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Stability Analysis of a Delayed SEIQR Epidemic Model with Diffusion and Elementary Saturated Incidence Rate

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ABSTRACT: The present study delves into the impact of delay and spatial diffusion on the dynamical behavior of the $SEIQR$ epidemic model. The inclusion of delay in this model renders it more realistic, modeling the latency period of the disease. Additionally, introducing diffusion into the $SEIQR$ model aims to provide better insight into the effects of spatial heterogeneity and individual mobility on disease persistence and extinction. Initially, we derived a threshold value $\mathcal{R}0$ for the delayed $SEIQR$ model with diffusion. Subsequently, using the theory of partial functional differential equations, we established that if $\mathcal{R}0 \leq 1$ and $\frac{\mu}{\alpha} > \Lambda$ and $\Lambda > \frac{1}{\alpha}$, the disease-free equilibrium is asymptotically stable, and no endemic equilibrium exists. In contrast, if $\mathcal{R}0 > \max\left(1, \frac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\,\eta id}\right)$ and $\frac{\mu}{\alpha} < \Lambda < \frac{1}{\alpha}$, a unique, asymptotically stable endemic equilibrium is present. Next, by constructing an appropriate Lyapunov function, we determined the threshold parameters that ensure the global asymptotic stability of the disease-free equilibrium. Finally, we illustrated the theoretical results through numerical simulations.

Key Words: SEIRQ epidemic model,Incidence rate, Ordinary differential equations, Delayed differential equations, Partial differential equations.

Contents

1. Introduction

Epidemiological models that incorporate a latent period or incubation period have been extensively studied in the literature due to the crucial role this period plays in the transmission dynamics of various diseases. During the incubation period, individuals are infected but not yet infectious. There are two widely used approaches to model this period: introducing a time delay (see [5]) or adding an exposed compartment (see [9]). Both approaches allow modeling the transmission process more realistically, and comparing their effects has been the subject of several works (see [11,1]).
In this work, we propose a delayed SEIQR epidemic model with spatial diffusion and saturated incidence

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function, extending and generalizing several previous delay and diffusion epidemic models presented in [2,3,1,27]. The model is governed by the following system of partial differential equations (PDEs):

$$\begin{cases} \dfrac{\partial \mathcal{S}(x,t)}{\partial t} = d\Delta \mathcal{S}(x,t) + A - \mu \mathcal{S}(x,t) - \dfrac{\beta \mathcal{S}(x,t)(\mathcal{I}(x,t) + q\mathcal{E}(x,t))}{1+\alpha \mathcal{S}(x,t)} \\ \dfrac{\partial \mathcal{E}(x,t)}{\partial t} = d\Delta \mathcal{E}(x,t) + \dfrac{\beta e^{-\mu\tau} \mathcal{S}(x,t-\tau)(\mathcal{I}(x,t-\tau) + q\mathcal{E}(x,t-\tau))}{1+\alpha \mathcal{S}(x,t-\tau)} - (\sigma+\mu)\mathcal{E}(x,t), \\ \dfrac{\partial \mathcal{I}}{\partial t}(x,t) = d\Delta \mathcal{I}(x,t) + \sigma \mathcal{E}(x,t) - (\gamma + \delta_q + \epsilon + \mu)\,\mathcal{I}(x,t), \\ \dfrac{\partial \mathcal{Q}}{\partial t}(x,t) = d\Delta \mathcal{Q}(x,t) + \delta_q \mathcal{I}(x,t) - (\gamma_q + \mu)\,\mathcal{Q}(x,t), \\ \dfrac{\partial \mathcal{R}}{\partial t}(x,t) = d\Delta \mathcal{R}(x,t) + \gamma \mathcal{I}(x,t) + \gamma_q \mathcal{Q}(x,t) - \mu \mathcal{R}(x,t). \end{cases} \tag{1.1}$$

Here, the variables $\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}$ denote, respectively, the number of susceptible, exposed, infectious symptomatic, quarantined (or isolated) infected, and recovered individuals at location x and time t. The Laplacian operator Δ models spatial diffusion, and the parameters d, μ, β, α, q, σ, γ, δ_q, γ_q, ϵ, A all have standard epidemiological meanings, as detailed in [1,11,12,22].
Several pioneering works form the theoretical foundation for our analysis. Classical epidemic models without delay were introduced in [12], while delay systems modeling incubation were developed by Cooke [5]. Diffusive epidemic models were treated in [6,23,24], and the combined effect of delay and diffusion was first investigated in [1,2,3,27]. Mathematical techniques related to the well-posedness of such systems in Banach spaces can be found in [7,8,13,14,16,17,19,25]. More recently, works such as [22] refined the understanding of the basic reproduction number $\mathcal{R}_0$ in structured populations.
The importance of saturated incidence rates has also been widely documented in the literature (see [4,26]). Such incidence functions account for behavioral changes in response to increasing disease prevalence, capturing saturation effects when susceptible individuals adjust their contact patterns. This approach has been particularly emphasized in models of vector-borne diseases [4].
The boundary and initial conditions associated with system (1.1) are given by:

$$\begin{cases} \mathcal{S}(x,t) = \varphi_1(x,t) \geq 0, \\ \mathcal{E}(x,t) = \varphi_2(x,t) \geq 0, \\ \mathcal{I}(x,t) = \varphi_3(x,t) \geq 0, \\ \mathcal{Q}(x,t) = \varphi_4(x,t) \geq 0, \\ \mathcal{R}(x,t) = \varphi_5(x,t) \geq 0, \end{cases} \quad \text{for } (x,t) \in \bar{\Omega} \times [-\tau, 0], \tag{1.2}$$

$$\frac{\partial \mathcal{S}}{\partial v} = \frac{\partial \mathcal{E}}{\partial v} = \frac{\partial \mathcal{I}}{\partial v} = \frac{\partial \mathcal{Q}}{\partial v} = \frac{\partial \mathcal{R}}{\partial v} = 0,\ t \geq 0,\ x \in \partial\Omega, \tag{1.3}$$

where $\frac{\partial}{\partial v}$ denotes the outward normal derivative, ensuring no flux across the boundary, meaning no individuals cross the domain boundary (see [15,19]).

The present work builds upon this extensive literature, combining delays, spatial diffusion, and saturated incidence functions into a unified SEIQR epidemic model. Our main focus is to establish conditions for:
- Well-posedness and existence of equilibrium (Section 2),
- Local stability of the disease-free and endemic equilibria (Section 3),
- Global stability of the disease-free equilibrium via a suitable Lyapunov functional (Section 4),
- Numerical simulations to illustrate theoretical results (Section 5).

This contribution extends prior works such as [1,27,2,3], incorporating spatial diffusion and delayed transmission within a fully nonlinear framework that can better capture spatial heterogeneity and mobility effects. By doing so, we contribute to the ongoing development of realistic epidemiological modeling for emerging infectious diseases.

2. The well-posedness and existence of equilibrium

2.1. The well-posedness

This subsection is concerned with the well-posedness of solutions for (1.1), which requires establishing the global existence, uniqueness, non-negativity, and boundedness of solutions. In order to do so, we introduce some notation. We let $\mathbb{X}$ denote the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^5$, and $\mathcal{C}\mathbb{X}$ represent the Banach space comprising continuous functions mapping $[-\tau, 0]$ to $\mathbb{X}$, where $\mathbb{X}$ denotes a specific mathematical object. This space is equipped with the supremum norm. Given $a \leq b$ and $t \in [a, b]$, and considering a continuous function $u : [a - \tau, b] \to \mathbb{X}$, we can define u_t as an element of $\mathcal{C}\mathbb{X}$. This element is obtained by evaluating u_t at $\theta \in [-\tau, 0]$, where $u_t(\theta)$ is given by $u(t + \theta)$. Furthermore, we identify any element $\varphi \in \mathcal{C}_{\mathbb{X}}$ as a function from $\overline{\Omega} \times [-\tau, 0]$ to $\mathbb{R}^5$ defined by $\varphi(x, t) = \varphi(t)(x)$.
To establish the existence and uniqueness of a global positive solution, we present the following:

Theorem 2.1 *The uniqueness, non-negativity, and global existence of a solution for the system (1.1)-(1.3) can be established for any initial condition $\varphi \in \mathcal{C}_{\mathbb{X}}$ that satisfies (1.2). The solution obtained is also nonnegative and remains so globally.*

