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Automatic Boundedness in Certain Bornological Algebras

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ABSTRACT: In this paper, we deal with the automatic boundedness of linear operators in some classes of bornological algebras, with a special emphasis on homomorphisms. In emphasizing one of the main results on multiplicatively p-convex and *-semi-simple algebras is the application of classical boundedness theorems to bornological algebras. Applications are given to spectral theory and operator analysis.

Key Words: Banach algebras, generalized derivations, bornological algebras, multiplicatively convex.

Contents

1. Introduction

In [5], A.M. Sinclair established fundamental results for the investigation of the continuity of homomorphisms, derivations, and operator pairs in Banach spaces. In [4], A. Tajmouati generalized some results of A.M. Sinclair to bornological vector spaces (b.v.s.), which gave a tool to answer questions concerning the boundedness of linear operators. He thus generalized the concept of the separator space $\mathfrak{S}(S)$ of a linear operator S, defined on a bornological space X and taking values in a bornological space Y (see Definition 3.1). The importance of the separator space comes from the fact that a linear operator is bounded if and only if its separator space reduces to the singleton $\{0\}$. In this paper, we deal with the automatic boundedness of homomorphisms in some complete bornological algebras. Motivated by the techniques of A.M. Sinclair [5], we transfer results from the normed case to bornological algebras. We then obtain results on the automatic boundedness of surjective (or dense-image) homomorphisms in complete $a.b.m.c_p$ *-simple (Theorem 3.5) and *-semi-simple (Theorem 4.1) algebras. The algebras in consideration are complex, associative but not unital, in general, and non-commutative.

2. Preliminaries

Definition 2.1
Let $\mathcal{X}$ be a set. A bornology on $\mathcal{X}$ is a family β of subsets of $\mathcal{X}$ which has the following properties:

(P_1) : *β forms a covering of $\mathcal{X}$,*

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(P_2) : *β is hereditary under inclusion ($\mathcal{B} \in \beta$ and $\mathcal{A} \subset \mathcal{B}$ imply $\mathcal{A} \in \beta$),*

(P_3) : *β is stable under finite unions.*

A bornological set is a couple $(\mathcal{X}, \beta)$ consisting of a set $\mathcal{X}$ and a bornology β on $\mathcal{X}$.

A basis of a bornology β' for β is a subfamily β' of β such that:

$$\forall \mathcal{B} \in \beta,\ \exists \mathcal{B}' \in \beta',\ \mathcal{B} \subset \mathcal{B}'.$$

In other words, β' covers $\mathcal{X}$, and any finite union of elements of β' is included in an element of β'.

Definition 2.2
A bornology β on a vector space $\mathcal{E}$ (over $\mathbb{R}$ or $\mathbb{C}$) is called vectorial if it satisfies for all $x \in \mathcal{E}$ and all scalar λ:

- *Stability under addition: for any $\mathcal{A}$, $\mathcal{B} \in \beta$, we have $\mathcal{A} + \mathcal{B} \in \beta$,*
- *Stability under homothety: for every $\mathcal{A} \in \beta$ and all $\lambda \in \mathbb{R}$ or $\mathbb{C}$ we have $\lambda\mathcal{A} \in \beta$,*
- *Closure under taking convex hulls: if $\mathcal{A} \in \beta$, then the convex hull of $\mathcal{A}$ also is an element of β.*

These properties are equivalent to saying that the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are bounded.

A bounded disk is a balanced, convex, and bounded set. A bornology is said to be convex when it can be defined by bounded disks. In that case, the space $\mathcal{E}$ is called a convex bornological space (c.b.s.).

A bornological vector space (b.v.s.) $\mathcal{E}$ is said to be of type M_1 if it satisfies the following condition, known as Mackey's countability condition: for every sequence $(B_k)_{k\in\mathbb{N}}$ of bounded sets in $\mathcal{E}$, there exists a sequence of positive scalars $(\lambda_k)_{k\in\mathbb{N}}$ such that the set $\bigcup_{k=0}^{\infty} \lambda_k B_k$ is bounded in $\mathcal{E}$.

3. p-Convex Bornological Vector Spaces

Let $0 < \mathrm{p} \leqslant 1$ and $\mathcal{E}$ be a vector space. A bornology β on a vector space $\mathcal{E}$ (over $\mathbb{R}$ or $\mathbb{C}$) is said to be vectorial if it satisfies the following conditions:

- Stability under addition: for all $\mathcal{A}, \mathcal{B} \in \beta$, we have $\mathcal{A} + \mathcal{B} \in \beta$,
- Stability under scalar multiplication: for all $\mathcal{A} \in \beta$ and all $\lambda \in \mathbb{R}$ or $\mathbb{C}$, we have $\lambda\mathcal{A} \in \beta$,
- Stability under taking the balanced hull: if $\mathcal{A} \in \beta$, then the balanced hull of $\mathcal{A}$ also belongs to β.

These properties are equivalent to stating that the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are bounded.

A bounded p-disk is a set that is balanced, p-convex, and bounded. A vectorial bornology is said to be p-convex if it can be defined by bounded p-disks. In this context, the space $\mathcal{E}$ is referred to as a p-convex bornological space ($e.b.c_p$).

