(a)$$

Theorem 2.2 *Given a function $f \in \mathcal{C}([a,b])$ and $\alpha > 0$, we have:*

$$^{C}D_{a+}^{\alpha-1;\psi} I_{a+}^{\alpha;\psi} f(x) = \int_a^x f(t)\psi'(t)dt$$

Proof: By definition,

$$^{C}D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(n-1-\alpha+1)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-1-\alpha+1-1} F_{\psi}^{[n-1]}(t)dt$$

with

$$F_{\psi}^{[n-1]}(x) = \frac{f(a)}{\Gamma(\alpha-n+2)}(\psi(x)-\psi(a))^{\alpha-n+1} + \frac{1}{\Gamma(\alpha-n+2)} \int_a^x (\psi(x)-\psi(t))^{\alpha-n+1} f'(t)dt.$$

Then,

$$\begin{aligned}
^{C}D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) &= \frac{f(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}(\psi(t)-\psi(a))^{\alpha-n+1}dt \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x \int_a^t \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}(\psi(t)-\psi(\tau))^{\alpha-n+1} f'(\tau)d\tau dt \\
&= \frac{f(a)\times(\psi(x)-\psi(a))^{n-\alpha-1}}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x \psi'(t)\left(1-\frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}\right)^{n-\alpha-1} (\psi(t)-\psi(a))^{\alpha-n+1}dt \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x \int_a^t \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}(\psi(t)-\psi(\tau))^{\alpha-n+1} f'(\tau)d\tau dt
\end{aligned}$$

By applying Dirichlet's formula and the change of variables $u = \frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}$, we arrive at the following conclusion:

$$\begin{aligned}
^{C}D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) &= \frac{f(a)\times(\psi(x)-\psi(a))}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_0^1 (1-u)^{n-\alpha-1} u^{\alpha-n+1} du \\
&+ \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x f'(t) \left\{ \int_t^x \psi'(\tau)(\psi(x)-\psi(\tau))^{n-\alpha-1}(\psi(\tau)-\psi(t))^{n-\alpha+1} d\tau \right\} dt \\
&= f(a)\times(\psi(x)-\psi(a)) + \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+2)} \int_a^x f'(t)(\psi(x)-\psi(t))^{n-\alpha-1} \\
&\times \int_t^x \psi'(\tau)\left(1-\frac{\psi(\tau)-\psi(t)}{\psi(x)-\psi(t)}\right)^{n-\alpha-1} (\psi(\tau)-\psi(t))^{\alpha-n+1} d\tau dt \\
&= f(a)\times(\psi(x)-\psi(a)) + \int_a^x f'(t)(\psi(x)-\psi(t))dt
\end{aligned}$$

Thus,

$$^{C}D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) = \int_a^x f(t)\psi'(t)dt$$

□

Theorem 2.3 *Given a function $f \in \mathcal{C}([a,b])$ and $\alpha > 0$, we have:*

$$^{C}D_{a+}^{\alpha-2;\psi} I_{a+}^{\alpha;\psi} f(x) = \int_a^x f(t)\psi'(t)(\psi(x)-(t))dt$$

Proof: Similar to proof of (2.2) □

Lemma 2.1 *Given $n \leq k \in \mathbb{N}$, we have:*

$$^{C}D_{a+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(t)-\psi(a))^{k-\alpha},$$

and

$$^{C}D_{b-}^{\alpha;\psi}(\psi(b)-\psi(t))^{k} = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(b)-\psi(t))^{k-\alpha}.$$

Theorem 2.4 *(Krasnselskii's fixed point theorem)*
Let S be a closed convex non-empty subset of a Banach space X. Suppose that A, B map S into X such that

1. *$Au + Bv \in S, \forall u, v \in S$,*
2. *A is a contraction mapping,*
3. *B is continuous and $B(S)$ is contained in a compact set.*

Then there exists $u \in S$ such that $Au + Bu = u$.

3. Existence result

We begin by defining the term "solution" for the boundary value problem (1.1).

