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On a New Predictor-Corrector Scheme for Solving Nonlinear Differential Equations with Conformable Fractional Derivative Operator

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ABSTRACT: This work proposes a conformable fractional predictor-corrector algorithm for solving conformable fractional differential equations. Fractional calculus is finding applications in various scientific fields. Therefore, developing numerical methods to solve fractional differential equations that model natural phenomena is of high importance, especially when finding analytical solutions is of high difficulty. Many authors have developed numerical methods for other fractional derivatives, such as the Caputo fractional derivative. In this article, our aim is to design Adams-Bashforth and Adams-Moulton methods specifically tailored for the conformable fractional derivative. Some examples are provided to illustrate the applicability of the numerical method.

Key Words: Fractional differential equations, conformable fractional derivative, Adams-Bashforth method, Adams-Moulton method, numerical solution.

Contents

1. Introduction

The theory of fractional derivatives is an ancient concept, originating from a conversation on September 30, 1695, between L'Hôpital and Leibniz. They discussed the definition of the operator $\frac{d^n(.)}{dt^n}$ for $n = \frac{1}{2}$, pondering "what if n is fractional?" [1]. Despite its long history, fractional calculus is both an old and novel topic. It has only been the subject of specialized conferences and treatises since the 1970s. Shortly after completing his Ph.D. dissertation on fractional calculus, B. Ross organized the First Conference on Fractional Calculus and its Applications at the University of New Haven in June 1974 and edited its proceedings [2].

Fractional calculus, or the calculus of non-integer order differentiation and integration, has garnered significant interest due to its ability to more accurately model and solve complex phenomena in various scientific fields [3,4,5,6]. For instance, in speech signal modeling, fractional calculus provides a superior alternative to the traditional Linear Predictive Coding (LPC) approach by employing fractional order integrals as basis functions, thus requiring fewer parameters and achieving more accurate signal representation [7]. One of the most extensive application areas of fractional calculus has been linear viscoelasticity [8,9,10], due to its effectiveness in modeling hereditary phenomena with long memory, for more details, refer to the book [11] and to the works [12,13,14]. Similarly, in the realm of acoustics, fractional derivatives have been successfully applied to describe sound wave propagation through porous materials, offering a new predictive method validated through experimental work on various media such

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as plastic foams and sandy sediments [15]. Furthermore, in image processing, edge detection can be significantly enhanced by fractional differentiation, which improves detection selectivity and robustness to noise, outperforming traditional integer-order methods [16]. Additionally, fluid mechanics has benefited from fractional calculus, where its application to time-dependent viscous-diffusion problems has yielded simpler and more powerful solutions compared to classical methods, thereby validating its effectiveness and broad applicability [17].

Over time, solving fractional differential equations and fractional differential systems has been an interesting challenge due to the difficulty of finding exact solutions. For this reason, multiple alternative methods have been proposed in the literature to design numerical schemes that provide approximate solutions to the given problems.

As an example, Adams methods [25,26,27] have proven their effectiveness in solving classical differential equations. In their work [28], N. J. Ford et al. introduced a new predictor-corrector algorithm to solve fractional differential equations involving the Caputo fractional derivative operator. In [29], the authors proposed a numerical scheme, which is essentially a predictor-corrector approach, to solve fractional differential equations with the Atangana–Baleanu fractional derivative operator of variable order. In the same spirit, the authors in [30] proposed a predictor-corrector scheme to solve fractional differential equations with the Caputo–Fabrizio fractional derivative operator. In 2021, C. W. H. Green et al. introduced a new predictor-corrector scheme [31], specifically designed for fractional differential equations with the Caputo–Hadamard fractional derivative.

This paper is organized to provide a comprehensive understanding of a new conformable fractional predictor-corrector algorithm and its numerical applications. In Section 2, we present the preliminaries necessary for understanding the subsequent sections, including fundamental concepts and definitions in conformable fractional calculus. Section 3 offers a detailed description of the Adams-Bashforth and Adams-Moulton methods, laying the groundwork for their extension to the fractional domain. In Section 4, we introduce the conformable fractional predictor-corrector algorithm, explaining its formulation and theoretical basis. Finally, Section 5 showcases the practical utility of the proposed algorithm through various examples, demonstrating its effectiveness and accuracy in solving conformable fractional differential equations.