Remark 3.1

- *A subset $\mathcal{B}$ of $\mathcal{E}$ is said to be p-disked if for all $x, y \in \mathcal{B}$, $\lambda x + \mu y \in \mathcal{B}$ for all positive λ and μ such that $|\lambda|^p + |\mu|^p \leq 1$. Clearly, any intersection of p-disks is p-disked. The p-disked hull of a subset $\mathcal{B}$ of $\mathcal{E}$, denoted by $\Gamma_p(\mathcal{B})$, is the intersection of all p-disks containing $\mathcal{B}$. It is shown that:*

$$\Gamma_p(\mathcal{B}) = \left\{ \sum_{1\leq i\leq n} \lambda_i x_i, \quad x_i \in \mathcal{B},\ \textit{and} \sum_{1\leq i\leq n} |\lambda_i|^p \leq 1,\ n \in \mathbb{N}^* \right\}.$$

Let $\mathcal{B} \subset \mathcal{E}$ and p a real number. Define:

$$J_{\mathcal{B},p}(x) = \inf \left\{ |\lambda|^p : x \in \lambda\mathcal{B} \right\},$$

with the convention that $\inf(\emptyset) = +\infty$. *The function* $J_{\mathcal{B},p}$ *is called the p-gauge of* $\mathcal{B}$. *If* $\mathcal{B}$ *is absorbing,* $J_{\mathcal{B},p}$ *is a finite function. It is then shown that the p-gauge of an absorbing p-disk* $\mathcal{B}$ *is a p-semi-norm.*

- *A subset* $\mathcal{B}$ *of* $\mathcal{E}$ *is said to be* ℓ^p*-disked if the sums of series* $\sum_{i=1}^{\infty} \lambda_i x_i$ *belong to* $\mathcal{B}$ *for* $(x_i) \in \mathcal{B}$ *and* (λ_i) *a sequence of scalars such that* $\sum_{i=1}^{\infty} |\lambda_i|^p \leq 1$. *The* ℓ^p*-disked hull of a subset* $\mathcal{B}$ *of* $\mathcal{E}$*, denoted by* $\nu_p(\mathcal{B})$*, is the intersection of all* ℓ^p*-disks containing* $\mathcal{B}$*. It is shown that:*

$$\nu_p(\mathcal{B}) = \left\{ \sum_{i=1}^{\infty} \lambda_i x_i \text{ where } (x_i) \in \mathcal{B} \text{ and } \sum_{i=1}^{\infty} |\lambda_i|^p \leq 1 \right\}.$$

- *When* $p = 1$*, the p-convex subsets of* $\mathcal{E}$ *coincide with the convex subsets of* $\mathcal{E}$*, leading to the notion of a convex bornological space.*

An endowed vector space (e.v.b.) $\mathcal{E}$ is said to be of type M_1 if it satisfies the following condition, called Mackey's countability condition: for every sequence $(\mathcal{B}_k)_{k\in\mathbb{N}}$ of bounded subsets of $\mathcal{E}$, there exists a sequence of positive scalars $(\lambda_k)_{k\in\mathbb{N}}$ such that the set $\bigcup_{k=0}^{\infty} \lambda_k \mathcal{B}_k$ is bounded in $\mathcal{E}$.

3.1. Canonical Bornology and Topology

Let $\mathcal{E}$ be a locally convex space (l.c.s.). The Von Neumann bornology on $\mathcal{E}$ is defined as the family β of bounded subsets of $\mathcal{E}$, i.e., the sets absorbed by every neighborhood of 0. This construction yields a convex bornological space (c.b.s.), denoted $(\mathcal{E}, \beta)$ or simply $\beta\mathcal{E}$.

Conversely, for a convex bornological space (c.b.s.) $\mathcal{E}$, one can associate a locally convex space (l.c.s.), denoted $T\mathcal{E}$, whose base of neighborhoods of 0 consists of bornivorous disks, i.e., sets that absorb all bounded sets. If $T\mathcal{E}$ is separated, $\mathcal{E}$ is said to be t-separated.

In general, the topology $TB\mathcal{E}$ defined on an l.c.s. is finer than its initial topology, while the bornology $BT\mathcal{E}$ associated with a c.b.s. is coarser than its initial bornology. An l.c.s. is called bornological if $TB\mathcal{E}$ coincides with the initial topology of $\mathcal{E}$, and a c.b.s. is called topological if $\mathcal{E} = BT\mathcal{E}$.

3.2. Mackey Convergence and the Topology of M-Closure

Let $\mathcal{E}$ be a bornological vector space (b.v.s.), and $(x_n)_n$ a sequence in $\mathcal{E}$. The sequence $(x_n)_n$ is said to converge in the Mackey sense to $x \in \mathcal{E}$ if there exists a bounded set $\mathcal{B}$ in $\mathcal{E}$ and a decreasing sequence of real numbers $(\lambda_n)_n$ tending to 0 such that for all n, $x_n - x \in \lambda_n \mathcal{B}$.
When $\mathcal{E}$ is regarded as a c.b.s., the following result holds:

Proposition 3.1
A sequence $(x_n)_n$ *in a c.b.s.* $\mathcal{E}$ *converges to* $x \in \mathcal{E}$ *in the Mackey sense if and only if there exists a bounded disk* $\mathcal{B}$ *in* $\mathcal{E}$ *such that* $(x_n)_n$ *converges to* x *in the semi-normed space* $\mathcal{E}_{\mathcal{B}}$.

