Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.75812

Third degree linear forms and quasi-antisymmetric semiclassical forms of class one

Mohamed Khalfallah *

ABSTRACT: In this contribution, we explore the characterizations of a family of quasi-antisymmetric semiclassical linear forms of class one, based on their third degree character. We show that there does not exist a strict third degree form that is simultaneously a quasi-antisymmetric semiclassical linear form of class one. By utilizing the Stieltjes function and the moments of these forms, we provide necessary and sufficient conditions for a regular form to be of second degree, quasi-antisymmetric, and semiclassical of class one. Our focus is on the connection between these forms and the Jacobi forms $\mathcal{T}_{0,q} = \mathcal{J}(-1/2, q-1/2), q \in \mathbb{N}$. All these forms are rational transformations of the first-kind Chebyshev form $\mathcal{T} = \mathcal{J}(-1/2, -1/2)$.

Key Words: Orthogonal polynomials, classical and semiclassical forms, Stieltjes function, quasi-antisymmetric forms, third degree forms.

Contents

1. Introduction

This work is based on the theory of semiclassical orthogonal polynomials. Since the pioneering paper by J. Shohat on semiclassical orthogonal polynomials [35], many authors have further explored this area. Specifically, an orthogonal polynomial sequence with respect to a regular linear form (or linear functional) w is called semiclassical if there exists a monic polynomial Φ and a polynomial Ψ with $\deg \Psi \geq 1$, such that $(\Phi w)' + \Psi w = 0$. These polynomials naturally generalize the well-known classical orthogonal polynomials, such as Hermite, Laguerre, Jacobi, and Bessel polynomials. Over the past four decades, the theory has been developed and thoroughly investigated by researchers. P. Maroni has made significant contributions to its development, particularly from an algebraic structural viewpoint (see [27] for a comprehensive survey on the topic, and [23], where the role of semiclassical linear forms in the analysis of polynomial sequences orthogonal with respect to Sobolev inner products is demonstrated).

The investigation of semiclassical forms with a class greater than or equal to one poses a challenging problem. In [4], the semiclassical linear forms w of class one are described by utilizing the Pearson equations they satisfy (also referred to in [12,31]). Examples of semiclassical forms with class two are found in [15,16,30,33], among others. Given the difficulties in solving the Laguerre-Freud equations, as discussed in [5], it becomes increasingly important to employ alternative tools for constructing semiclassical forms (see [1,2,7,32,33], among others), either based on algebraic equations satisfied by the associated Stieltjes function or in terms of their integral representation.

* ORCID ID: https://orcid.org/0000-0003-3624-8879
2010 *Mathematics Subject Classification:* Primary: 33C45; Secondary: 42C05.
Submitted February 20, 2025. Published September 25, 2025

On the other hand, examining regular forms for which the corresponding formal Stieltjes function

$$S(w)(z) := -\sum_{n\geq 0} \frac{\langle w, x^n \rangle}{z^{n+1}}$$

satisfies an algebraic equation provides an interesting approach to analyzing their properties.

Second degree linear forms are defined via a quadratic equation with polynomial coefficients

$$M(z)S^2(w)(z) + N(z)S(w)(z) + R(z) = 0.$$

These forms were first introduced in [29]. Examples of second degree linear forms have been examined in various contexts, including the classical case (as seen in [3]), in the semiclassical case with class $s = 1$ (see [1,2,8,10,15,32], among others), and in the semiclassical case of class $s = 2$ in [15,19,33].