Definition 3.1 *A function $u \in \mathcal{C}(J, \mathbb{R})$ is said to be a solution of problem (1.1) if, u satisfies the equation*

$$^{C}D_{0+}^{\alpha;\psi}u(t) = f(t, u(t)), t \in J$$

and the conditions

$$u(0) = u'(0) = 0 \text{ and } {}^{C}D_{0+}^{\alpha-1;\psi}u(1) = {}^{C}D_{0+}^{\alpha-2;\psi}u(1) = 0$$

Lemma 3.1 *For a given $h : J \to \mathbb{R}$ continuous, the unique solution of the nonlinear fractional differential equation*

$$\begin{cases} {}^{C}D_{0+}^{\alpha;\psi}u(t) = h(t), t \in J \\ u(0) = u'(0) = 0 \text{ and } {}^{C}D_{0+}^{\alpha-1;\psi}u(1) = {}^{C}D_{0+}^{\alpha-2;\psi}u(1) = 0 \end{cases} \tag{3.1}$$

is given by:

$$u(t) = c_2(\psi(t)-\psi(0))^2 + c_3(\psi(t)-\psi(0))^3 + I_{0+}^{\alpha;\psi}h(t)dt \tag{3.2}$$

with:

$$C_2 = -\left(\int_0^1 h(s)\psi'(s)(\psi(1)-\psi(s))ds\right)\frac{\Gamma(5-\alpha)}{2}(\psi(1)-\psi(0))^{\alpha-4}$$
$$+\frac{\Gamma(5-\alpha)}{(6-\alpha)\times 2}(\psi(1)-\psi(0))^{\alpha-3} \times \int_0^1 h(s)\psi'(s)ds$$

and

$$c_3 = -\left(\int_0^1 h(s)\psi'(s)ds\right)\frac{\Gamma(5-\alpha)}{6}(\psi(1)-\psi(0))^{\alpha-4}$$

Proof: The following is obtained by applying the ψ-Riemann-Liouville fractional integral of order α to the first equation of (3.1)

$$u(t) = c_0 + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 + c_3(\psi(t)-\psi(0))^3 + I_{0+}^{\alpha;\psi}h(t)dt$$

Given that $u(0) = 0$ as well as $u'(0) = 0$, We conclude that $c_0 = c_1 = 0$.. Then,

$$u(t) = c_2(\psi(t)-\psi(0))^2 + c_3(\psi(t)-\psi(0))^3 + I_{0+}^{\alpha;\psi}h(t)dt$$

With the condition ${}^{C}D_{0^+}^{\alpha-1;\psi}u(1)=0$ and theorem(2.2) and lemma(2.1), we have:

$$ {}^{C}D_{0^+}^{\alpha-1;\psi}u(t)=\frac{6c_3}{\Gamma(5-\alpha)}(\psi(t)-\psi(0))^{4-\alpha}+\int_0^t h(s)\psi'(s)ds $$

so,

$$ c_3=-\left(\int_0^1 h(s)\psi'(s)ds\right)\frac{\Gamma(5-\alpha)}{6}(\psi(1)-\psi(0))^{\alpha-4} $$

Additionally, under the prerequisite ${}^{C}D_{0^+}^{\alpha-2;\psi}u(1)=0$ and lemma(2.1) and theorem(2.2), we have:

$$ {}^{C}D_{0^+}^{\alpha-2;\psi}u(1)=\frac{2c_2}{\Gamma(5-\alpha)}(\psi(1)-\psi(0))^{4-\alpha}+\frac{6\times\Gamma(5-\alpha)\times c_3}{\Gamma(6-\alpha)\times 6}(\psi(1)-\psi(0))^{1}\times\int_0^1 h(s)\psi'(s)ds $$

$$ +\int_0^1 h(s)\psi'(s)(\psi(1)-\psi(s))ds $$

so,

$$ c_2=-\left(\int_0^1 h(s)\psi'(s)(\psi(1)-\psi(s))ds\right)\frac{\Gamma(5-\alpha)}{2}(\psi(1)-\psi(0))^{\alpha-4} $$

$$ +\frac{\Gamma(5-\alpha)}{(6-\alpha)\times 2}(\psi(1)-\psi(0))^{\alpha-3}\times\int_0^1 h(s)\psi'(s)ds $$

The integral equation (3.2) is thus obtained, and the proof is completed by the direct computation of the opposite. □