If $(x_n)_n$ and $(y_n)_n$ converge bornologically to x and y, respectively, and λ is a scalar, then $(x_n + y_n)_n$ and $(\lambda x_n)_n$ converge to $x + y$ and λx, respectively.

A bornological vector space (b.v.s.) is said to be separated if it contains no bounded lines. This is equivalent to stating that the limit of any M-convergent sequence is unique.

If $\mathcal{E}$ is a convex bornological space (c.b.s.), it is separated if, for every bounded disk $\mathcal{B}$, the space $\mathcal{E}_{\mathcal{B}}$ is a normed space.

If $\mathcal{E}$ is a topological vector space (t.v.s.), it is separated if and only if its Von Neumann bornology is separated.

Let $\mathcal{F}$ be a subspace of $\mathcal{E}$. The traces of bounded sets of $\mathcal{E}$ on $\mathcal{F}$ form a bornology on $\mathcal{F}$, called the induced bornology. In this case, $\mathcal{F}$ is called a bornological subspace of $\mathcal{E}$.

On the quotient $\mathcal{E}/\mathcal{F}$, the family $\{q(\mathcal{B}) \mid \mathcal{B} \text{ bounded in } \mathcal{E}\}$, where q is the canonical projection, forms a vector bornology on $\mathcal{E}$, called the quotient bornology. This bornology is separated if and only if $\mathcal{F}$ is b-closed.

A subset $\mathcal{B}$ of a b.v.s. is said to be M-closed or b-closed if $\mathcal{B} = \mathcal{B}^1$, where $\mathcal{B}^1$ is the set of Mackey limits in $\mathcal{E}$ of sequences in $\mathcal{B}$.

The collection of M-closed subsets of $\mathcal{E}$ defines a topology on $\mathcal{E}$, denoted $\tau\mathcal{E}$, called the topology of M-closure (or b-closure).

Let $\mathcal{E}$ be a b.v.s. The bornological closure of a subset $\mathcal{B}$ of $\mathcal{E}$ is the intersection of all bornologically closed subsets of $\mathcal{E}$ containing $\mathcal{B}$, denoted $\bar{\mathcal{B}}$. An element $x \in \bar{\mathcal{B}}$ is not necessarily the bornological limit of a sequence of points in $\mathcal{B}$.

A convex bornological space $\mathcal{E}$ is said to satisfy the M-closure property if, for every subset $\mathcal{B}$ of $\mathcal{E}$, we have $\mathcal{B}^{(1)} = \bar{\mathcal{B}}$.

3.3. Bounded Mappings

A mapping $f : (\mathcal{X}, \beta) \to (\mathcal{Y}, \beta_1)$ is said to be bounded if it satisfies $f(\beta) \subseteq \beta_1$. Furthermore, if β and β_1 are two bornologies on the set $\mathcal{X}$, β is said to be finer than β_1 when $\beta \subseteq \beta_1$. This is equivalent to stating that the identity mapping $\text{id} : (\mathcal{X}, \beta) \to (\mathcal{X}, \beta_1)$ is bounded.

3.3.1. Network Bornology.

Let $\mathcal{F}$ be a vector space. A network in $\mathcal{F}$ is a family R of disks in $\mathcal{F}$, indexed by $\mathbb{N}^{\mathbb{N}} : e_{n_1,n_2,\ldots,n_k}$, where $k, n_1, n_2, \ldots, n_k \in \mathbb{N}$, satisfying the following condition R:

$$r = \bigcup_{n_1=0}^{\infty} e_{n_1,n_2,\ldots,n_k}, \quad \text{and} \quad e_{n_1,n_2,\ldots,n_{k-1}} = \bigcup_{n_k=1}^{\infty} e_{n_1,n_2,\ldots,n_k}, \quad \forall k \geq 1.$$

If β is a convex and separated bornology on $\mathcal{F}$, β and R are said to be compatible if the following two properties hold:

- (βR_1): For every sequence of natural numbers $(n_k)_k$, there exists a sequence of positive real numbers $(v_k)_k$ such that for any $f_k \in e_{n_1,n_2,\ldots,n_k}$ and $\mu_k \in [0, v_k]$, the series $\sum \mu_k f_k$ is M-convergent in $(\mathcal{F}, \beta)$. Moreover, the sum of this series satisfies:

$$\sum_{k=k_0}^{\infty} \mu_k f_k \in e_{n_1,n_2,\ldots,n_0}, \quad \forall k_0 \in \mathbb{N}.$$

- (βR_2): For every pair $[(n_k), (v_k)]$, where $(n_k)_k \subset \mathbb{N}$ and $(v_k)_k \in (\mathbb{R}^+)^*$, the following set is bounded in $(\mathcal{F}, \beta)$:

$$\bigcap_{k=1}^{\infty} v_k e_{n_1,n_2,\ldots,n_k}.$$

Example 3.1
Every complete bornological vector space (c.b.s.) with a countable bornology basis is a network space.

Let $\mathcal{F}$ be a convex bornological space (c.b.s.) that is separated. $\mathcal{F}$ is said to have the M-closed graph property if any linear map from a Banach space $\mathcal{E}$ to $\mathcal{F}$, whose graph is M-closed in $\mathcal{E} \times \mathcal{F}$, is bounded. This is equivalent to assuming that $\mathcal{F}$ is a complete convex bornological space.
The main types of separated convex bornological spaces (c.b.s.) that possess the M-closed graph property are as follows:

1. *Network c.b.s. (in particular, complete c.b.s. with a countable basis).*

2. *Souslin c.b.s.*

Theorem 3.1
Let $\mathcal{E}$ be a complete convex bornological space (c.b.s.), and let $\mathcal{F}$ be a networked c.b.s. Any linear mapping $u : \mathcal{E} \to \mathcal{F}$ that is both bijective and bounded is a bornological isomorphism.