Third degree linear forms (TDRF, in short)) are characterized by the fact that their formal Stieltjes function satisfies a cubic equation with polynomial coefficients

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0.$$

A regular form w is called a strict third degree form (STDRF) if it is a TDRF and its Stieltjes function does not satisfy a quadratic equation with polynomial coefficients, meaning it is not a second degree form. It is important to note that third degree forms belong to the Laguerre-Hahn class [11]. In [6] (and [3] for second degree forms), all classical forms that are STDRFs (and second degree forms) have been identified. Notably, the only classical STDRFs (and second degree forms) are the Jacobi forms $\mathcal{J}(k + q/3, l - q/3)$, where $q \in \{1, 2\}$ and k, l are integers with $k + l \geq -1$ (see [6]), and $\mathcal{J}(p - 1/2, q - 1/2)$, where p, q are integers with $p+q \geq 0$ (see [3]). The properties and examples of TDRFs are provided in [6]. Additionally, methods for constructing certain families of TDRFs are discussed, either through spectral perturbations of the linear form (see [9]) or by using a cubic decomposition of the associated sequences of orthogonal polynomials (see [7], [16]).

Characterizations of certain families of semiclassical linear forms of class $s \leq 2$ that are TDRFs are provided in the following references: [1,2,3,6,7,8,10,14,15,18,20,22,32,33]. Additionally, Table 1 in [20] presents a summary of the Pearson equations for some useful examples of third degree semiclassical forms of class at most 2.

This work is centered on analyzing semiclassical forms of class $s = 1$ that are third degree forms. In particular, we focus on describing a large family of such forms, using their third degree character, where the corresponding sequences of orthogonal polynomials $\{W_n\}_{n\geq 0}$ satisfy the three-term recurrence relation (TTRR)

$$W_{n+2}(x) = (x - (\alpha_n - \alpha_{n+1}))\, W_{n+1}(x) + \alpha_n^2 W_n(x), \quad n \geq 0, \tag{1.1}$$

with initial conditions $W_0(x) = 1,\ W_1(x) = x + \alpha_0$, where $\alpha_n \neq 0, n \geq 0$.

The goal of this contribution is not only to describe all the semiclassical forms of class one that are of second degree and quasi-antisymmetric, but also to extend this by providing a new identification of this family of forms, highlighting their connection with the classical forms of third degree mentioned earlier. Our primary tool is the representation of the corresponding Stieltjes functions, which allows us to directly deduce the moments of the forms. As a result, we obtain explicit expressions for these Stieltjes functions, and we also derive the quadratic equation, as well as the first-order ordinary linear differential equation, that they satisfy.

This paper is organized as follows. In Section 2, we introduce the notation and basic background that will be used in the subsequent sections. In Section 3, we first recall the definitions and main properties of third degree forms. We then present results concerning third degree classical forms, which are needed for the following sections. In Sections 4, we state our main results. We begin by showing that there are no quasi-antisymmetric forms that are strict third degree forms. Using second degree forms, we provide an identification of the family of quasi-antisymmetric regular forms. Specifically, we give necessary and sufficient conditions for a regular form to simultaneously be quasi-antisymmetric, second degree, and semiclassical of class one. Thus, we establish the connection between all these forms and the Jacobi forms $\mathcal{T}_{0,q} = \mathcal{J}\left(-1/2, q-1/2\right), q \in \mathbb{N}$, and we show that all these forms are rational transformations of the Chebyshev form of the first kind $\mathcal{T} = \mathcal{J}\left(-1/2, -1/2\right)$.

2. Notation and basic background

Let $\mathcal{P}$ be the linear space of algebraic polynomials with complex coefficients. $\langle w, p\rangle$ will denote the action of the form (linear functional) $w \in \mathcal{P}'$ over the polynomial $p \in \mathcal{P}$, where $\mathcal{P}'$ denotes the algebraic dual of the linear space $\mathcal{P}$. In particular, $\langle w, x^n\rangle := (w)_n,\ n \geq 0$, represent the moments of w.