Proof: Let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathcal{C}_{\mathbb{X}}$, and let $x \in \overline{\Omega}$. Define the nonlinear term $g = (g_1, g_2, g_3, g_4, g_5) : \mathcal{C}_{\mathbb{X}} \to \mathbb{X}$ by:

$$\begin{aligned}
g_1(\varphi)(x) &= A - \mu\varphi_1(x,0) - \frac{\beta\varphi_1(x,0)\left(\varphi_3(x,0) + q\varphi_2(x,0)\right)}{1 + \alpha\varphi_1(x,0)},\\
g_2(\varphi)(x) &= \frac{\beta\varphi_1(x,-\tau)\left(\varphi_3(x,-\tau) + q\varphi_2(x,-\tau)\right)}{1 + \alpha\varphi_1(x,-\tau)} - (\mu+\sigma)\varphi_2(x,0),\\
g_3(\varphi)(x) &= \sigma\varphi_2(x,0) - (\mu + \delta_q + \epsilon + \gamma)\varphi_3(x,0),\\
g_4(\varphi)(x) &= \delta_q\varphi_3(x,0) - (\mu + \gamma_q)\varphi_4(x,0),\\
g_5(\varphi)(x) &= \gamma\varphi_3(x,0) + \gamma_q\varphi_4(x,0) - \mu\varphi_5(x,0).
\end{aligned}$$

The system (1.1)-(1.3) can then be reformulated as an abstract differential equation in the phase space $\mathcal{C}_{\mathbb{X}}$:

$$\begin{cases} \dot{u} = Bu + g\left(u_t\right), \quad t \geq 0,\\ u(0) = \varphi \in \mathcal{C}_{\mathbb{X}}, \end{cases} \tag{2.1}$$

where:

$$u(t) = \begin{bmatrix} \mathcal{S}(.,t)\\ \mathcal{E}(.,t)\\ \mathcal{I}(.,t)\\ \mathcal{Q}(.,t)\\ \mathcal{R}(.,t) \end{bmatrix}, \quad Bu = \begin{bmatrix} d\Delta\mathcal{S}\\ d\Delta\mathcal{E}\\ d\Delta\mathcal{I}\\ d\Delta\mathcal{Q}\\ d\Delta\mathcal{R} \end{bmatrix}.$$

It can be shown that g is locally Lipschitz in $\mathcal{C}_{\mathbb{X}}$. By the theory of semilinear parabolic equations with delay (see [7,13,14,21,25]), system (2.1) admits a unique local solution on a maximal interval $[0, t_{\max})$.
Nonnegativity of the Solution
The zero function:

$$0 = (0, 0, 0, 0, 0)$$

acts as a lower solution for system (1.1)-(1.3). By the comparison principle for parabolic systems (see [9]), the solution components satisfy:

$$\mathcal{S}(x,t), \mathcal{E}(x,t), \mathcal{I}(x,t), \mathcal{Q}(x,t), \mathcal{R}(x,t) \geq 0 \quad \forall (x,t) \in \overline{\Omega} \times [0, t_{\max}).$$

A Priori Estimates (Uniform Boundedness)
Step 1: Upper bound for $\mathcal{S}$
From the first equation in (1.1):

$$\frac{\partial \mathcal{S}}{\partial t} - d\Delta\mathcal{S} \leq A - \mu\mathcal{S}$$

This can be compared to the ordinary differential equation:

$$\frac{d\tilde{\mathcal{S}}}{dt} = A - \mu\tilde{\mathcal{S}}, \quad \tilde{\mathcal{S}}(0) = \max_{x\in\overline{\Omega}} \varphi_1(x,0)$$

which has the explicit solution:

$$\tilde{\mathcal{S}}(t) = \tilde{\mathcal{S}}(0)e^{-\mu t} + \frac{A}{\mu}(1 - e^{-\mu t})$$

This implies:

$$\mathcal{S}(x,t) \leq \max\left\{\frac{A}{\mu}, \max_{x\in\overline{\Omega}} \varphi_1(x,0)\right\}$$

Step 2: Upper bound for the total population

Define:

$$L(x,t) = e^{-\mu\tau}\mathcal{S}(x,t-\tau) + \mathcal{E}(x,t) + \mathcal{I}(x,t) + \mathcal{Q}(x,t) + \mathcal{R}(x,t)$$

which satisfies:

$$\frac{\partial L}{\partial t} - d\Delta L \leq e^{-\mu\tau}A - \mu L$$

This can be compared to the simpler ODE:

$$\frac{d\tilde{L}}{dt} = e^{-\mu\tau}A - \mu\tilde{L}, \quad \tilde{L}(0) = \max_{x\in\overline{\Omega}} L(x,0)$$

This gives:

$$L(x,t) \leq \max\left\{\frac{e^{-\mu\tau}A}{\mu}, \max_{x\in\overline{\Omega}} L(x,0)\right\}$$

Since all components are uniformly bounded, the solution can be extended globally, i.e., $t_{\max} = +\infty$. This completes the proof. □

2.2. Existence of equilibrium

In this subsection, we aim to determine the equilibrium of the $\mathcal{SEIRQ}$ models. To achieve this, we will solve the following system:

$$\begin{cases} \Lambda - \dfrac{\beta\mathcal{S}(I+q\mathcal{E})}{1+\alpha\mathcal{S}} - \mu\mathcal{S} = 0, \\ \dfrac{\beta\mathcal{S}(\mathcal{I}+q\mathcal{E})e^{-\mu\tau}}{1+\alpha\mathcal{S}} - (\sigma+\mu)\mathcal{E} = 0, \\ \sigma\mathcal{E} - (\gamma+\delta_q+\epsilon+\mu)\,\mathcal{I} = 0, \\ \gamma\mathcal{I} + \gamma_q\mathcal{Q} - \mu\mathcal{R} = 0, \\ \delta_q\mathcal{I} - (\gamma_q+\mu)\,\mathcal{Q} = 0. \end{cases}$$

Then the disease-free equilibrium is defined as follows:

$$\mathcal{P}_0 = \left(\mathcal{S}^0, \mathcal{E}^0, \mathcal{I}^0, \mathcal{Q}^0, \mathcal{R}^0\right)$$

where $\mathcal{E}^0 = \mathcal{I}^0 = \mathcal{Q}^0 = \mathcal{R}^0 = 0$ and $\mathcal{S}^0 = \dfrac{\Lambda}{\mu}$.

Furthermore, the system (1.1) has a unique endemic equilibrium

$$\mathcal{P}^* = \left(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*\right),$$

where

$$\begin{cases} \mathcal{S}^* = \dfrac{\Lambda}{\mu\mathcal{R}_0 + \alpha\Lambda(\mathcal{R}_0 - 1)}, \\ \mathcal{I}^* = \dfrac{\sigma((1-\alpha\Lambda)(1+\alpha\Lambda) + \mathcal{R}_0(\mu - \alpha\Lambda)(\mu + \alpha\Lambda))}{\beta(\mu\mathcal{R}_0 + \alpha\Lambda(\mathcal{R}_0 - 1))(\sigma + q(\mu + \epsilon + \gamma + \delta_q))}, \\ \mathcal{E}^* = \dfrac{\mu + \gamma + \epsilon + \delta_q}{\sigma}\mathcal{I}^* \\ \mathcal{Q}^* = \dfrac{\delta_q}{\gamma_q + \mu}\mathcal{I}^*, \\ \mathcal{R}^* = \dfrac{\gamma\left(\gamma_q + \mu\right) + \gamma_q\delta_q}{\mu\left(\mu + \gamma_q\right)}\mathcal{I}^*. \end{cases} \tag{2.2}$$

Where the basic reproduction number is giving by:

$$\mathcal{R}_0 = \frac{\beta\Lambda e^{-\mu\tau}\left(q\left(\gamma + \delta_q + \mu\right) + \sigma\right)}{\left(\mu + \alpha\Lambda\right)\left(\sigma + \mu\right)\left(\gamma + \delta_q + \mu\right)}.$$

Then, $\mathcal{P}^* = (\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*)$ exist if $\mathcal{R}_0 > 1$ and $\dfrac{\mu}{\alpha} < \Lambda < \dfrac{1}{\alpha}$.