We will now discuss our main result concerning the existence of solutions of problem (1.1) . In order to establish the existence result we make the following assumption:
(H_h) : There exists a constant $L>0$, such that

$$ |h(t,u(t))-h(t,v(t))|\leq L|u(t)-v(t)|, \forall u,v\in\mathbb{R}, \forall t\in J $$

Theorem 3.1 *Let $h: J\times[0,\infty)\to\mathbb{R}$ be a continuous function such that (H_h) holds, and if we have, that*

$$ L(\psi(1)-\psi(0))^{\alpha}\left\{\Gamma(5-\alpha)\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right\}<1 $$

Then problem (1.1) has a unique solution on J.

Proof: Suppose that:

$$ \begin{aligned} Pu(t)=&\frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{2}(\psi(t)-\psi(0))^2\times\left(\int_0^1 h(s,u(s))\psi'(s)(\psi(1)-\psi(s))ds\right)\\ &+\frac{\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-3}}{2(6-\alpha)}(\psi(t)-\psi(0))^2\times\left(\int_0^1 h(s,u(s))\psi'(s)ds\right)\\ &+\frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{6}(\psi(t)-\psi(0))^3\times\left(\int_0^1 h(s,u(s))\psi'(s)ds\right)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s,u(s))ds \end{aligned} $$

Then,

$$\begin{aligned}|Pu(t)-Pv(t)| &\leq \frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{2}(\psi(t)-\psi(0))^2\\ &\times\left(\int_0^1 |h(s,u(s))-h(s,v(s))|\psi'(s)(\psi(1)-\psi(s))ds\right)\\ &+\frac{\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-3}}{2(6-\alpha)}(\psi(t)-\psi(0))^2\times\left(\int_0^1 |h(s,u(s))-h(s,v(s))|\psi'(s)ds\right)\\ &+\frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{6}(\psi(t)-\psi(0))^3\times\left(\int_0^1 |h(s,u(s))-h(s,v(s))|\psi'(s)ds\right)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|h(s,u(s))-h(s,v(s))|ds\\ &\leq \frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{4}\|u-v\|_\infty+\frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{2(6-\alpha)}\|u-v\|_\infty\\ &+\frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{6}\|u-v\|_\infty+\frac{L}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha}\|u-v\|_\infty\\ &=L(\psi(1)-\psi(0))^{\alpha}\left(\Gamma(5-\alpha)\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right)\|u-v\|_\infty.\end{aligned}$$

because P is a contraction. The unique solution to problem (1.1) is P's unique fixed point, according to the Banach fixed point theorem. □

Example 3.1 Consider the following nonlinear fractional boundary value problem with $\alpha = 3.5$ (so $3 < \alpha \leq 4$) and $\psi(t) = t$ (the identity function):

$$\mathcal{C}D_{0+}^{3.5;t}u(t) = h(t,u(t)), \quad t \in [0,1],$$

with boundary conditions:

$$u(0) = u'(0) = 0, \quad \mathcal{C}D_{0+}^{2.5;t}u(1) = \mathcal{C}D_{0+}^{1.5;t}u(1) = 0.$$

Here, $\mathcal{C}D_{0+}^{3.5;t}$ is the ψ-Caputo fractional derivative with $\psi(t) = t$, which simplifies to the standard Caputo derivative.

Define the Function $h(t,u(t))$ Let's choose:

$$h(t,u(t)) = \frac{1}{10}\sin(u(t)).$$

This function is continuous and satisfies the Lipschitz condition with $L = \frac{1}{10}$, because:

$$|h(t,u(t))-h(t,v(t))| = \left|\frac{1}{10}\sin(u(t))-\frac{1}{10}\sin(v(t))\right| \leq \frac{1}{10}|u(t)-v(t)|.$$

Verify the Condition of Theorem (3.1) We need to check if:

$$L(\psi(1)-\psi(0))^{\alpha}\left\{\Gamma(5-\alpha)\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right\} < 1.$$

Substitute $L = \frac{1}{10}$, $\alpha = 3.5$, and $\psi(1)-\psi(0) = 1$:

$$\frac{1}{10}\cdot(1)^{3.5}\left\{\Gamma(1.5)\left(\frac{5}{12}+\frac{1}{2(2.5)}\right)+\frac{1}{\Gamma(4.5)}\right\} < 1.$$

Calculate the terms:

- $\Gamma(1.5) = \frac{\sqrt{\pi}}{2} \approx 0.8862$,
- $\Gamma(4.5) \approx 11.6317$,
- $\frac{5}{12} + \frac{1}{2(2.5)} = \frac{5}{12} + \frac{1}{5} = \frac{25}{60} + \frac{12}{60} = \frac{37}{60} \approx 0.6167$.