Proposition 3.2 *([3])*
Let $\mathcal{E}$ and $\mathcal{F}$ be two complete p-convex bornological spaces with $\mathcal{F}$ having a countable basis. Any linear mapping $u : \mathcal{E} \to \mathcal{F}$ whose graph is M-closed is bounded.

Proposition 3.3 *([3])*
Let $\mathcal{E}$ and $\mathcal{F}$ be two complete p-convex bornological spaces with $\mathcal{E}$ having a countable basis. Any bounded linear bijection $u : \mathcal{E} \to \mathcal{F}$ is a bornological isomorphism.

Theorem 3.2
Suppose $\mathcal{E}$ is a p-convex bornological space $(0 < p \leqslant 1)$. The following statements are equivalent:

- *$\mathcal{E}$ is complete;*
- *$\mathcal{E}$ is Mackey-complete, and its bornology is l^p-disked.*

3.4. Bornological Algebras

Let $\mathcal{E}$ be an algebra, and let β be a bornology on $\mathcal{E}$ such that $(\mathcal{E}, \beta)$ is a bornological vector space (b.v.s.).

$(\mathcal{E}, \beta)$ is called a bornological algebra if the multiplication map $(x, y) \mapsto x \cdot y$ is bounded. In other words, this means that the product of two bounded subsets of $(\mathcal{E}, \beta)$ is also bounded in β.

If $(\mathcal{E}, \beta)$ is a complete convex bornological space (c.b.s.) and the multiplication is bounded, then $(\mathcal{E}, \beta)$ is called a convex bornological algebra (c.b.a.).

A subset $A \subset \mathcal{E}$ of an algebra $\mathcal{E}$ is said to be idempotent if it satisfies $A^2 = A \cdot A \subset A$.

Recall that if β is a vectorial bornology defined on a vector space $\mathcal{E}$, a pseudo-basis of β is a subfamily $\beta' \subset \beta$ such that every element of β is absorbed by an element of β'.

A bornological algebra on $\mathcal{E}$ is said to be multiplicatively convex if it admits a pseudo-basis consisting of idempotent disks. A multiplicatively convex bornological algebra (abbreviated as m.c.b.a.) is a bornological algebra whose bornology is multiplicatively convex.

Let $\mathcal{E}$ be a bornological algebra, and let $\mathcal{E}^{\#}$ denote the unital algebra obtained by adjoining a unit to $\mathcal{E}$. Recall that $\mathcal{E}^{\#} = \mathbb{K} \times \mathcal{E}$, equipped with the usual addition and a multiplication defined by:

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab),$$

for all $(\lambda, a), (\mu, b) \in \mathbb{K} \times \mathcal{E}$.

The product bornology on $\mathcal{E}^{\#}$ is a bornological algebra, meaning it is compatible with the multiplication defined above. The algebra $\mathcal{E}^{\#}$, equipped with this bornology, is called the bornological algebra obtained by the bornological adjunction of a unit to $\mathcal{E}$.

If $\mathcal{E}$ is the inductive limit bornological algebra of the system $(\mathcal{E}_i, \pi_{ji})$, then $\mathcal{E}^{\#}$ is the inductive limit bornological algebra of the system $\left(\mathcal{E}_i^{\#}, \pi_{ji}\right)$.

Theorem 3.3
Let $\mathcal{E}$ be a c.b.a.
$\mathcal{E}$ is a multiplicatively convex bornological algebra (m.c.b.a.) if and only if it is the bornological inductive limit of semi-normed algebras.

Theorem 3.4
Let $\mathcal{E}$ be a a.b.c$_p$.
$\mathcal{E}$ is a multiplicatively p-convex bornological algebra (a.b.m.c$_p$) if and only if it is the inductive limit of p-semi-normable spaces.

Notations:
Let β be a pseudo-basis consisting of p-idempotent disks. We endow β with the following order: For $\mathcal{A}, \mathcal{B} \in \beta$, we say that $\mathcal{A} \geq \mathcal{B}$ if and only if there exists $\alpha > 0$ such that $\mathcal{A} \subset \alpha\mathcal{B}$. With this order, β is a preordered and upward-filtering set.

Furthermore, if $\pi_{\mathcal{B}\mathcal{A}}$ denotes the canonical injection from $\mathcal{E}_{\mathcal{A}}$ into $\mathcal{E}_{\mathcal{B}}$ for $\mathcal{A} \geq \mathcal{B}$, then:

$$\mathcal{E} = \varinjlim \left(\mathcal{E}_{\mathcal{A}}, \pi_{\mathcal{B}\mathcal{A}}\right).$$

Consider now a separated multiplicatively p-convex bornological algebra ($a.b.m.c_p$) $\mathcal{E} = \varinjlim \left(\mathcal{E}_{\mathcal{A}}, \pi_{\mathcal{B}\mathcal{A}}\right)$. In this case, for every $\mathcal{B} \in \beta$, $(\mathcal{E}_{\mathcal{B}}, P_{\mathcal{B}})$ is a p-normed algebra.

Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. The spectrum of a, denoted $\mathrm{Sp}(a)$, is defined as the subset of $\mathbb{C}$ given by:

$$\mathrm{Sp}(a) = \left\{\lambda \in \mathbb{C} \mid a - \lambda e_{\mathcal{A}} \text{ is not invertible}\right\}.$$

Let $\mathcal{E}$ be a complete and unital $a.b.m.c_p$ with unit e. One can choose a pseudo-basis β such that, for every $\mathcal{B} \in \beta$, $\mathcal{E}_{\mathcal{B}}$ is a unital Banach algebra with unit e. For any $x \in \mathcal{E}$, define:

$$\mathcal{I}(x) = \left\{\mathcal{B} \in \beta \mid x \in \mathcal{E}_{\mathcal{B}}\right\}.$$

Then:

$$\mathrm{Sp}_{\mathcal{E}}(x) = \bigcap_{\mathcal{B} \in \mathcal{I}(x)} \mathrm{Sp}_{\mathcal{E}_{\mathcal{B}}}(x),$$

where:

$$\mathrm{Sp}_{\mathcal{E}}(x) = \left\{\lambda \in \mathbb{C} \mid x - \lambda e \text{ is not invertible in } \mathcal{E}\right\},$$

and:

$$\mathrm{Sp}_{\mathcal{E}_{\mathcal{B}}}(x) = \left\{\lambda \in \mathbb{C} \mid x - \lambda e \text{ is not invertible in } \mathcal{E}_{\mathcal{B}}\right\}.$$

3.5. Separator Space

Definition 3.1 *([4])*
Let $\mathcal{X}$ and $\mathcal{Y}$ be two bornological vector spaces, and T a linear map from $\mathcal{X}$ to $\mathcal{Y}$. The separator space of T, denoted $\mathfrak{S}(T)$, is the subset of $\mathcal{Y}$ defined by:

$$\mathfrak{S}(T) = \left\{y \in \mathcal{Y} \mid \exists (x_n)_n \subset \mathcal{X} : x_n \xrightarrow{M} 0 \text{ and } T(x_n) \xrightarrow{M} y\right\}$$

Definition 3.2
Let $\mathcal{X}$ and $\mathcal{Y}$ be two bornological vector spaces, and T a linear map from $\mathcal{X}$ to $\mathcal{Y}$. The separator space of T in $\mathcal{X}$, denoted $\mathfrak{S}'(T)$, is the subset of $\mathcal{X}$ defined by:

$$\mathfrak{S}'(T) = \left\{x \in \mathcal{X} \mid \exists (x_n)_n \subset \mathcal{X} : x_n \xrightarrow{M} 0 \text{ and } T(x_n) \xrightarrow{M} T(x)\right\}$$

Proposition 3.4 *([4])*
Let $\mathcal{X}$ and $\mathcal{Y}$ be two p-convex bornological spaces. Then, the separator space $\mathfrak{S}(T)$ of any linear map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a b-closed vector subspace of $\mathcal{Y}$.

Proposition 3.5 *([4])*
Let $\mathcal{X}$ and $\mathcal{Y}$ be two bornological vector spaces and $T : \mathcal{X} \longrightarrow \mathcal{Y}$ a linear map. Then, we have:
i) $\mathfrak{S}(T) = \{0\}$ if and only if the graph of T is b-closed.
ii) If R and S are two bounded operators on $\mathcal{X}$ and $\mathcal{Y}$ respectively and if $TR = ST$, then:

$$S(\mathfrak{S}(T)) \subset \mathfrak{S}(T).$$

Corollary 3.1 *([4]) Let $\mathcal{X}$ be a complete e.b.c$_p$, $\mathcal{Y}$ a lattice e.b.c$_p$, and $T : \mathcal{X} \longrightarrow \mathcal{Y}$ a linear map. Then, T is bounded if and only if $\mathfrak{S}(T) = \{0\}$.*

3.6. Characterizations of Bounded Operators:

Proposition 3.6 *([4])*
Let $\mathcal{X}$ and $\mathcal{Y}$ be two e.b.c$_p$ of type M_1 and $\mathcal{Z}$ a separated bornological space. Assume that $\mathcal{X}$ is complete and that $\mathcal{Y}$ is a lattice. Let $S : \mathcal{X} \longrightarrow \mathcal{Y}$ and $R : \mathcal{Y} \longrightarrow \mathcal{Z}$ be bounded linear maps. Then, we have:
i) RS is bounded if and only if $R\mathfrak{S}(S) = \{0\}$.
ii) $[R\mathfrak{S}(S)]^{(1)} = \mathfrak{S}(RS)$.

Proposition 3.7
Let $\mathcal{X}$ and $\mathcal{Y}$ be two p-convex bornological vector spaces. Then, the separator space $\mathfrak{S}'(S)$ of any linear map $S : \mathcal{X} \longrightarrow \mathcal{Y}$ is a b-closed vector subspace of $\mathcal{X}$.