2. Preliminaries

Over time, multiple approaches have been presented in the literature to define the concept of fractional derivative, such as the Riemann-Liouville definition and the Caputo definition. For $n-1 \leq \alpha < n$ with $n \in \mathbb{Z}_+^*$, the fractional derivative of a function f in the Riemann-Liouville sense is given in [18] by the following formula

$$ {}^{RL}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx, \tag{2.1} $$

where Γ is the Euler Gamma function. We note that the Riemann-Liouville derivative of a constant function is not necessarily zero, but this property holds true for the fractional derivative in the Caputo sense, defined in [18] by the following formula

$$ {}^{C}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. \tag{2.2} $$

In their work, Khalil et al. introduce a novel definition of the fractional derivative and integral, presenting it as the most natural and fruitful formulation to date. This new definition for $0 < \alpha \leq 1$ aligns with classical definitions when applied to polynomials, differing only by a constant. Moreover, when $\alpha = 1$, it seamlessly matches the classical definition of the first derivative. This innovative approach not only offers theoretical elegance but also proves to be highly practical [19].

Definition 2.1 *[19] Let $\alpha \in (0,1]$, the conformable fractional derivative of order α of a function $f : [0,+\infty) \longrightarrow \mathbb{R}$ is defined as*

$$T_\alpha f(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad \text{for } t > 0 \text{ and } T_\alpha f(0) = \lim_{t \longrightarrow 0^+} T_\alpha f(t), \tag{2.3}$$

provided that the limits exist. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

The conformable fractional derivative has become an essential tool in extending classical calculus, attracted significant research interest. Key contributions include generalized Lyapunov-type inequalities [20], conformable fractional semigroups of operators [21], and results such as the fundamental properties of conformable Sturm-Liouville eigenvalue problems [22] and Lyapunov-type inequalities for mixed non-linear forced differential equations [23]. These works highlight the derivative's potential to address diverse problems in mathematical analysis and its applications.

Recalling briefly some preliminary facts on the conformable fractional calculus.

Theorem 2.1 *[19] If a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Let $\alpha \in (0,1]$, we give the conformable fractional derivative of certain usual functions [19].

$$T_\alpha(t^p) = pt^{p-\alpha}, \quad \text{for all } p \in \mathbb{R}; \qquad T_\alpha(\lambda) = 0, \quad \text{for all } \lambda \in \mathbb{R};$$

$$T_\alpha\left(e^{ct}\right) = ct^{1-\alpha}e^{ct}, \quad \text{for all } c \in \mathbb{R}; \qquad T_\alpha(\sin bt) = bt^{1-\alpha}\cos bt, \quad \text{for all } b \in \mathbb{R};$$

$$T_\alpha(\cos bt) = -bt^{1-\alpha}\sin bt, \quad \text{for all } b \in \mathbb{R}; \qquad T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1;$$

$$T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha; \qquad T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin\frac{1}{\alpha}t^\alpha;$$

$$T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}.$$

Theorem 2.2 *[19] Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then*

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
2. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
3. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
4. *If, in addition, f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 2.2 *[19] The conformable fractional integral I^α of a function f is defined by*

$$I^\alpha f(t) = \int_0^t x^{\alpha-1} f(x)dx, \quad \text{for } t \geq 0. \tag{2.4}$$

Theorem 2.3 *[19] If f is a continuous function in the domain of I^α, then we have*

$$T_\alpha I^\alpha f(t) = f(t). \tag{2.5}$$

Theorem 2.4 *[24] If f is a differentiable function, then we have*

$$I^\alpha T_\alpha f(t) = f(t) - f(0). \tag{2.6}$$

3. Description of Adams-Bashforth and Adams-Moulton methods

The calssical Adams-Bashforth and Adams-Moulton methods.

The numerical solution of ordinary differential equations (ODEs) is a fundamental aspect of computational mathematics. Adams-Bashforth and Adams-Moulton methods are widely used for this purpose, providing efficient and accurate solutions.