Now, substitute these values:

$$\frac{1}{10}\left\{0.8862 \cdot 0.6167 + \frac{1}{11.6317}\right\} = \frac{1}{10}\{0.5467 + 0.0859\} = \frac{1}{10} \cdot 0.6326 = 0.06326.$$

Since $0.06326 < 1$, the condition of Theorem(3.1) is satisfied.
Conclusion, By Theorem(3.1), the boundary value problem:

$$\mathcal{C}D_{0+}^{3.5;t}u(t) = \frac{1}{10}\sin(u(t)), \quad t \in [0,1],$$

with the given boundary conditions, has a **unique solution** on the interval $[0,1]$.

Example 3.2 Consider the following nonlinear fractional boundary value problem with $\alpha = 3.8$ (so $3 < \alpha \leq 4$) and $\psi(t) = t$ (the identity function):

$$\mathcal{C}D_{0+}^{3.8;t}u(t) = h(t, u(t)), \quad t \in [0,1],$$

with boundary conditions:

$$u(0) = u'(0) = 0, \quad \mathcal{C}D_{0+}^{2.8;t}u(1) = \mathcal{C}D_{0+}^{1.8;t}u(1) = 0.$$

Here, $\mathcal{C}D_{0+}^{3.8;t}$ is the ψ-Caputo fractional derivative with $\psi(t) = t$, which simplifies to the standard Caputo derivative.
Define the Function $h(t, u(t))$**:** Let's choose:

$$h(t, u(t)) = \frac{1}{15}\left(u(t) - \frac{u(t)^3}{3}\right).$$

This function is continuous and satisfies the Lipschitz condition with $L = \frac{1}{15}$, because:

$$|h(t, u(t)) - h(t, v(t))| = \left|\frac{1}{15}\left(u(t) - \frac{u(t)^3}{3}\right) - \frac{1}{15}\left(v(t) - \frac{v(t)^3}{3}\right)\right| \leq \frac{1}{15}|u(t) - v(t)|.$$

Verify the Condition of Theorem (3.1)**:** We need to check if:

$$L(\psi(1) - \psi(0))^{\alpha}\left\{\Gamma(5-\alpha)\left(\frac{5}{12} + \frac{1}{2(6-\alpha)}\right) + \frac{1}{\Gamma(\alpha+1)}\right\} < 1.$$

Substitute $L = \frac{1}{15}$, $\alpha = 3.8$, and $\psi(1) - \psi(0) = 1$:

$$\frac{1}{15} \cdot (1)^{3.8}\left\{\Gamma(1.2)\left(\frac{5}{12} + \frac{1}{2(2.2)}\right) + \frac{1}{\Gamma(4.8)}\right\} < 1.$$

Calculate the terms:

- $\Gamma(1.2) \approx 0.9182$,
- $\Gamma(4.8) \approx 17.837$,
- $\frac{5}{12} + \frac{1}{2(2.2)} = \frac{5}{12} + \frac{1}{4.4} \approx 0.4167 + 0.2273 = 0.6440$.

Now, substitute these values:

$$\frac{1}{15}\left\{0.9182\cdot 0.6440+\frac{1}{17.837}\right\}=\frac{1}{15}\left\{0.5913+0.0561\right\}=\frac{1}{15}\cdot 0.6474\approx 0.0432.$$

Since $0.0432<1$, the condition of Theorem (3.1) is satisfied.
Conclusion: By Theorem (3.1), the boundary value problem:

$$\mathcal{C}D_{0+}^{3.8;t}u(t)=\frac{1}{15}\left(u(t)-\frac{u(t)^3}{3}\right),\quad t\in[0,1],$$

with the given boundary conditions, has a **unique solution** on the interval $[0,1]$.