3. Local stability of the equilibria for the SEIRQ models

We define the perturbations of the variables as $\tilde{\mathcal{S}} = \mathcal{S} - \mathcal{S}^*$, $\tilde{\mathcal{E}} = \mathcal{E} - \mathcal{E}^*$, $\tilde{\mathcal{I}} = \mathcal{I} - \mathcal{I}^*$, $\tilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}^*$ and $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}^*$, where $(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*)^\top$ is any arbitrary equilibrium point. For simplicity, we will drop the bars. By doing so, the system (1.1) can be expressed in the following form:

$$\begin{cases} \dfrac{\partial\mathcal{S}}{\partial t}(x,t) = d\Delta\mathcal{S}(x,t) + \Lambda - \beta\dfrac{(\mathcal{S}(x,t) + \mathcal{S}^*)((\mathcal{I}(x,t) + \mathcal{I}^*) + q(\mathcal{E}(x,t) + \mathcal{E}^*))}{1 + \alpha\left(\mathcal{S}(x,t) + \mathcal{S}^*\right)} - \mu(\mathcal{S}(x,t) + \mathcal{S}^*), \\ \dfrac{\partial\mathcal{E}}{\partial t}(x,t) = d\Delta\mathcal{E}(x,t) + e^{-\mu\tau}\beta\dfrac{(\mathcal{S}(x,t-\tau) + \mathcal{S}^*)((\mathcal{I}(x,t-\tau) + \mathcal{I}^*) + q(\mathcal{E}(x,t-\tau) + \mathcal{E}^*))}{1 + \alpha\left(\mathcal{S}(x,t-\tau) + \mathcal{S}^*\right)} \\ \qquad -(\sigma + \mu)(\mathcal{E}(x,t) + \mathcal{E}^*), \\ \dfrac{\partial\mathcal{I}}{\partial t}(x,t) = d\Delta\mathcal{I}(x,t) + \sigma(\mathcal{E}(x,t) + \mathcal{E}^*) - \left(\gamma + \delta_q + \epsilon + \mu\right)\left(\mathcal{I}(x,t) + \mathcal{I}^*\right), \\ \dfrac{\partial\mathcal{Q}}{\partial t}(x,t) = d\Delta\mathcal{Q}(x,t) + \delta_q(\mathcal{I}(x,t) + \mathcal{I}^*) - \left(\gamma_q + \mu\right)\left(\mathcal{Q}(x,t) + \mathcal{Q}^*\right), \\ \dfrac{\partial\mathcal{R}}{\partial t}(x,t) = d\Delta\mathcal{R}(x,t) + \gamma(\mathcal{I}(x,t) + \mathcal{I}^*) + \gamma_q(\mathcal{Q}(x,t) + \mathcal{Q}^*) - \mu(\mathcal{R}(x,t) + \mathcal{R}^*). \end{cases} \tag{3.1}$$

Thus, the arbitrary equilibrium point $\mathcal{P}^* = (\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*)^\top$ of the system (1.1) can be transformed into the zero equilibrium point $(0,0,0,0,0)^\top$ of the system (3.1).
Next, we will investigate the stability of the zero equilibrium point of the system (3.1). Let $u(t) = (\mathcal{S}(.,t), \mathcal{E}(.,t), \mathcal{I}(.,t), \mathcal{Q}(.,t), \mathcal{R}(.,t))^\top$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathcal{C}\mathrm{X}$. The system (3.1) can be reformulated as an abstract differential equation in the phase space $\mathcal{C}\mathrm{X}$ with the following structure:

$$\dot{u}(t) = D\Delta u(t) + L\left(u_t\right) + g\left(u_t\right), \tag{3.2}$$

where $D = \mathrm{diag}\{d,d,d,d,d\}$, and the linear operator $L : \mathcal{C}_{\mathbb{X}} \to \mathbb{X}$ as well as the function $g : \mathcal{C}_\chi \to \mathbb{X}$ are given, respectively, by

$$L(\varphi)(x) = \begin{pmatrix} -\left(\mu + \dfrac{\beta\left(\mathcal{I}^* + q\mathcal{E}^*\right)}{\left(1 + \alpha\mathcal{S}^*\right)^2}\right)\varphi_1(x,0) - \dfrac{\beta\mathcal{S}^*\left(\varphi_3(x,0) + q\varphi_2(x,0)\right)}{\left(1 + \alpha\mathcal{S}^*\right)} \\ \dfrac{\beta e^{-\mu\tau}\left(\mathcal{I}^* + q\mathcal{E}^*\right)}{\left(1 + \alpha\mathcal{S}^*\right)^2}\varphi_1(x,-\tau) + \dfrac{\beta e^{-\mu\tau}\mathcal{S}^*\left(\varphi_3(x,-\tau) + q\varphi_2(x,-\tau)\right)}{\left(1 + \alpha S^*\right)} - (\mu + \sigma)\varphi_2(x,0) \\ \sigma\varphi_2(x,0) - (\mu + \delta_q + \epsilon + \gamma)\varphi_3(x,0) \\ \delta_q\varphi_3(x,0) - (\mu + \gamma_q)\varphi_4(x,0) \\ \gamma\varphi_3(x,0) + \gamma_q\varphi_4(x,0) - \mu\varphi_5(x,0) \end{pmatrix},$$

and

$$g(\varphi)(x)=\begin{pmatrix} g_1(\varphi)(x)\\ g_2(\varphi)(x)\\ g_3(\varphi)(x)\\ g_4(\varphi)(x)\\ g_5(\varphi)(x)\end{pmatrix},$$

where

$$\begin{cases} g_1(\varphi)(x) = & \dfrac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\varphi_1(x,0)+\dfrac{\beta\mathcal{S}^*\left(\varphi_3(x,0)+q\varphi_2(x,0)\right)}{\left(1+\alpha\mathcal{S}^*\right)} \\ & +\Lambda-\dfrac{\beta\left(\varphi_1(x,0)+\mathcal{S}^*\right)\left(\left(\varphi_3(x,0)+\mathcal{I}^*\right)+q\left(\varphi_2(x,0)+\mathcal{E}^*\right)\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}-\mu\mathcal{S}^*, \\ g_2(\varphi)(x) = & -\dfrac{\beta e^{-\mu\tau}\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\varphi_1(x,0)-\dfrac{\beta e^{-\mu\tau}\mathcal{S}^*\left(\varphi_3(x,0)+q\varphi_2(x,0)\right)}{\left(1+\alpha\mathcal{S}^*\right)} \\ & +\dfrac{\beta e^{-\mu\tau}\left(\varphi_1(x,0)+\mathcal{S}^*\right)\left(\left(\varphi_3(x,0)+I^*\right)+q\left(\varphi_2(x,0)+\mathcal{E}^*\right)\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}-(\mu+\sigma)\mathcal{E}^*, \\ g_3(\varphi)(x) = & \sigma\mathcal{E}^*-(\mu+\gamma+\epsilon+\delta_q)\mathcal{I}^*, \\ g_4(\varphi)(x) = & \delta_q\mathcal{I}^*-(\mu+\gamma_q)\mathcal{Q}^*, \\ g_5(\varphi)(x) = & \gamma\mathcal{I}^*+\gamma_q\mathcal{Q}^*-\mu\mathcal{R}^*. \end{cases} \tag{3.3}$$

For $\varphi=u_t, \varphi=\left(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right)^\top\in\mathcal{C}_\times$. The linearized system of (3.2) at the zero equilibrium point is given by

$$\dot{u}=D\Delta u(t)+L\left(u_t\right),$$

and its characteristic equation is

$$\lambda\omega-D\Delta\omega-L\left(e^{\lambda\cdot}\omega\right)=0, \tag{3.4}$$

where $\omega\in\operatorname{dom}(\Delta)$, $\omega\neq 0$, and $\operatorname{dom}(\Delta)\subset\mathbb{X}$.