In this section, we motivate our work by recalling the classical one-step Adams-Bashforth and Adams-Moulton methods for first-order equations, also known as the predictor-corrector algorithm. We are interested in the numerical solution of the initial value problem:

$$\begin{cases} \frac{dy}{dt}(t) &= f\left(t, y(t)\right), \quad \text{with} \quad t \in [0, \tau], \quad \tau \in \mathbb{R}_+^*, \\ y(0) &= y_0. \end{cases} \tag{3.1}$$

Let $(t_n)_{n \in \{0,1,2,\ldots,N+1\}}$ be a subdivision of the interval $[0, \tau]$ with $N \in \mathbb{N}^*$, $t_0 = 0$ and $t_{N+1} = \tau$, by a simple integration over the interval $[t_n, t_{n+1}]$, we get

$$\int_{t_n}^{t_{n+1}} dy(t) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{3.2}$$

Thus one get

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{3.3}$$

By letting y_n denotes an approximation of $y(t_n)$ we find the following

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{3.4}$$

The main idea behind Adams methods is approximating the integral $\int_{t_n}^{t_{n+1}} f(t, y(t))dt$ using numerical methods.

By using the two-point trapezoidal quadrature formula [32] and considering $h = (t_{n+1} - t_n)$, we obtain the following approximation

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt = \frac{h}{2} \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]. \tag{3.5}$$

Thus one get the following scheme

$$\begin{cases} y_0 &= y(t_0), \\ y_{n+1} &= y_n + \frac{h}{2} \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]. \end{cases} \tag{3.6}$$

Formula (3.6) refers to the Adams–Moulton [25,26] one-step method of order 2, note that it is an implicit scheme.

Now if we replace the trapezoidal quadrature formula by the rectangle rule one get

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt = hf(t_n, y_n). \tag{3.7}$$

Therefore the following scheme is obtained

$$\begin{cases} y_0 &= y(t_0), \\ y_{n+1} &= y_n + hf(t_n, y_n). \end{cases} \tag{3.8}$$

Formula (3.8) refers to the Adams–Bashforth one-step method of order 2 (also known as the forward Euler method), note that it is an explicit scheme.

Consequently if we note y_{n+1}^p the predicted value of y_{n+1}, we get the one-step predictor–corrector algorithm, which is a hybrid method that uses the explicit advantage of Adams–Bashforth method (3.8) and the stability [27] of Adams–Moulton method (3.6), it is given as follows

$$\begin{cases} y_0 &= y(t_0), \\ y_{n+1}^p &= y_n + hf(t_n, y_n), \\ y_{n+1} &= y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}^p)\right]. \end{cases} \tag{3.9}$$

A predictor-corrector scheme for Caputo fractional derivative operator.

In 2002, K. Diethelm et al. construct an Adams-Bashforth and Adams-Moulton method to solve the following initial value problem with Caputo type fractional derivative operator of order α [28].

$$\begin{cases} {}^C D_0^\alpha y(t) &= f\left(t, y(t)\right), \\ y^{(k)}(0) &= y_0^{(k)}, \qquad k = 0, 1, ..., m-1, \end{cases} \tag{3.10}$$

where without loss of generality $t \in [0, \tau]$, $\tau \in \mathbb{R}_+^*$, ${}^C D_0^\alpha$ is the Caputo type fractional derivative of order $\alpha > 0$ defined in formula (2.2), $m = \lceil \alpha \rceil$ is just the value α rounded up to the nearest integer and $y^{(k)}$ is the ordinary kth derivative of y. One can readily observe that the requisite number of initial conditions to determine a unique solution is denoted by m.
A complete discussion elucidating why the authors opt to use Caputo type fractional derivative of order α and to use the classical ordinary derivative of y as initial conditions in (3.10) can be found in [28,33,34].

Note that if, (a) f is continues with respect to both its arguments and (b) f has a Lipschitz condition with respect to the second argument, we can indeed say that a solution exists and that this solution is uniquely determined [33], for the initial value problem (3.10).