Theorem 3.2 *Assuming that h fulfills (H_h) and that $\beta>0$ exists; such that:*

$$|\ f(t,u)\ |\leq\beta,\quad \forall t\in J,\forall x\geq 0$$

And if there exists $\gamma>0$ such that:

$$\beta(\psi(1)-\psi(0))^{\alpha}\left(\Gamma(5-\alpha)\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right)\leq\gamma$$

then the problem (1.1) has at least one solution on J.

Proof: We define a subset S of X by:

$$S=\{u\in\mathcal{C}(J,\mathbb{R}),\|u\|_{\infty}\leq\gamma\}$$

Define two operators $A:S\to X$ and $B:S\to X$ by:

$$\begin{aligned}Au(t)=&\frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{2}(\psi(t)-\psi(0))^2\times\left(\int_0^1 h(s,u(s))\psi'(s)(\psi(1)-\psi(s))ds\right)\\&+\frac{\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-3}}{2(6-\alpha)}(\psi(t)-\psi(0))^2\times\left(\int_0^1 h(s,u(s))\psi'(s)ds\right)\\&+\frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{6}(\psi(t)-\psi(0))^3\times\left(\int_0^1 h(s,u(s))\psi'(s)ds\right)\\&+\frac{1}{2\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s,u(s))ds\end{aligned}$$

and

$$Bu(t)=\frac{1}{2\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,u(s))ds$$

Then, the equation (3) is transformed into the operator equation as

$$u(t)=Au(t)+Bu(t)$$

We demonstrate in multiple phases that the operators A and B meet all the requirements of theorem (2.3).

Step 1. Let $u, v \in S$. Then:

$$\begin{aligned}
|Au(t)+Bv(t)| &= \bigg| \frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{2}(\psi(t)-\psi(0))^2 \\
&\times \left(\int_0^1 h(s,u(s))\psi'(s)(\psi(1)-\psi(s))ds\right) \\
&+ \frac{\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-3}}{2(6-\alpha)}(\psi(t)-\psi(0))^2 \times \left(\int_0^1 h(s,u(s))\psi'(s)ds\right) \\
&+ \frac{-\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha-4}}{6}(\psi(t)-\psi(0))^3 \times \left(\int_0^1 h(s,u(s))\psi'(s)ds\right) \\
&+ \frac{1}{2\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s,u(s))ds \\
&+ \frac{1}{2\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s,u(s))ds \bigg| \\
&\le \frac{\beta\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{4} + \frac{\beta\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{2(6-\alpha)} \\
&+ \frac{\beta\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{6} + \frac{\beta(\psi(1)-\psi(0))^{\alpha}}{2\Gamma(\alpha+1)} + \frac{\beta(\psi(1)-\psi(0))^{\alpha}}{2\Gamma(\alpha+1)} \\
&= \beta\left((\Gamma(5-\alpha))\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right)(\psi(1)-\psi(0))^{\alpha} \\
&\le \gamma
\end{aligned}$$

Step 2. Let $u, v \in S$. Then

$$\begin{aligned}
|Au(t)-Av(t)| &\le \frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{4}\|u-v\|_\infty + \frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{2(6-\alpha)}\|u-v\|_\infty \\
&+ \frac{L\Gamma(5-\alpha)(\psi(1)-\psi(0))^{\alpha}}{6}\|u-v\|_\infty + \frac{L}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha}\|u-v\|_\infty \\
&= L(\psi(1)-\psi(0))^{\alpha}\left(\Gamma(5-\alpha)\left(\frac{5}{12}+\frac{1}{2(6-\alpha)}\right)+\frac{1}{\Gamma(\alpha+1)}\right)\|u-v\|_\infty \\
&< \|u-v\|_\infty \quad (\text{ by condition } (4)).
\end{aligned}$$

Step 3. Let $(u_n)_n$ be a sequence such that $u_n \to u \in \mathcal{C}(J,\mathbb{R})$. For $t \in J$, we have:

$$\begin{aligned}
|Bu_n(t)-Bu(t)| &\le \frac{1}{2\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\Big| h((s,u_n(s))-h(s,u(s)) \mid ds \\
&\le \frac{L(\psi(1)-\psi(0))^{\alpha}}{2\Gamma(\alpha+1)}\|u_n-u\|_\infty
\end{aligned}$$