Let $0=\eta_0<\eta_1<\cdots$ be the sequence of eigenvalues for the elliptic operator $-\Delta$ with Neumann boundary condition on Ω, and $E\left(\eta_i\right)$ be the eigenspace corresponding to η_i in $L^2(\Omega)$.
We can find an orthonormal basis $\left\{\phi_{ij}, j=1,\ldots,\dim E\left(\eta_i\right)\right\}$ of $E\left(\eta_i\right)$, and $\mathbb{Y}_{ij}=\left\{a\phi_{ij}, a\in\mathbb{R}\right\}$. Then, we have

$$L^2(\Omega)=\bigoplus_{i=0}^{+\infty}\mathbb{Y}_i \text{ and } \mathbb{V}_i=\bigoplus_{j=1}^{\dim E(\eta_i)}\mathbb{Y}_{ij}.$$

Moreover, we put

$$\begin{cases} \beta_{ij}^1=\begin{pmatrix}\phi_{ij}\\0\\0\\0\\0\end{pmatrix}, \quad \beta_{ij}^2=\begin{pmatrix}0\\\phi_{ij}\\0\\0\\0\end{pmatrix}, \quad \beta_{ij}^3=\begin{pmatrix}0\\0\\\phi_{ij}\\0\\0\end{pmatrix}, \quad \beta_{ij}^4=\begin{pmatrix}0\\0\\0\\\phi_{ij}\\0\end{pmatrix}, \\ \textit{and} \qquad \beta_{ij}^5=\begin{pmatrix}0\\0\\0\\0\\\phi_{ij}\end{pmatrix}, i=0,1,2,\ldots, j=1,2,\ldots,\dim E\left(\eta_i\right). \end{cases} \tag{3.5}$$

It is evident that the set $\left(\beta_{ij}^1,\beta_{ij}^2,\beta_{ij}^3,\beta_{ij}^4,\beta_{ij}^5\right)$ forms a basis for $\left(L^2(\Omega)\right)^5$. Hence, any element ω of $\mathbb{X}$ can be written in the in the following form

$$\begin{aligned} \omega &= \left(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\right) \\ &= \sum_{i=0}^{+\infty}\sum_{j=1}^{\dim E(\eta_i)}\left\langle\omega_1,\phi_{ij}\right\rangle\beta_{ij}^1+\left\langle\omega_2,\phi_{ij}\right\rangle\beta_{ij}^2+\left\langle\omega_3,\phi_{ij}\right\rangle\beta_{ij}^3+\left\langle\omega_4,\phi_{ij}\right\rangle\beta_{ij}^4+\left\langle\omega_5,\phi_{ij}\right\rangle\beta_{ij}^5. \end{aligned} \tag{3.6}$$

Subsequently, through a simple analysis and by utilizing (3.5) and (3.6), we demonstrate that (3.4) is equivalent to

$$(\lambda I_5 + \eta_i D - M)\begin{pmatrix} \langle \omega_1, \phi_{ij}\rangle \\ \langle \omega_2, \phi_{ij}\rangle \\ \langle \omega_3, \phi_{ij}\rangle \\ \langle \omega_4, \phi_{ij}\rangle \\ \langle \omega_5, \phi_{ij}\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, i = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, \dim E(\eta_i), \tag{3.7}$$

where M is given by

$$M = \begin{pmatrix} -\mu - \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2} & -\frac{\beta q\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)} & -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)} & 0 & 0 \\ \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}e^{-\mu\tau}e^{-\lambda\tau} & -(\mu+\sigma) + \frac{\beta q\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}e^{-\lambda\tau} & \frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}e^{-\lambda\tau} & 0 & 0 \\ 0 & \sigma & -(\mu+\gamma+\epsilon+\delta_q) & 0 & 0 \\ 0 & 0 & \delta_q & -(\mu+\gamma_q) & 0 \\ 0 & 0 & \gamma & \gamma_q & -\mu \end{pmatrix}.$$

Thus the characteristic equation is

$$(\lambda + d\eta_i + \mu)(\lambda + d\eta_i + \mu + \delta_q)\left(\lambda^3 + a\lambda^2 + b\lambda + c + (x\lambda^2 + y\lambda + z)e^{-\lambda\tau}\right) = 0, \quad i = 0, 1, \ldots \tag{3.8}$$

where

$$\begin{aligned}
a &= 3(\eta_i d + \mu) + \sigma + \gamma + \epsilon + \delta_q + \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}, \\
b &= (\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}) + (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(2(\eta_i d + \mu) + \sigma + \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}), \\
c &= (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}), \\
x &= -\frac{q\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}, \\
y &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma + q(\eta_i d + \mu)), \\
z &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(\eta_i d + \mu)(q(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma).
\end{aligned}$$

3.1. Stability of disease-free equilibrium P

Based on the aforementioned analysis, in this section, we set $(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*) = P = \left(\frac{A}{\mu}, 0, 0, 0, 0\right)$. As a result, the characteristic equation (3.8) takes the form of $i = 0, 1, \ldots$

$$(\lambda + d\eta_i + \mu + \delta_q)(\lambda + d\eta_i + \mu)^2\left[\lambda^2 + \lambda(B + C) + BC - e^{-\lambda\tau}\frac{\beta\Lambda e^{-\mu\tau}}{\mu + \alpha\Lambda}(\lambda q + qC + \sigma)\right] = 0, \tag{3.9}$$

where

$$\begin{cases} B = d\eta_i + \mu + \sigma, \\ C = d\eta_i + \mu + \delta_q + \gamma + \epsilon. \end{cases}$$

Theorem 3.1 *Assuming $\mathcal{R}_0 \le 1$, the disease-free equilibrium P is said to be locally asymptotically stable for all $\tau \ge 0$.*

Proof: When τ is equal to zero, the equation (3.9) can be expressed as a cubic equation as follows:

$$(\lambda + d\eta_i + \mu)^2(\lambda + d\eta_i + \mu + \delta_q)\left[\lambda^2 + \lambda(C + B - \frac{q\beta\Lambda}{\mu + \alpha\Lambda}) + BC - \frac{\beta\Lambda}{\mu + \alpha\Lambda}(qC + \sigma)\right] = 0, \ i = 0, 1, \ldots \tag{3.10}$$

where

$$\begin{cases} B = d\eta_i + \mu + \sigma, \\ C = d\eta_i + \mu + \delta_q + \gamma + \epsilon. \end{cases}$$

As $\mathcal{R}_0 \leq 1$, we have

$$C + B - \frac{q\beta\Lambda}{\mu + \alpha\Lambda} = C + d\eta_i + \frac{q(\mu+\sigma)(\mu+\delta_q+\gamma+\epsilon)(1-\mathcal{R}_0) + (\mu+\sigma)\sigma}{q(\mu+\delta_q+\gamma+\epsilon)+\sigma} > 0,$$

and

$$\begin{aligned} CB - \frac{\beta\Lambda}{\mu+\alpha\Lambda}(qC+\sigma) = (d\eta_i)^2 + d\eta_i \left[\mu+\gamma+\epsilon+\delta_q + \frac{q(\mu+\sigma)(\mu+\delta_q+\gamma+\epsilon)(1-\mathcal{R}_0)+(\mu+\sigma)\sigma}{q(\mu+\delta_q+\gamma+\epsilon)+\sigma}\right] \\ + (\mu+\sigma)(\mu+\gamma+\epsilon+\delta_q)(1-\mathcal{R}_0) > 0. \end{aligned}$$

Based on the Routh-Hurwitz criteria, it can be concluded that all the roots of equation (3.10) possess negative real values. Hence, the disease-free equilibrium point P is deemed locally asymptotically stable when τ is equal to zero.

Furthermore, if instability occurs for a specific value of the delay τ, it signifies that a characteristic root of (3.9) intersects with the imaginary axis. In case (3.9) contains a purely imaginary root $i\omega$, where $\omega > 0$, separating the real and imaginary parts of (3.9) yields:

$$\begin{cases} \omega^2 - BC = q\omega\dfrac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\sin(\omega\tau) + \dfrac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}(qC+\sigma)\cos(\omega\tau), \\ -\omega(C+B) = q\omega\dfrac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\cos(\omega\tau) - \dfrac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}(qC+\sigma)\sin(\omega\tau). \end{cases} \tag{3.11}$$

By squaring both sides of the equations in (3.11) and subsequently adding them together, we obtain:

$$\omega^4 + \left(C^2 + B^2 - \left(\frac{q\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\right)^2\right)\omega^2 + (BC)^2 - \left(\frac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}(qC+\sigma)\right)^2 = 0. \tag{3.12}$$

It is easy to see that $BC - \dfrac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}(qC+\sigma) > 0$, and as $\mathcal{R}_0 \leq 1$, we deduce that

$$(BC)^2 - \left(\frac{\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}(qC+\sigma)\right)^2 > 0.$$