Let $(t_n)_{n\in\{0,1,2,...,N+1\}}$ be a subdivision of the interval $[0, \tau]$ with $N \in \mathbb{N}^*$, $t_0 = 0$, $t_{N+1} = \tau$ and $h = t_{n+1} - t_n$. According to [28] a predictor-corrector algorithm in the framework of Caputo type fractional derivative of order α can be suggested as follows

$$\begin{cases} y^{(k)}(0) &= y_0^{(k)} \in \mathbb{R}, \qquad k = 0, 1, ..., m-1, \\ y_{n+1}^p &= \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^n \left((n+1-j)^\alpha - (n-j)^\alpha\right) f(t_j, y_j), \\ y_{n+1} &= \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_j f\left(t_j, y_j\right) + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_{n+1}^p\right), \end{cases} \tag{3.11}$$

where the coefficients a_j, $j = 0, 1, ..., n$ are given as

$$a_j = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j+2)^{\alpha+1}, & \text{if } 1 \le j \le n. \end{cases} \tag{3.12}$$

4. A new predictor-corrector scheme for Conformable fractional derivative operator

Let us consider the following Initial value problem

$$\begin{cases} T_\alpha y(t) &= f\left(t, y(t)\right) \text{ with } t \in [0, \tau], \quad \tau \in \mathbb{R}_+^* \quad \text{and} \quad \alpha \in (0, 1], \\ y(0) &= y_0. \end{cases} \tag{4.1}$$

where T_α denotes the conformable fractional derivative of order α and $f : [0, \tau] \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function.

In [35] the existence of at least one solution to the aforementioned problem was established by applying Schauder's fixed point theorem. Now if we suppose that y is differentiable, then from theorem (2.4) we get

$$I_\alpha T_\alpha(y)(t) = y(t) - y(0). \tag{4.2}$$

To give an equivalent predictor-corrector algorithm in the framework of conformable fractional derivative we need to introduce a similar formula as in (3.4) with some unavoidable modification, in fact such formula can be obtained simply by applying the operator I_α defined in (2.4) to both sides of the initial value problem (4.1). Thus one obtain

$$y(t) = y(0) + \int_0^t x^{\alpha-1} f(x, y(x))dx. \tag{4.3}$$

This equation exhibits a slight variation from Equation (3.4), as the integration range now initiates from 0 rather than t_j, however, this does not pose significant challenges in our efforts to generalize the Adams method to the conformable fractional differential equations framework. The idea is to approximate the integral $\int_0^t x^{\alpha-1} g(x)dx$ using numerical methods.
Let $(t_n)_{n\in\{0,1,2,\ldots,N+1\}}$ be a subdivision of the interval $[0, \tau]$ with $N \in \mathbb{N}^*$, $t_0 = 0$, $t_{N+1} = \tau$ and $h = t_{n+1} - t_n$, note that we consider a regular subdivision of the interval $[0, \tau]$ which makes h a constant in this situation.

One can simply use the product trapezoidal quadrature formula to approximate the integral

$$\int_0^{t_{n+1}} x^{\alpha-1} g(x)\mathrm{d}x \approx \int_0^{t_{n+1}} x^{\alpha-1} \tilde{g}(x)\mathrm{d}x, \tag{4.4}$$

here, $\tilde{g}$ represents the piecewise linear interpolant for g, with nodes and knots selected at t_n, $n = 0, 1, 2, \ldots, N + 1$ such that $t_n = nh$. Consequently we get the following lemma.

Lemma 4.1 *The integral $\int_0^{t_{n+1}} x^{\alpha-1} g(x)dx$ can be approximated as follows*

$$\int_0^{t_{n+1}} x^{\alpha-1} g(x)\mathrm{d}x \approx \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_j g(t_j), \tag{4.5}$$

where

$$a_j = \begin{cases} 1, & \text{if } j = 0, \\ (j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ (\alpha+1)(n+1)^\alpha + n^{\alpha+1} - (n+1)^{\alpha+1}, & \text{if } j = n+1. \end{cases} \tag{4.6}$$