Therefore

$$\|Bu_n(t)-Bu(t)\|_\infty \to 0 \text{ as } \|u_n-u\|_\infty \to 0$$

to demonstrate the compactness of B. Let $\Omega \subset S$ be a bounded set. In $\mathcal{C}(J,\mathbb{R})$, we must demonstrate that $B(\Omega)$ is substantially compact. For random $t \in J$ and $u \in \Omega$. We possess:

$$\|Bu\| \le \frac{\beta(\psi(1)-\psi(0))^{\alpha}}{2\Gamma(\alpha+1)} = \text{ cste.}$$

Now, for equi-continuity of B take $t_1, t_2 \in J$ with $t_1 < t_2$, and let $u \in \Omega$. Thus, we get

$$|Bu(t_2)-Bu(t_1)| \le \frac{\beta}{2\Gamma(\alpha+1)}\left\{(\psi(t_2)-\psi(0))^{\alpha}+(\psi(t_1)-\psi(0))^{\alpha}\right\}$$

The last estimate leads us to the conclusion that when $t_2 \to t_1$, we have $\|Bu(t_2) - Bu(t_1)\| \to 0$. then B is equicontinuous as a consequence. Accordingly, the operator B is compact by the Ascoli-Arzela theorem. Thus, on J, there is at least one solution to issue (1.1).

□

Example 3.3 Consider the following nonlinear fractional boundary value problem with $\alpha = 3.5$ (so $3 < \alpha \leq 4$) and $\psi(t) = t$ (the identity function):

$$\mathcal{C}D_{0+}^{3.5;t} u(t) = h(t, u(t)), \quad t \in [0,1],$$

with boundary conditions:

$$u(0) = u'(0) = 0, \quad \mathcal{C}D_{0+}^{2.5;t} u(1) = \mathcal{C}D_{0+}^{1.5;t} u(1) = 0.$$

Here, $\mathcal{C}D_{0+}^{3.5;t}$ is the ψ-Caputo fractional derivative with $\psi(t) = t$, which simplifies to the standard Caputo derivative.

Define the Function $h(t, u(t))$: Let's choose:

$$h(t, u(t)) = \frac{1}{10} \arctan(u(t)).$$

This function is continuous and satisfies:

1. **Lipschitz condition**: The derivative of $\arctan(u(t))$ is bounded by 1, so:

$$|h(t, u(t)) - h(t, v(t))| = \left| \frac{1}{10} \arctan(u(t)) - \frac{1}{10} \arctan(v(t)) \right| \leq \frac{1}{10} |u(t) - v(t)|.$$

   Thus, $L = \frac{1}{10}$.

2. **Boundedness condition**: Since $|\arctan(u(t))| \leq \frac{\pi}{2}$, we have:

$$|h(t, u(t))| = \left| \frac{1}{10} \arctan(u(t)) \right| \leq \frac{\pi}{20}.$$

   Thus, $\beta = \frac{\pi}{20}$.

Verify the Inequality Condition:
We need to check if:

$$\beta(\psi(1) - \psi(0))^\alpha \left\{ \Gamma(5-\alpha) \left(\frac{5}{12} + \frac{1}{2(6-\alpha)} \right) + \frac{1}{\Gamma(\alpha+1)} \right\} \leq \gamma.$$

Substitute $\beta = \frac{\pi}{20}$, $\alpha = 3.5$, and $\psi(1) - \psi(0) = 1$:

$$\frac{\pi}{20} \cdot (1)^{3.5} \left\{ \Gamma(1.5) \left(\frac{5}{12} + \frac{1}{2(2.5)} \right) + \frac{1}{\Gamma(4.5)} \right\} \leq \gamma.$$

Calculate the terms:

- $\Gamma(1.5) = \frac{\sqrt{\pi}}{2} \approx 0.8862$,
- $\Gamma(4.5) \approx 11.6317$,
- $\frac{5}{12} + \frac{1}{2(2.5)} = \frac{5}{12} + \frac{1}{5} = \frac{25}{60} + \frac{12}{60} = \frac{37}{60} \approx 0.6167$.