Moreover, as $\mathcal{R}_0 \leq 1$, we have

$$\begin{aligned} C^2 + B^2 - \left(\frac{q\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\right)^2 &= C^2 + \left(B + \left(\frac{q\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\right)\right)\left(B - \left(\frac{q\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\right)\right) \\ &= C^2 \\ &+ \left(B + \left(\frac{q\beta\Lambda e^{-\mu\tau}}{\mu+\alpha\Lambda}\right)\right)\left(d\eta_i + \frac{q(\mu+\sigma)(\mu+\delta_q+\epsilon+\gamma)(1-\mathcal{R}_0)+(\mu+\sigma)\sigma}{q(\mu+\delta_q+\epsilon+\gamma)+\sigma}\right) \\ &> 0. \end{aligned}$$

Thus, it can be deduced that equation (3.11) has no positive roots and the characteristic equation (3.9) does not have any purely imaginary root for all values of τ. Given that P is asymptotically stable for $\tau = 0$, it will remain asymptotically stable for all $\tau \geq 0$. □

3.2. Stability of endemic equilibrium P*

In this part, we will examine the local stability of the endemic equilibrium P^*. Initially, we set $(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*) = P^*$. Thus, the characteristic equation (3.8) becomes

$$(\lambda + d\eta_i + \mu)\,(\lambda + d\eta_i + \mu + \delta_q)\left(\lambda^3 + a\lambda^2 + b\lambda + c + (x\lambda^2 + y\lambda + z)e^{-\lambda\tau}\right) = 0, \quad i = 0, 1, \ldots \tag{3.13}$$

where

$$\begin{aligned}
a &= 3(\eta_i d + \mu) + \sigma + \gamma + \epsilon + \delta_q + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2},\\
b &= (\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}) + (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(2(\eta_i d + \mu) + \sigma + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}),\\
c &= (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}),\\
x &= -\frac{q\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau},\\
y &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q\,(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma + q\,(\eta_i d + \mu)),\\
z &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}\,(\eta_i d + \mu)\,(q\,(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma).
\end{aligned}$$

Theorem 3.2 *If* $\mathcal{R}_0 > \max\left(1, \frac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\,\eta_i d}\right)$ *and* $\frac{\mu}{\alpha} < \Lambda < \frac{1}{\alpha}$ *then the endemic equilibrium* P^* *is locally asymptotically stable for all* $\tau \geq 0$.

Proof: When τ is equal to zero, the characteristic equation (3.15) can be expressed in the following form

$$(\lambda + d\eta_i + \mu + \delta_q)\,(\lambda + d\eta_i + \mu)\left(\lambda^3 + (a+x)\lambda^2 + (b+y)\lambda + c + z\right) = 0, \quad i = 0, 1, \ldots \tag{3.14}$$

where

$$\begin{aligned}
a &= 3(\eta_i d + \mu) + \sigma + \gamma + \epsilon + \delta_q + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2},\\
b &= (\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}) + (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(2(\eta_i d + \mu) + \sigma\\
&\quad + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}),\\
c &= (\eta_i d + \mu + \gamma + \epsilon + \delta_q)(\eta_i d + \mu + \sigma)(\eta_i d + \mu + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}),\\
x &= -\frac{q\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)},\\
y &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}(q\,(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma + q\,(\eta_i d + \mu)),\\
z &= -\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}\,(\eta_i d + \mu)\,(q\,(\eta_i d + \mu + \gamma + \epsilon + \delta_q) + \sigma).
\end{aligned}$$

As $\mathcal{R}_0 > 1$, we deduce that

$$a + x = 3\eta_i d + 2\mu + + \gamma + \epsilon + \delta_q + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2} + \frac{\sigma\,(\mu+\sigma)}{\sigma + q\,(\mu + \gamma + \epsilon + \delta_q)} > 0,$$

$$\begin{aligned}
b + y = &(\eta_i d)^2 + \eta_i d\,(\mu + \gamma + \epsilon + \delta_q) + (\eta_i d + \mu + \gamma + \epsilon + \delta_q)\left(\eta_i d + \mu + \frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)\\
&+ (\eta_i d + \mu + \sigma)\left(\frac{\beta\,(\mathcal{I}^* + q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right) + \eta_i d\left(\frac{\sigma\,(\mu+\sigma)}{\sigma + q\,(\mu + \gamma + \delta_q)}\right)\\
&+ (\eta_i d + \mu)\left(\eta_i d + \frac{\sigma\,(\mu+\sigma)}{\sigma + q\,(\mu + \gamma + \epsilon + \delta_q)}\right) > 0,
\end{aligned}$$

$$
\begin{aligned}
c+z =& \eta_i d\left(\eta_i d+\mu\right)\left(\mu+\gamma+\epsilon+\delta_q\right)+\left(\eta_i d+\mu+\sigma\right)\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\\
&+\eta_i d\left(\eta_i d+\mu\right)\left(\eta_i d\frac{\sigma\left(\mu+\sigma\right)}{\sigma+q\left(\mu+\gamma+\epsilon+\delta_q\right)}\right)>0,
\end{aligned}
$$

and as $\mathcal{R}_0 \geq \dfrac{\beta\Lambda e^{-\mu\tau}}{\left(\mu+\alpha\Lambda\right)\eta_i d}$, we have

$$
\begin{aligned}
(a+x)(b+y)-(c+z) =& \left(\eta_i d+\mu+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)\left(2(\eta_i d+\mu)+\gamma+\epsilon+\delta_q+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)\\
&\times\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)+\frac{\beta\mathcal{S}^* e^{-\mu\tau}}{\left(1+\alpha\mathcal{S}^*\right)}\left(\eta_i d+\mu\right)\left(q\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)+\sigma\right)\\
&+\eta_i d\left(\left(\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)^2+\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)\left(\eta_i d+\mu+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)\right.\\
&\left.+\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)^2\right)+\left(\eta_i d+\mu+\sigma\right)\left(\eta_i d+\mu+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)\\
&+\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)\left(2(\eta_i d+\mu)+\sigma+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)\times\left[\frac{\eta_i d}{\mathcal{R}_0}\left(\mathcal{R}_0-\frac{\beta\Lambda e^{-\mu\tau}}{\left(\mu+\alpha\Lambda\right)\eta_i d}\right)\right]\\
&+\left(\frac{\beta\mathcal{S}^*}{\left(1+\alpha\mathcal{S}^*\right)}e^{-\mu\tau}\left(q\left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)+\sigma+q\left(\eta_i d+\mu\right)\right)\right)\times\left(\frac{\beta\mathcal{S}^*}{\left(1+\alpha\mathcal{S}^*\right)}\right)\\
&>0.
\end{aligned}
$$

Based on the Routh-Hurwitz criteria, all roots of equation (3.14) possess negative real parts. Consequently, when τ is zero, the endemic equilibrium point P^* is locally asymptotically stable.

Furthermore, since all roots of equation (3.14) have negative real parts for τ equal to zero, it implies that if instability arises for a specific value of the delay τ, then a characteristic root of (3.13) must intersect the imaginary axis. If (3.13) has a purely imaginary root $i\omega$, where $\omega > 0$, then by segregating real and imaginary components in (3.13), we get:

$$
\begin{cases}
\omega^3-b\omega=y\omega\cos(\omega\tau)-(z-x\omega^2)\sin(\omega\tau),\\
a\omega^2-c=y\omega\sin(\omega\tau)+(z-x\omega^2)\cos(\omega\tau).
\end{cases}
\tag{3.15}
$$

Taking square on both sides of the equations of (3.15) and summing them up, we obtain

$$
\omega^6+\left(a^2-2b-x^2\right)\omega^4+(b^2-2ac-y^2+2xz)\omega^2+c^2-z^2=0. \tag{3.16}
$$