Proof: Let g_j denotes $g(t_j)$ then we get

$$\begin{aligned}
\int_0^{t_{n+1}} x^{\alpha-1} \tilde{g}(x)\mathrm{d}x &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} x^{\alpha-1} \left[g_j + \frac{g_{j+1} - g_j}{t_{j+1} - t_j}(x - t_j) \right] dx \\
&= \sum_{j=0}^{n} \left[g_j \int_{t_j}^{t_{j+1}} x^{\alpha-1} dx + \frac{g_{j+1} - g_j}{h} \int_{t_j}^{t_{j+1}} x^{\alpha-1}(x - t_j)dx \right] \\
&= \sum_{j=0}^{n} \left[g_j \left[\frac{1}{\alpha} x^\alpha \right]_{t_j}^{t_{j+1}} + \frac{g_{j+1} - g_j}{h} \left(\left[\frac{1}{\alpha} x^\alpha (x - t_j) \right]_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}} \frac{1}{\alpha} x^\alpha dx \right) \right] \\
&= \sum_{j=0}^{n} \left[\frac{1}{\alpha} g_j \left(t_{j+1}^\alpha - t_j^\alpha \right) + \frac{g_{j+1} - g_j}{\alpha h} \left(t_{j+1}^\alpha (t_{j+1} - t_j) - \left[\frac{1}{\alpha+1} x^{\alpha+1} \right]_{t_j}^{t_{j+1}} \right) \right] \\
&= \sum_{j=0}^{n} \left[\frac{1}{\alpha} g_j \left((j+1)^\alpha - j^\alpha \right) h^\alpha + \frac{g_{j+1} - g_j}{\alpha} \left((j+1)^\alpha h^\alpha + \frac{1}{\alpha+1} \left(j^{\alpha+1} - (j+1)^{\alpha+1} \right) h^\alpha \right) \right].
\end{aligned}$$

After simplification we get

$$\int_0^{t_{n+1}} x^{\alpha-1}\tilde{g}(x)\mathrm{d}x = \frac{h^\alpha}{\alpha(\alpha+1)}\left[g_0 + \left((\alpha+1)(n+1)^\alpha + n^{\alpha+1} - (n+1)^{\alpha+1}\right)g_{n+1}\right] + \frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=1}^{n}\left((j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}\right)g_j.$$

□

Hence if we note y_n an approximation of $y(t_n)$ for t_n, $n = 0, 1, 2, \dots, N+1$ one get the following scheme

$$y_{n+1} = y_0 + \frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n} a_j f(t_j, y_j) + \frac{h^\alpha}{\alpha(\alpha+1)}\left((\alpha+1)(n+1)^\alpha + n^{\alpha+1} - (n+1)^{\alpha+1}\right) f(t_{n+1}, y_{n+1}). \tag{4.7}$$

As a result formula (4.7) refers to the conformable fractional Adams–Moulton method, note that it is an implicit scheme, which means we need to predict the value of y_{n+1} an approximation of $y(t_{n+1})$, to do that it is a must to introduce a conformable fractional Adams–Bashforth method.

By using the product rectangle rule we can approximate the integral on the left-hand side of Equation (4.4) as

$$\int_0^{t_{n+1}} x^{\alpha-1}g(x)\mathrm{d}x \approx \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} x^{\alpha-1}g(t_j)\mathrm{d}x = \frac{h^\alpha}{\alpha}\sum_{j=0}^{n}\left((j+1)^\alpha - j^\alpha\right)g(t_j). \tag{4.8}$$

Consequently if we note y^p_{n+1} the predicted value of y_{n+1}, one get the conformable fractional Adams–Bashforth method

$$y^p_{n+1} = y_0 + \frac{h^\alpha}{\alpha}\sum_{j=0}^{n}\left((j+1)^\alpha - j^\alpha\right)f(t_j, y_j). \tag{4.9}$$

As a result we get a conformable fractional predictor-corrector algorithm

$$\begin{cases} y_0 &= y(0), \\ y^p_{n+1} &= y_0 + \frac{h^\alpha}{\alpha}\sum_{j=0}^{n}\left((j+1)^\alpha - j^\alpha\right)f(t_j, y_j), \\ y_{n+1} &= y_0 + \frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n} a_j f(t_j, y_j) \\ & \quad + \frac{h^\alpha}{\alpha(\alpha+1)}\left((\alpha+1)(n+1)^\alpha + n^{\alpha+1} - (n+1)^{\alpha+1}\right) f\left(t_{n+1}, y^p_{n+1}\right). \end{cases} \tag{4.10}$$