Now, substitute these values:

$$\frac{\pi}{20} \left\{ 0.8862 \cdot 0.6167 + \frac{1}{11.6317} \right\} = \frac{\pi}{20} \{0.5467 + 0.0859\} = \frac{\pi}{20} \cdot 0.6326 \approx 0.0995.$$

Thus, we can choose $\gamma = 0.1$, since $0.0995 \leq 0.1$.

Conclusion: By Theorem(3.2), the boundary value problem:

$$\mathcal{C}D_{0+}^{3.5;t} u(t) = \frac{1}{10} \arctan(u(t)), \quad t \in [0,1],$$

with the given boundary conditions, has **at least one solution** on the interval $[0,1]$.

4. Conclusion

This article addresses the existence and uniqueness of solutions for a class of nonlinear fractional boundary value problems involving ψ-Caputo derivatives. Fractional calculus, particularly fractional differential equations (FDEs), has gained significant attention in recent years due to its applications in various scientific and engineering fields. The authors extend the classical Caputo derivative by introducing the ψ-Caputo fractional derivative, which provides greater flexibility and accuracy in modeling real-world phenomena by incorporating an arbitrary function ψ. This generalization allows for a more nuanced analysis of fractional differential equations.

The main focus of the paper is to establish sufficient conditions under which the nonlinear fractional boundary value problem admits a unique or at least one solution. The authors employ powerful mathematical tools, including **Krasnoselskii's fixed point theorem** and the **Banach contraction principle**, to derive these conditions. The use of ψ-Riemann-Liouville fractional integrals and derivatives plays a crucial role in the analysis, providing a solid foundation for the theoretical results.

Two key theorems are presented: **Theorem** (3.1) guarantees the existence of a unique solution under a Lipschitz condition on the nonlinear function $h(t, u(t))$, while **Theorem** (3.2) ensures the existence of at least one solution under boundedness and continuity assumptions. These results are significant as they generalize previous work on fractional boundary value problems and provide a framework for solving more complex problems involving ψ-Caputo derivatives.

The theoretical findings are complemented by illustrative examples, which demonstrate the practical applicability of the theorems. These examples show how the conditions of the theorems can be verified for specific functions and parameters, highlighting the versatility of the approach.

In conclusion, this article makes a valuable contribution to the field of fractional calculus by extending the classical theory to include ψ-Caputo derivatives and providing new insights into the solvability of nonlinear fractional boundary value problems. The results open up new avenues for research and have potential applications in areas such as physics, biology, and engineering, where fractional models are increasingly used to describe complex systems.

References

1. Almeida, R., A.B. Malinowska, and M.M.T. Monteiro. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. Math. Meth. Appl, 41 (2018), 336-352.
2. Almeida, R. A Caputo fractional derivative of a function with respect to another function. Nonlinear Science and Numerical Simulation, 44 (2017), 460-481.
3. Baitiche, Z., C. Derbazi, and M. Benchohra. ψ-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory. RNA, 3 (2020), 167-178.
4. Baleanu, D., H. Jafari, H. Khan, and S.J. Johnston. Results for mild solution for fractional coupled hybrid boundary value problem. Open Math, 13 (2005), 601-608.
5. Baleanu, D., H. Jafari, H. Khan, and R.A. Alipour. On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions. Adv. Differ. Equ, 1 (2015), 1-14.
6. Benlabess, A., M. Benbachir, and M. Lakrib. Boundary value problems for nonlinear fractional differential equations. 30 (2015), 157-168.
7. Daftardar-Gejji, V. Fractional calculus and fractional differential equations. Springer Nature Singapore, 2019.
8. Dhage, B.C. On α-condensing mappings in Banach algebras. Math Student, 63 (1994), 146-152.
9. K.Hilal , A.Kajouni: Boundary value problems for hybrid differential equations with fractional order. Advances in Difference Equations (2015).
10. Podlubny, I. Fractional differential equations. Mathematics in Science and Engineering, 198 (1999), 41-119.
11. S. Solhi, A. Kajouni and K. Hilal. Boundary value problems for hybrid differential equations with fractional order. Boletim da Sociedade Paranaense de Matemática,Vol 43 (2025)

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