It is easy to see that $c-z>0$, we deduce that $c^2-z^2>0$. Moreover, we have

$$
\begin{aligned}
a^2-2b-x^2 =& \left(\eta_i d+\mu+\gamma+\epsilon+\delta_q\right)^2+\left(\eta_i d+\mu+\frac{\beta\left(\mathcal{I}^*+q\mathcal{E}^*\right)}{\left(1+\alpha\mathcal{S}^*\right)^2}\right)^2\\
&+\left(\eta_i d+\mu+\sigma+\frac{q\beta\mathcal{S}^*}{\left(1+\alpha\mathcal{S}^*\right)}e^{-\mu\tau}\right)\left(\eta_i d+\frac{\sigma\left(\mu+\sigma\right)}{\sigma+q\left(\mu+\gamma+\epsilon+\delta_q\right)}\right)>0,
\end{aligned}
$$

$$
\begin{aligned}
b^2-2ac-y^2+2xz &= \left[(\eta_i d+\mu+\gamma+\epsilon+\delta_q)\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)\right]^2+\left[(\eta_i d+\mu+\gamma+\epsilon+\delta_q)\right.\\
&\times(\eta_i d+\mu+\sigma)+\left(\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)\Bigg]\times\left[\eta_i d(\eta_i d+\mu+\gamma+\epsilon+\delta_q)\right.\\
&\left.+\eta_i d\left(\frac{\sigma(\mu+\sigma)}{\sigma+q(\mu+\gamma+\epsilon+\delta_q)}\right)\right]\\
&+\left[\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)(\eta_i d+\mu+\sigma)+\left(\frac{\beta q(\eta_i d+\mu)\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}\right)\right]\\
&\times\left[\eta_i d\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)+(\mu+\sigma)\left(\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)+(\eta_i d+\mu)\left(+\frac{\sigma(\mu+\sigma)}{\sigma+q(\mu+\gamma+\epsilon+\delta_q)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
(a^2-2b-x^2)(b^2-2ac-y^2+2xz)-c^2+z^2 &= (\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2\\
&\times\left((\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2+\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2\right)+\eta_i d(\eta_i d+\mu+\sigma)\left[(\eta_i d+\mu+\gamma+\epsilon+\delta_q)^4\right.\\
&\left.+\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^4+(\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2\times\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2\right]\\
&+\left[(\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2+\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2+(\eta_i d+\mu+\sigma)^2\right]\\
&\times\left[\eta_i d\left((\eta_i d+\mu+\gamma+\epsilon+\delta_q)\left(\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)\right.\right.\\
&\left.+\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)\left(\frac{\beta e^{-\mu\tau}q(\eta_i d+\mu)\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}\right)\right)\\
&+\left(\eta_i d\left(\frac{\sigma(\mu+\sigma)}{\sigma+q(\mu+\gamma+\epsilon+\delta_q)}\right)+\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)\\
&\times\left(\left(\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)+(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\gamma+\epsilon+\delta_q)\right)\\
&+\left((\eta_i d+\mu)\left(+\frac{\sigma(\mu+\sigma)}{\sigma+q(\mu+\gamma+\epsilon+\delta_q)}\right)+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)(\mu+\sigma)}{(1+\alpha\mathcal{S}^*)^2}\right)\\
&\times\left[(\eta_i d+\mu+\sigma)\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)+\left(\frac{\beta e^{-\mu\tau}q(\eta_i d+\mu)\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}\right)\right]\\
&+\left[\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2\times(\eta_i d+\mu+\sigma)^2+(\eta_i d+\mu+\sigma)^2(\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2\right.\\
&\left.+(\eta_i d+\mu+\gamma+\epsilon+\delta_q)^2\left(\eta_i d+\mu+\frac{\beta(\mathcal{I}^*+q\mathcal{E}^*)}{(1+\alpha\mathcal{S}^*)^2}\right)^2\right]\times\left[\eta_i d(\mu+\sigma)+\left(\eta_i d+\frac{q\beta e^{-\mu\tau}\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}\right)\right.\\
&\left.\left.\times\left(\frac{\eta_i d}{\mathcal{R}_0}\left(\mathcal{R}_0-\frac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\eta_i d}\right)\right)\right]+\left(\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(\eta_i d+\mu)(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)^2\\
&+\left(\frac{q\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}\right)^2\times\left[\left(\frac{\beta e^{-\mu\tau}q(\eta_i d+\mu)\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}\right)^2+\left(\frac{\beta\mathcal{S}^*}{(1+\alpha\mathcal{S}^*)}e^{-\mu\tau}(q(\eta_i d+\mu+\gamma+\epsilon+\delta_q)+\sigma)\right)^2\right]\\
&>0.
\end{aligned}
$$

Thus, it can be concluded that for all values of η_i, equation (3.16) does not have any positive roots and the characteristic equation (3.13) does not have any purely imaginary roots. As P^* is locally asymptotically stable at $\tau=0$, it will remain asymptotically stable for all $\tau \geq 0$. □

4. Global stability

This section aims to demonstrate the global asymptotic stability of the disease-free equilibrium P when $\mathcal{R}_0 \leq 1$, through the construction of a suitable Lyapunov function.

Theorem 4.1 *If $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium P of system* (1.1) − (1.3) *is globally asymptotically stable for all $\tau \geq 0$.*

Proof: We consider the following Lyapunov functional

$$L_1 = \int_\Omega \left[e^{-\mu\tau} \int_{\frac{\Lambda}{\mu}}^{\mathcal{S}(x,t-\tau)} \left(1 - \frac{\Lambda(1+\alpha u)}{(\mu+\alpha\Lambda)u}\right) du + \mathcal{E}(x,t) + (\mu+\sigma)\int_0^\tau \mathcal{E}(x,t-u)du \right.$$
$$\left. + \frac{(\mu+\sigma)}{q(\mu+\gamma+\epsilon+\delta_q)+\sigma} I(x,t-\tau) \right] dx$$

Calculating the time derivative of L_1 along solution of system (1.1)-(1.3), we obtain

$$\begin{aligned}
\frac{dL_1(t)}{dt} &= \int_\Omega \left[e^{-\mu\tau}\left(1 - \frac{\Lambda(1+\alpha\mathcal{S}(x,t-\tau))}{(\mu+\alpha\Lambda)\mathcal{S}(x,t-\tau)}\right)(d\Delta\mathcal{S}(x,t-\tau) + \Lambda - \mu\mathcal{S}(x,t-\tau) \right.\\
&\quad - \frac{\beta\mathcal{S}(x,t-\tau)(\mathcal{I}(x,t-\tau)+q\mathcal{E}(x,t-\tau))}{1+\alpha\mathcal{S}(x,t-\tau)}\Bigg) + d\Delta\mathcal{E}(x,t) - (\mu+\sigma)\mathcal{E}(x,t)\\
&\quad + \frac{e^{-\mu\tau}\beta\mathcal{S}(x,t-\tau)(\mathcal{I}(x,t-\tau)+q\mathcal{E}(x,t-\tau))}{1+\alpha\mathcal{S}(x,t-\tau)} + (\mu+\sigma)[\mathcal{E}(x,t) - \mathcal{E}(x,t-\tau)]\\
&\quad \left. + \frac{(\mu+\sigma)}{q(\mu+\gamma+\epsilon+\delta_q)+\sigma}(d\Delta\mathcal{I}(x,t-\tau) + \sigma\mathcal{E}(x,t-\tau) + (\mu+\gamma+\epsilon+\delta_q)\mathcal{I}(x,t-\tau)) \right] dx\\
&= \int_\Omega \left\{ e^{-\mu\tau} d\Delta\mathcal{S}(x,t-\tau) - \frac{\Lambda(1+\alpha\mathcal{S}(x,t-\tau))e^{-\mu\tau}}{(\mu+\alpha\Lambda)\mathcal{S}(x,t-\tau)} d\Delta\mathcal{S}(x,t-\tau) \right.\\
&\quad + \left(\frac{-e^{-\mu\tau}(\Lambda-\mu\mathcal{S}(x,t-\tau))^2}{(\mu+\alpha\Lambda)\mathcal{S}(x,t-\tau)}\right) + d\Delta\mathcal{E}(x,t) + d\Delta\mathcal{I}(x,t-\tau)\\
&\quad \left. + \frac{(\mu+\sigma)(\mu+\gamma+\epsilon+\delta_q)}{q(\mu+\gamma+\epsilon+\delta_q)+\sigma}\left(\frac{(1+\alpha\mathcal{S}(x,t-\tau))\mathcal{R}_0}{1+\alpha\mathcal{S}(x,t-\tau)} - 1\right)(\mathcal{I}(x,t-\tau)+q\mathcal{E}(x,t-\tau)) \right\} dx
\end{aligned}$$