Proof:
$\mathfrak{S}'(S)$ is obviously a vector subspace of $\mathcal{X}$. Let $\mathfrak{S}(S)$ be the separator space of S in $\mathcal{Y}$ and $Q : \mathcal{Y} \longrightarrow \mathcal{Y}/\mathfrak{S}(S)$ be the canonical surjection. Since $Q(\mathfrak{S}(S)) = \{0\}$, it follows from the previous proposition that QS is bounded. We have:

$$\mathfrak{S}'(S) = S^{-1}|\mathfrak{S}(S)| = \mathrm{Ker}(QS) = (QS)^{-1}(\{0\})$$

Since $\mathfrak{S}(S)$ is b-closed, $\mathcal{Y}\mathfrak{S}(S)$ is separated. Therefore, $\mathfrak{S}'(S) = S^{-1}|\mathfrak{S}(S)|$ is b-closed in $\mathcal{X}$. □

Proposition 3.8 *([4])*
Let $\mathcal{X}$ and $\mathcal{Y}$ be two e.b.c$_p$ of type M_1 and $S : \mathcal{X} \longrightarrow \mathcal{Y}$ a linear map. Assume that $\mathcal{X}$ is complete and that $\mathcal{Y}$ is a lattice. Let $\mathcal{X}_0$ and $\mathcal{Y}_0$ be two b-closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$ respectively, with: $S(\mathcal{X}_0) \subset \mathcal{Y}_0$. Let $S_0 : \mathcal{X}/\mathcal{X}_0 \longrightarrow \mathcal{Y}/\mathcal{Y}_0$ be defined by: $S_0(x + \mathcal{X}_0) = S(x) + \mathcal{Y}_0$. Then, S_0 is bounded if and only if $\mathfrak{S}(S) \subset \mathcal{Y}_0$.

3.7. Automatic Boundedness

This section is dedicated to the study of the automatic boundedness of surjective (or dense image) homomorphisms in complete *a.b.m.c$_p$*s. In what follows, the algebras considered are assumed to be of type M_1.

Proposition 3.9
Let $\mathcal{X}$ and $\mathcal{Y}$ be two e.b.c$_p$ and $S : \mathcal{X} \longrightarrow \mathcal{Y}$ a linear map. Assume that $\mathcal{X}$ is complete and that $\mathcal{Y}$ is a lattice.
Let $\mathfrak{S}(S)$ be the separator space of S in $\mathcal{Y}$ (resp. $\mathfrak{S}'(S)$ be the separator space of S in $\mathcal{X}$).
Consider the map $S_0 : \mathcal{X}/\mathfrak{S}'(S) \longrightarrow \mathcal{Y}/\mathfrak{S}(S)$ defined by: $S_0(x + \mathfrak{S}'(S)) = S(x) + \mathfrak{S}(S)$ for all $x \in \mathcal{X}$. Thus, S_0 is bounded.

Proof:
Since $S^{-1}(\mathfrak{S}(S)) = \mathfrak{S}'(S)$, it follows that $S(\mathfrak{S}'(S)) \subset \mathfrak{S}(S)$. Moreover, since $\mathfrak{S}(S)$ (resp. $\mathfrak{S}'(S)$) is a b-closed vector subspace of $\mathcal{Y}$ (resp. of $\mathcal{X}$), it follows that S_0 is a bounded map (Proposition 3.8). □

Proposition 3.10
Let $\mathcal{A}$ and $\mathcal{B}$ be two e.b.c$_p$ spaces, and $S : \mathcal{A} \longrightarrow \mathcal{B}$ a linear map. Assume that $\mathcal{A}$ is complete and $\mathcal{B}$ is a lattice. Then, S is continuous if and only if $\mathfrak{S}(S) = \{0\}$.

Proof:
We prove that if S is continuous, then $\mathfrak{S}(S) = \{0\}$. If S is continuous, its graph is b-closed, so by Proposition 3.5, we have $\mathfrak{S}(S) = \{0\}$. Conversely, if $\mathfrak{S}(S) = \{0\}$, the graph of S is b-closed, implying that S is continuous by the closed graph theorem. □

Proposition 3.11
Let $\mathcal{A}$ and $\mathcal{B}$ be two complete a.b.m.c_p spaces, and let T be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If $b \in \mathfrak{S}(T)$, then $\mathrm{Sp}(b)$ *is a subset of* $\mathbb{C}$ *containing 0.*

Proof:
By contradiction, assume that $0 \notin \mathrm{Sp}(b)$.
Since $b \in \mathfrak{S}(T)$, there exists a sequence $(a_n)_n \subset \mathcal{A}$ such that:
$a_n \xrightarrow{M} 0$ in $\mathcal{A}$ and $T(a_n) \xrightarrow{M} b$ in $\mathcal{B}$.
Choose a compact neighborhood V of 0 in $\mathbb{C}$ such that $0 \notin \mathrm{Sp}(b) + V$.
Thus, for large enough n, we have $\mathrm{Sp}(a_n) \cap \mathrm{Sp}(b) + V = \emptyset$.
Now, $\mathrm{Sp}(T(a_n)) \subset \mathrm{Sp}(a_n)$, so for large enough n,
$\mathrm{Sp}(T(a_n)) \cap \mathrm{Sp}(b) + V = \emptyset$.
This contradicts Proposition (**??**). □

Definition 3.3
Let $\mathcal{E}$ and $\mathcal{F}$ be two complete e.b.c_p spaces, and let T be a linear map from $\mathcal{E}$ to $\mathcal{F}$. We say that T has a dense image in $\mathcal{F}$ if $(T(\mathcal{E}))^{(1)} = \mathcal{F}$.

Proposition 3.12
Let $\mathcal{A}$ and $\mathcal{B}$ be two complete a.b.m.c_p spaces, and let T be a surjective (or dense image) homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If $\mathcal{B}$ is simple and has a countable basis, then T is bounded.