Where the coefficients a_j, $j = 0, 1, ..., n$ are given as

$$a_j = \begin{cases} 1, & \text{if } j = 0, \\ (j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}, & \text{if } 1 \le j \le n. \end{cases} \tag{4.11}$$

5. Numerical application

Example 1: To give a concrete example of the application of our algorithm, one can consider the following conformable fractional initial value problem

$$\begin{cases} T_\alpha y(t) &= ty(t) \text{ with } t \in [0, \tau], \quad \tau \in \mathbb{R}^*_+ \quad \text{and} \quad \alpha \in (0, 1], \\ y(0) &= 1. \end{cases} \tag{5.1}$$

Lemma 5.1 *Let $\alpha \in (0, 1]$ and y be α-differentiable at a point $t > 0$, then if, in addition, y is differentiable [19], one get*

$$T_\alpha y(t) = t^{1-\alpha}\frac{dy}{dt}(t). \tag{5.2}$$

We can easily use lemma (5.1), to verify that the exact solution of this initial value problem, has the form

$$y(t) = \exp(\frac{t^{\alpha+1}}{\alpha+1}). \tag{5.3}$$

By using the algorithm (4.10), we can have the plot in (Figure 1) of the numerical and exact solution, of the conformable fractional initial value problem (5.1).

Example 2: Consider the following non-linear conformable fractional differential equation

$$\begin{cases} T_\alpha y(t) &= 1 + y^2(t) \quad \text{with} \quad t \in [0, \tau], \quad \tau \in \mathbb{R}_+^* \quad \text{and} \quad \alpha \in (0, 1], \\ y(0) &= 0. \end{cases} \tag{5.4}$$

Recalling lemma (5.1), separating variables and integrating, one get $\arctan(y(t)) = \frac{t^\alpha}{\alpha}$. Using the fact that the range of the inverse tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, we get $0 \leq \frac{t^\alpha}{\alpha} < \frac{\pi}{2}$. Therefore the domain of the solution to the initial value problem (5.4) is the interval $0 \leq t < \left(\alpha\frac{\pi}{2}\right)^{\frac{1}{\alpha}}$.

For this reason it follows that the exact solution of (5.4) is

$$y(t) = \tan(\frac{t^\alpha}{\alpha}), \qquad \text{with} \quad t \in \left[0, \left(\alpha\frac{\pi}{2}\right)^{\frac{1}{\alpha}}\right). \tag{5.5}$$

Note that the solution cannot jump over the vertical asymptote. To avoid the singularity point of the solution y, one can plot the numerical and exact solution of the conformable fractional initial value problem (5.4) over the interval $[0, \frac{1}{2}]$ for $\alpha = 0.5$, note that here $\tau = \frac{1}{2}$, see (Figure 2).

Example 3: It is worth providing another example to demonstrate the proficiency of our algorithm in generating numerical solutions that closely approximate the exact solution with a negligible error that is imperceptible.

We can consider the following non-linear conformable fractional differential equation.

$$\begin{cases} T_\alpha y(t) &= -\alpha y^2(t) \quad \text{with} \quad t \in [0, \tau], \quad \tau \in \mathbb{R}_+^* \quad \text{and} \quad \alpha \in (0, 1], \\ y(0) &= 1. \end{cases} \tag{5.6}$$

We can easily verify that the exact solution of the conformable fractional differential equation (5.6) can be written as

$$y(t) = \frac{1}{1 + t^\alpha}. \tag{5.7}$$

The plot of the numerical solution and the exact solution of the conformable fractional differential equation (5.6) is given in (Figure 3), for as an example $\alpha = 0.7$.