Recall that $\int_\Omega \Delta\mathcal{S}(x,t-\tau)dx = 0$, $\int_\Omega \Delta\mathcal{E}(x,t)dx = 0$, $\int_\Omega \Delta\mathcal{I}(x,t-\tau)dx = 0$ and using Green's formula, we have

$$\begin{aligned}
\frac{dL_1(t)}{dt} &\leq \int_\Omega \left\{ -e^{-\mu\tau}\frac{dA}{(\mu+\alpha A)}\frac{\|\nabla\mathcal{S}(x,t-\tau)\|^2}{\mathcal{S}^2(x,t-\tau)} - \frac{e^{-\mu\tau}(A-\mu\mathcal{S}(x,t-\tau))^2}{(\mu+\alpha A)\mathcal{S}(x,t-\tau)} \right.\\
&\quad \left. + \left(\frac{(\mu+\sigma)(\mu+\gamma+\epsilon+\delta_q)}{q(\mu+\gamma+\epsilon+\delta_q)+\sigma}\right)(\mathcal{R}_0 - 1)(\mathcal{I}(x,t-\tau)+q\mathcal{E}(x,t-\tau)) \right\} dx
\end{aligned}$$

Thus, if $\mathcal{R}_0 \leq 1$, it guarantees that $\frac{dL_1}{dt} \leq 0$ for all $t \geq 0$. Furthermore, it has been demonstrated that the most extensive compact invariant set within $\left\{(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) \mid \frac{dL_1}{dt} = 0\right\}$ is the singleton $\{P\}$. As a result, according to LaSalle's invariant principle [8], P is deemed globally asymptotically stable under the condition that $\mathcal{R}_0 \leq 1$. □

5. Numerical

In this section, we conduct numerical simulations to demonstrate the theoretical findings. To keep things simple, we limit ourselves to a spatial domain $\Omega = [0,1]$ that is confined to one dimension. As a result, we suggest employing the system (1.1) alongside Neumann boundary conditions

$$\frac{\partial\mathcal{S}}{\partial v} = \frac{\partial\mathcal{E}}{\partial v} = \frac{\partial\mathcal{I}}{\partial v} = \frac{\partial\mathcal{Q}}{\partial v} = \frac{\partial\mathcal{R}}{\partial v} = 0, \quad t \geq 0,\ x = 0,1$$

and initial conditions

$$\begin{aligned}
&\mathcal{S}(x,t) = |\cos(3\pi x)| \geq 0,\ \mathcal{E}(x,t) = |\cos(3\pi x)| \geq 0,\ I(x,t) = |\sin(2\pi x)| \geq 0,\\
&\mathcal{R}(x,t) = |\sin(2\pi x)| \geq 0,\ \mathcal{Q}(x,t) = |\cos(3\pi x)| \geq 0, \quad (x,t) \in [0,1]\times[-\tau,0].
\end{aligned}$$

Furthermore, in order to utilize a numerical algorithm to solve system (1.1), we must discretize each equation of the system as a finite difference equation. The Crank-Nicolson method [6] is an effective finite difference method used for numerically solving partial differential equations. This method is second-order in both time and space and offers numerical stability. Below, we provide a brief description of how the Crank-Nicolson method can be applied to our problem. We start by dividing the spatial interval $[0,1]$ and the temporal interval $[0,t_f]$ into finite grids:

$$\begin{cases} t_j = (j-1)\Delta t, \quad j = 1,2,\dots,N_t+1 \text{ where } \Delta t := \dfrac{t_f}{N_t}, \\ x_i = (i-1)\Delta x, \quad i = 1,2,\dots,N_x+1 \text{ where } \Delta x := \dfrac{1}{N_x}. \end{cases}$$

Hence, by utilizing discretization, we can represent $\mathcal{S}(x,t)$ as $\mathcal{S}_{i,j}\,(i=1,\dots,N_x+1, j=1,\dots,N_t+1)$, $\mathcal{E}(x,t)$ as $E_{i,j}\,(i=1,\dots,N_x+1, j=1,\dots,N_t+1)$, $I(x,t)$ as $I_{i,j}\,(i=1,\dots,N_x+1, j=1,\dots,N_t+1)$, $\mathcal{Q}(x,t)$ as $Q_{i,j}\,(i=1,\dots,N_x+1, j=1,\dots,N_t+1)$, and $\mathcal{R}(x,t)$ as $R_{i,j}\,(i=1,\dots,N_x+1, j=1,\dots,N_t+1)$. Furthermore, we can discretize system (1.1) as follows:

$$\begin{cases} \dfrac{\mathcal{S}_{i,j+1}-\mathcal{S}_{i,j}}{\Delta t} = \dfrac{d}{2}\left(\dfrac{\mathcal{S}_{i+1,j+1}-2\mathcal{S}_{i,j+1}+\mathcal{S}_{i-1,j+1}}{\Delta x^2} + \dfrac{\mathcal{S}_{i+1,j}-2\mathcal{S}_{i,j}+\mathcal{S}_{i-1,j}}{\Delta x^2}\right) \\ \qquad + A - \mu\mathcal{S}_{i,j} - \dfrac{\beta\mathcal{S}_{i,j}(\mathcal{I}_{i,j}+q\mathcal{E}_{i,j})}{1+\alpha\mathcal{S}_{i,j}}, \\ \dfrac{\mathcal{E}_{i,j+1}-\mathcal{E}_{i,j}}{\Delta t} = \dfrac{d}{2}\left(\dfrac{\mathcal{E}_{i+1,j+1}-2\mathcal{E}_{i,j+1}+\mathcal{E}_{i-1,j+1}}{\Delta x^2} + \dfrac{\mathcal{E}_{i+1,j}-2\mathcal{E}_{i,j}+\mathcal{E}_{i-1,j}}{\Delta x^2}\right) \\ \qquad + \dfrac{e^{-\mu\tau}\beta\mathcal{S}_{i,j-\tau/\Delta t}(\mathcal{I}_{i,j-\tau/\Delta t}+q\mathcal{E}_{i,j-\tau/\Delta t})}{1+\alpha\mathcal{S}_{i,j-\tau/\Delta t}} - (\mu+\sigma)\mathcal{E}_{i,j}, \\ \dfrac{\mathcal{I}_{i,j+1}-\mathcal{I}_{i,j}}{\Delta t} = \dfrac{d}{2}\left(\dfrac{\mathcal{I}_{i+1,j+1}-2\mathcal{I}_{i,j+1}+\mathcal{I}_{i-1,j+1}}{\Delta x^2} + \dfrac{\mathcal{I}_{i+1,j}-2\mathcal{I}_{i,j}+\mathcal{I}_{i-1,j}}{\Delta x^2}\right) \\ \qquad + \sigma\mathcal{E}_{i,j} - (\gamma+\mu+\epsilon+\delta_q)\mathcal{I}_{i,j}, \\ \dfrac{\mathcal{R}_{i,j+1}-\mathcal{R}_{i,j}}{\Delta t} = \dfrac{d}{2}\left(\dfrac{\mathcal{R}_{i+1,j+1}-2\mathcal{R}_{i,j+1}+\mathcal{R}_{i-1,j+1}}{\Delta x^2} + \dfrac{\mathcal{R}_{i+1,j}-2\mathcal{R}_{i,j}+R_{i-1,j}}{\Delta x^2}\right) \\ \qquad + \gamma\mathcal{I}_{i,j} + \gamma_q\mathcal{Q}_{i,j} - \mu\mathcal{R}_{i,j}, \\ \dfrac{\mathcal{Q}_{i,j+1}-\mathcal{Q}_{i,j}}{\Delta t} = \dfrac{d}{2}\left(\dfrac{\mathcal{Q}_{i+1,j+1}-2\mathcal{Q}_{i,j+1}+\mathcal{Q}_{i-1,j+1}}{\Delta x^2} + \dfrac{\mathcal{Q}_{i+1,j}-2\mathcal{Q}_{i,j}+Q_{i-1,j}}{\Delta x^2}\right) \\ \qquad + \delta_q\mathcal{I}_{i,j} - (\gamma_q+\mu)\mathcal{Q}_{i,j}. \end{cases} \tag{5.1}$$

Consequently, we obtain a recursive scheme that is stable from a numerical perspective. The parameters used in our numerical simulations are presented in both Table (1) and Table (2).

Assuming the parameter values from Table(1), we obtain $\mathcal{R}_0 = 0.2765$. According to Theorem (3.1), the disease-free equilibrium $P(1,0,0,0,0)$ is locally asymptotically stable, which implies that the disease will vanish (as depicted in Fig. 1). Furthermore, assuming the parameter values from Table (2), we have $0.9921 > \alpha = 0.004 > 0.000396$ and $\mathcal{R}_0 = 27.5406 > \dfrac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\eta_i d} = 11.3355 > 1$. By Theorem (3.2), the endemic equilibrium $P^*(8.6454, 898.8622, 237.6111, 377.7837, 960.334)$ is locally asymptotically stable, which indicates that the disease will vanish (as illustrated in Fig. 2).