Proof:
Let $\mathfrak{S}(T)$ be the separator ideal of T in $\mathcal{B}$, which is simple.
Thus, $\mathfrak{S}(T) = \{0\}$ or $\mathfrak{S}(T) = \mathcal{B}$.
If $\mathfrak{S}(T) = \mathcal{B}$, then $1_{\mathcal{B}} \in \mathfrak{S}(T)$.
Therefore, by Proposition 3.10, we have $0 \in \mathrm{Sp}(1_{\mathcal{B}})$, which is impossible.
Thus, $\mathfrak{S}(T) = \{0\}$.
Consequently, T is bounded. □

Theorem 3.5
*Let $\mathcal{A}$ and $\mathcal{B}$ be two complete a.b.m.c_p spaces, and let T be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ with a countable basis. If B is *-simple and if T is surjective (or has a dense image), then T is bounded.*

Proof:
Let β be a pseudo-base formed by completing and idempotent p-disks.
Since $\mathcal{B}$ is a *-simple algebra, there exists a simple unitary subalgebra $\mathcal{I}$ of $\mathcal{B}$ such that: $\mathcal{B} = \mathcal{I} \oplus \mathcal{I}^*$ (Proposition **??**).
From the following algebraic isomorphism: $\mathcal{I} \simeq \mathcal{B}/\mathcal{I}^*$, we deduce that $\mathcal{I}$ is a maximal ideal of $\mathcal{B}$.
Thus, $\mathcal{I}$ (resp. $\mathcal{I}^*$) is M-closed in $\mathcal{B}$ ([1] Proposition II-1.3).
Consequently, by Proposition (1, [2]), $\mathcal{I}$ (resp. $\mathcal{I}^*$) is a complete $e.b.c_p$ of type M_1 in $\mathcal{B}$.
Let $\beta_{\mathcal{I}}$ be the set defined by: $\beta_{\mathcal{I}} = \{\mathcal{B} \cap \mathcal{I} / \mathcal{B} \in \beta\}$.
$\beta_{\mathcal{I}}$ is a base of $a.b.m.c_p$ over $\mathcal{I}$.
Thus, $\mathcal{I}$ (resp. $\mathcal{I}^*$) is a complete sub-$a.b.m.c_p$ of type M_1 in $\mathcal{B}$.
Consider $Pr_1 : \mathcal{B} \longrightarrow \mathcal{I}$, the canonical projection of $\mathcal{B}$ onto $\mathcal{I}$.
And $Pr_2 : \mathcal{B} \longrightarrow \mathcal{I}^*$, the canonical projection of $\mathcal{B}$ onto $\mathcal{I}^*$.
Since Pr_1 (resp. Pr_2) is a bounded epimorphism (or has a dense image if T does),
by Proposition (3.6), we have that $Pr_1 \circ T$ (resp. $Pr_2 \circ T$) is bounded.
Consequently, $T = (Pr_1 + Pr_2) \circ T = Pr_1 \circ T + Pr_2 \circ T$ is bounded. □

Proposition 3.13
*Let $\mathcal{A}$ be a complete *-a.b.m.c_p space and $\mathcal{M}$ a *-maximal ideal of $\mathcal{A}$. Then, $\mathcal{M}$ is an M-closed ideal of $\mathcal{A}$.*

Proof:
If $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$, then $\mathcal{M}$ is M-closed (Proposition II-1.3 [1]).
Otherwise, there exists a maximal ideal $\mathcal{N}$ of $\mathcal{A}$ such that $\mathcal{M} = \mathcal{N} \cap \mathcal{N}^*$ (Proposition **??**).
Now, $\mathcal{N}$ (resp. $\mathcal{N}^*$) is M-closed.
Thus, $\mathcal{M}$ is M-closed in $\mathcal{A}$. □

4. Main Results

Theorem 4.1
*Let $\mathcal{A}$ and $\mathcal{B}$ be two complete a.b.m.c_p algebras and let T be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If $\mathcal{B}$ is *-semi-simple and if T is surjective (or has dense image), then T is bounded.*

Proof:
Let β be a pseudo-base consisting of completing and idempotent p-disks.
Let $\mathcal{M}$ be a *-maximal ideal of $\mathcal{B}$.
Then the involution $*$ induces an involution on $\mathcal{B}/\mathcal{M}$, also denoted by $*$, defined by: $(a+\mathcal{M})^* = a^* + \mathcal{M}$.
Let $\beta_{\mathcal{M}}$ be the set defined by: $\beta_{\mathcal{M}} = \{\mathcal{B} + \mathcal{M}/\mathcal{B} \in \beta\}$.
$\beta_{\mathcal{M}}$ is a pseudo-base formed of completing and idempotent p-disks on $\mathcal{B}/\mathcal{M}$.
Thus, $\mathcal{B}/\mathcal{M}$ is a *-$a.b.m.c_p$ *-semi-simple algebra.
Since $\mathcal{M}$ is M-closed (proposition 3.13), it follows that $\mathcal{B}/\mathcal{M}$ is a complete $e.b.c_p$ algebra ([2] proposition 2).
Consider the canonical surjection $Q : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{M}$.
Since Q is bounded, it follows from Theorem (3.5) that the homomorphism $Q \circ T$ is bounded.
Thus, $\mathfrak{S}(Q \circ T) = \{0\}$. Since $\mathfrak{S}(Q \circ T) = |Q(\mathfrak{S}(T))|^{(1)}$ (proposition 3.9),
we deduce that $Q(\mathfrak{S}(T)) = \{0\}$. Hence, $\mathfrak{S}(T) \subseteq \mathcal{M}$.
Since $\mathcal{M}$ is arbitrary, it follows that $\mathfrak{S}(T) \subseteq \bigcap \mathcal{M} = \mathrm{Rad}_*(\mathcal{B}) = \{0\}$.
Therefore, T is bounded. □