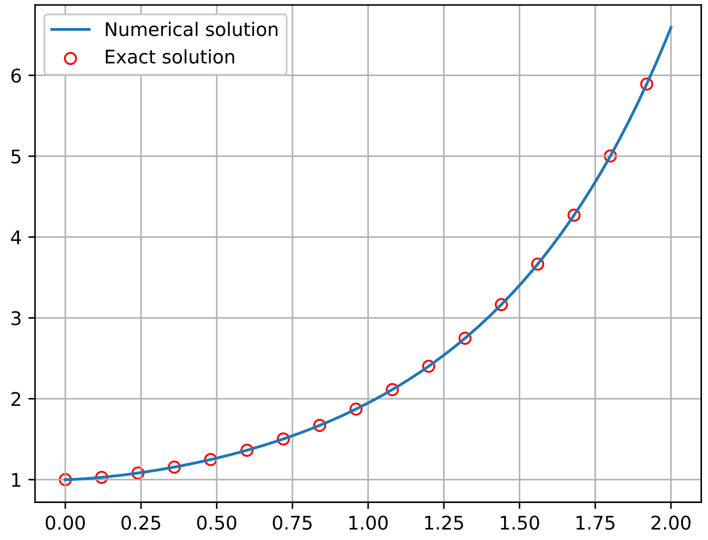

Figure 1: The plot of $y(t)$, the solution of the conformable fractional initial value problem (5.1) with $\alpha = 0.5$.

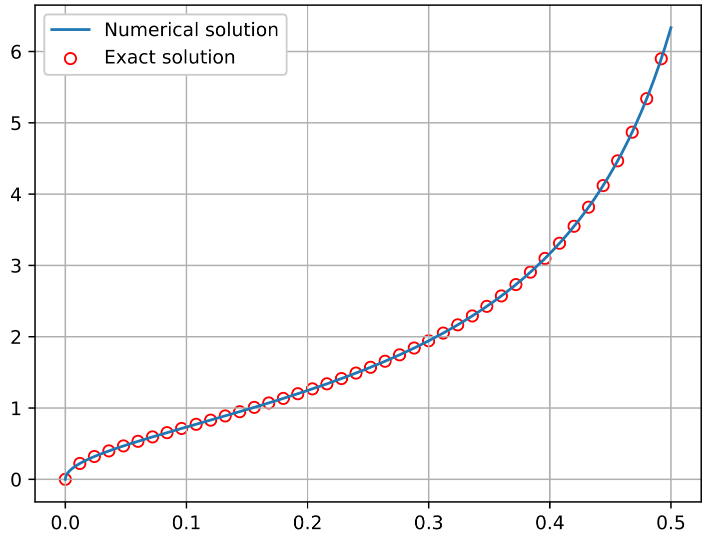

Figure 2: The plot of $y(t)$, the solution of the conformable fractional initial value problem (5.4) with $\alpha = 0.5$.

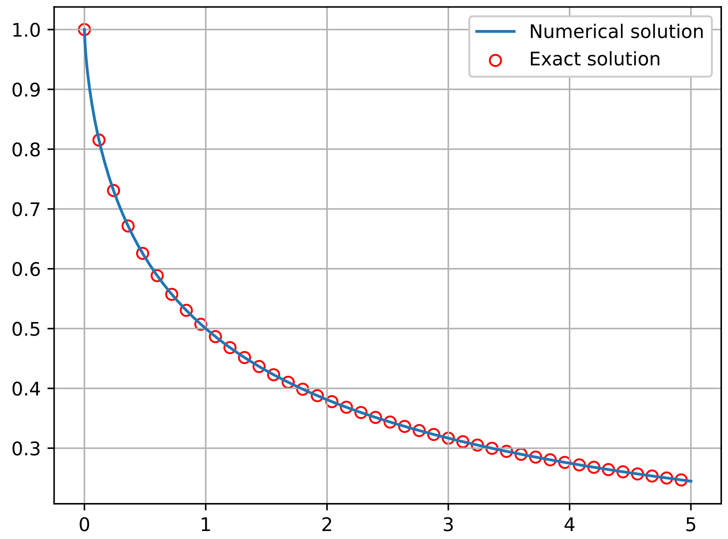

Figure 3: The plot of $y(t)$, the solution of the conformable fractional initial value problem (5.6) with $\alpha = 0.7$.

6. Conclusion

In this paper, we designed a new numerical scheme to solve conformable fractional differential equations. Our approach was based on Adams methods to construct a predictor-corrector algorithm capable of providing a numerical solution, especially when finding an exact solution is highly challenging. The algorithm was tested in multiple situations and performed very well. Our future perspective is to develop more approaches capable of solving these equations more efficiently and rapidly.

Declaration of competing interest

The authors state that there are no financial interests or personal relationships that could have influenced the work presented in this paper.

Data availability

No datasets were utilized for the research presented in this article.

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