Finally, to numerically demonstrate the global stability of equilibrium points, we need to perturb the initial conditions of system (1.1). To accomplish this, we present the solutions of system (1.1) for four distinct initial conditions. We begin with the disease-free equilibrium, using the same values as the first simulation, resulting in $\mathcal{R}_0 = 0.2765$. According to Theorem (4.1), the disease-free equilibrium $P(1,0,0,0,0)$ is globally asymptotically stable, which implies that regardless of the initial densities of susceptible, exposed, infectious, quarantined, and recovered individuals, the disease will ultimately die out, leaving only susceptible individuals(see Fig. 3).

Table 1: The parameters and their corresponding values utilized in the numerical simulations.

Parameter	Description	Value
Λ	Recruitment rate of the population	0.004
μ	Natural death of the population	0.004
ϵ	the death rate due to the disease	0.04
δ_q	isolation Rate of infected	0.04
σ	Rate of exposed individuals to the infected	0.04
β	Transmission rate	0.04
q	the fraction of transmission rate for exposed	0.004
γ	Recovery rate	0.04
γ_q	Recovery rate of isolated infected	0.004
d	Rate of diffusion	0.00004
α	the parameters that measure the inhibitory effect	0.04
τ	Time incubation	8

Table 2: The parameters and their corresponding values utilized in the numerical simulations.

Parameter	Description	Value
Λ	Recruitment rate of the population	1.008
μ	Natural death of the population	0.0004
ϵ	the death rate due to the disease	0.00004
δ_q	isolation Rate of infected	0.001547
σ	Rate of exposed individuals to the infected	0.000714
β	Transmission rate	0.0004
q	the fraction of transmission rate for exposed	0.07
γ	Recovery rate	0.000714
γ_q	Recovery rate of isolated infected	0.000573
d	Rate of diffusion	0.008
α	the parameters that measure the inhibitory effect	0.004
τ	Time incubation	8

Figure 1: The spatiotemporal solution was obtained through numerical integration of system (1.1) subject to conditions (1.2) and (1.3), with $\mathcal{R}_0 = 0.2765$.

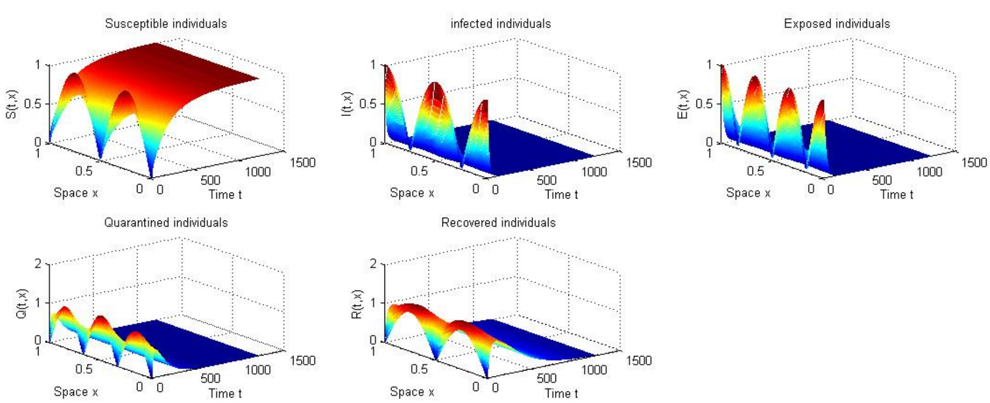


Figure 2: Spatiotemporal solution found by numerical integration of system (1.1) under conditions (1.2) and (1.3) when $\mathcal{R}_0 = 27.5406$ and $\dfrac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\,\eta_i d} = 11.3355$

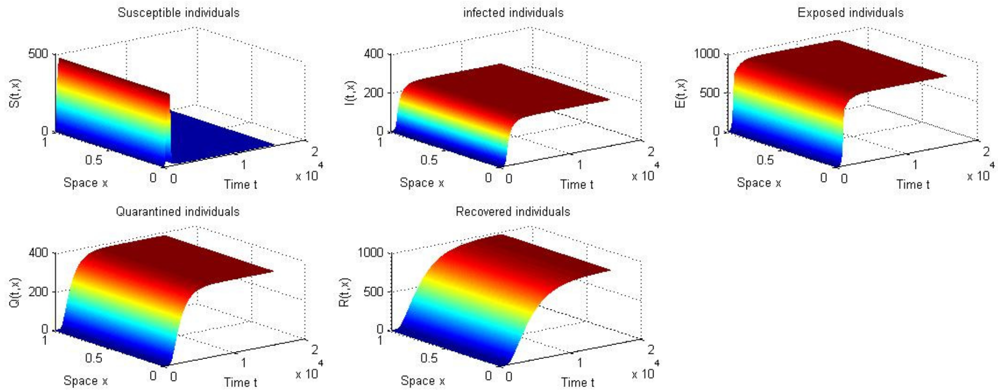
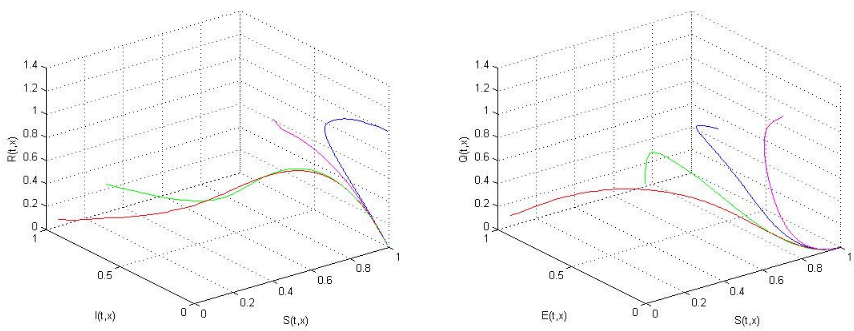


Figure 3: The disease-free equilibrium point $P^*(1,0,0,0,0)$ is globally asymptotically stable.



6. Conclusion

In this paper, we have formulated and analyzed a delayed SEIQR epidemic model with spatial diffusion and a saturated incidence function, which generalizes several previously studied models (see [1,11,2,3]). By incorporating both time delay and spatial heterogeneity, the proposed system reflects more realistic transmission dynamics, especially for diseases where individuals undergo an incubation period before becoming infectious.

We first established the local stability of the disease-free equilibrium P and the endemic equilibrium P^* by analyzing the corresponding characteristic equations under homogeneous Neumann boundary conditions. One key observation, as highlighted in Theorem **??**, is that the basic reproduction number $\mathcal{R}_0$ is independent of the diffusion coefficient d, meaning that spatial diffusion does not influence the local stability of the disease-free equilibrium. This confirms and generalizes the findings of previous works on non-spatial epidemic models (see [1,11]).

However, for the endemic equilibrium P^*, the diffusion coefficient d plays a more intricate role. As shown in our analysis, if $\mathcal{R}_0$ is reformulated to account for diffusion in a specific manner:

$$\mathcal{R}_0 \leq \frac{\beta\Lambda e^{-\mu\tau}}{(\mu+\alpha\Lambda)\eta_i d},$$

the transition from ordinary differential equations (ODE) to reaction-diffusion equations (PDE) significantly influences the stability of the endemic equilibrium. This highlights how spatial mobility can impact long-term disease persistence in heterogeneous environments.

We further demonstrated the global stability of the disease-free equilibrium P when $\mathcal{R}_0 \leq 1$, using a Lyapunov functional approach. This generalizes classical global stability results (e.g., Proposition 2.2 in [1]) to our spatially explicit delayed model, extending the theoretical understanding of epidemic models with spatial diffusion.

Finally, numerical simulations were presented to illustrate and validate our theoretical results, showing how different parameter combinations and initial conditions influence disease dynamics, confirming both local and global stability properties.

In summary, this work bridges delay differential equations (DDEs), reaction-diffusion systems (RDEs), and epidemiological modeling, offering a unified framework for understanding how incubation periods, spatial heterogeneity, and saturated incidence jointly shape the spread and control of infectious diseases. Future research could further investigate the role of spatially varying coefficients, heterogeneous boundary conditions, or even nonlocal diffusion operators, to better capture the complexity of disease spread in real-world settings.

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