Corollary 4.1
*Let $\mathcal{A}$ be a *-semi-simple algebra. If β_1 and β_2 are two complete a.b.m.c_p bornologies with countable bases on $\mathcal{A}$, then $\beta_1 = \beta_2$.*

Proof:
It is sufficient to apply the previous theorem to the identity on $\mathcal{A}$. □

Corollary 4.2
*Let $(\mathcal{A}, \beta)$ be a complete *-a.b.m.c_p algebra with a countable *-simple basis. Then the involution * is automatically bounded.*

Proof:
Let the set β^* be defined by: $\beta^* = \{\mathcal{B}^*/\mathcal{B} \in \beta\}$.
β^* is a complete $a.b.m.c_p$ bornology on $\mathcal{A}$.
Indeed, let x be an element of $\mathcal{A}$, we can write $x = y + iz$, where y and z are self-adjoint elements of $\mathcal{A}$.
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be elements of β such that $y \in \mathcal{B}_1$ and $z \in \mathcal{B}_2$. We have, $y \in \mathcal{B}_1^*$ and $z \in \mathcal{B}_2^*$.
Thus, $x \in \mathcal{B}_1^* + i\mathcal{B}_2^* = (\mathcal{B}_1 - i\mathcal{B}_2)^*$, and since $\mathcal{B}_1 - i\mathcal{B}_2 \in \beta$, there exists an element $\mathcal{B}$ of β such that $\mathcal{B}_1 - i\mathcal{B}_2 = \mathcal{B}$.
This implies that: $x \in \mathcal{B}^*$.
Therefore, β^* covers $\mathcal{A}$.
Let $\mathcal{B}_1^*$ be an element of β^* and $\mathcal{B}_2$ a subset of $\mathcal{A}$ such that: $\mathcal{B}_2 \subset \mathcal{B}_1^*$.
We have $\mathcal{B}_2^* \subset \mathcal{B}_1$, and since $\mathcal{B}_1 \in \beta$ it follows that $\mathcal{B}_2^* \in \beta$ hence $\mathcal{B}_2 \in \beta^*$.
Let $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ be two elements of β^*. We show that $\mathcal{B}_1^* \cup \mathcal{B}_2^* \in \beta^*$.
We have $\mathcal{B}_1^* \cup \mathcal{B}_2^* \subseteq (\mathcal{B}_1 \cup \mathcal{B}_2)^*$.
Since $\mathcal{B}_1 \cup \mathcal{B}_2 \in \beta$ it follows that $(\mathcal{B}_1^* \cup \mathcal{B}_2^*)^* \in \beta$ thus $\mathcal{B}_1^* \cup \mathcal{B}_2^* \in \beta^*$.

We now show that β^* is a vector bornology.
Let $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ be two elements of β^* and $\lambda \in \mathbb{C}$.
We have: $\mathcal{B}_1^* + \mathcal{B}_2^* = (\mathcal{B}_1 + \mathcal{B}_2)^*$ and $\lambda \mathcal{B}_1^* = (\bar{\lambda}\mathcal{B}_1)^*$.
Since $\mathcal{B}_1 + \mathcal{B}_2$ and $\bar{\lambda}\mathcal{B}_2$ are in β, it follows that $\mathcal{B}_1^* + \mathcal{B}_2^*$ and $\lambda\mathcal{B}_1^*$ are in β^*. Thus, β^* is a vector bornology.
Let $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ be two elements of β^*. Then, $\mathcal{B}_1^*\mathcal{B}_2^* = (\mathcal{B}_2\mathcal{B}_1)^*$.
Since $\mathcal{B}_2\mathcal{B}_1 \in \beta$, it follows that $\mathcal{B}_1^*\mathcal{B}_2^* \in \beta^*$.
We still need to show that β^* is a complete $a.b.m.c_p$ bornology with a countable basis.
Let $\mathcal{B}$ be a completing and idempotent p-disk of β, then $\mathcal{B}^*$ is also a *-idempotent p-disk.
If $x \in \mathcal{A}_\mathcal{B}$ then we have:

$$P_\mathcal{B}(x) = \inf\{\lambda^p(\lambda \geqslant 0)/x \in \lambda\mathcal{B}\} = \inf\{\lambda^p(\lambda \geqslant 0)/x^* \in \bar{\lambda}\mathcal{B}^* = \lambda\mathcal{B}^*\} = P_{\mathcal{B}^*}(x^*)$$

Since $(\mathcal{A}_\mathcal{B}, P_\mathcal{B})$ is a Banach algebra, it follows that $(\mathcal{A}_{\mathcal{B}^*}, P_{\mathcal{B}^*})$ is also a Banach algebra.
It is easy to verify that β^* has a countable basis on $\mathcal{A}$.
By the previous theorem, $\beta = \beta^*$, therefore the involution * is bounded. □

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