Let us define the following operations in the algebraic dual space of the polynomials: the left product of w by a polynomial, defined as $\langle fw, p\rangle = \langle w, fp\rangle, p \in \mathcal{P}$; the derivative Dw of the linear form w is defined as $\langle Dw, p\rangle = -\langle w, p'\rangle, p \in \mathcal{P}$; the dilations and shifted forms $h_a w$ and $\tau_b w$ are defined, respectively, as $\langle h_a w, p\rangle = \langle w, h_a p\rangle = \langle w, p(ax)\rangle$, $\langle \tau_b w, p\rangle = \langle w, \tau_{-b} p\rangle = \langle w, p(x+b)\rangle$; the form $x^{-1}u$ defined as $\langle x^{-1}w, p\rangle = \langle w, \theta_0 p\rangle = \left\langle w, \frac{p(x)-p(0)}{x}\right\rangle$; the Cauchy product of two forms, v and w, defined as $\langle vw, f\rangle := \langle v, wf\rangle, f \in \mathcal{P}$, where the right product of a linear form by a polynomial is given by

$$(wp)(x) := \left\langle w, \frac{xp(x) - \zeta p(\zeta)}{x - \zeta}\right\rangle = \sum_{i=0}^{n}\left(\sum_{j=i}^{n}(w)_{j-i}a_j\right)x^i,$$

being $p(x) = \sum_{i=0}^{n} a_i x^i$.

In $\mathcal{P}'$, we have the well-known formula

$$\tau_b \circ h_a = h_a \circ \tau_{a^{-1}b},\ a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}. \tag{2.1}$$

The linear form $w \in \mathcal{P}'$ is said to be a rational perturbation of $v \in \mathcal{P}'$, if there exist polynomials p and q, such that

$$q(x)w = p(x)v.$$

The even part of a form w is given by

$$\langle \sigma(w), p\rangle = \langle w, \sigma(p)\rangle, \quad p \in \mathcal{P}.$$

where the linear operator $\sigma : \mathcal{P} \to \mathcal{P}$ is defined by $\sigma(p)(x) := p(x^2)$ for every $p \in \mathcal{P}$.

We introduce the so-called anti-symmetrization operator $\boldsymbol{\alpha} : \mathcal{P}' \to \mathcal{P}'$ defined by, for $\varpi \in \mathcal{P}'$ [27]

$$\big(\boldsymbol{\alpha}(\varpi)\big)_{2n} = 0, \qquad \big(\boldsymbol{\alpha}(\varpi)\big)_{2n+1} = (\varpi)_n, \quad n \geq 0. \tag{2.2}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$ that is defined by [13,27]

$$S(w)(z) = -\sum_{n\geq 0}\frac{(w)_n}{z^{n+1}}.$$

Remark 2.1 *For any $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, $S(w)(z) = p(z)$ if and only if $w = 0$ and $f = 0$.*

For any $p \in \mathcal{P}$ and $u \in \mathcal{P}'$, the following property holds [27]

$$S(pu)(z) = p(z)S(u)(z) + (u\theta_0 p)(z). \tag{2.3}$$

Let us recall that a form w is called regular (quasi-definite) if there exists a monic polynomial sequence $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that [13]

$$\langle w, W_n W_m\rangle = r_n \delta_{n,m}, \quad n, m \geq 0,$$

where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is the Kronecker symbol. $\{W_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS, in short) with respect to the form w. It is characterized by the following three-term recurrence relation

$$\begin{aligned} &W_0(x) = 1, \quad W_1(x) = x - \beta_0, \\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \; n \geq 0. \end{aligned} \tag{2.4}$$

Here $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are sequences of complex numbers such that $\gamma_{n+1} \neq 0$ for all n. This is the so called Favard's theorem (see [13,27,28]). The form w is said to be normalized if $(w)_0 = 1$. In the sequel, we only consider normalized forms.

In this work, we will consider a (MOPS) $\{W_n\}_{n\geq 0}$ with respect to the form w fulfilling a second-order recurrence relation (2.4) with coefficients

$$\beta_0 = -\alpha_0, \quad \beta_{n+1} = \alpha_n - \alpha_{n+1}, \quad \gamma_{n+1} = -\alpha_n^2, \quad n \geq 0,$$

with $\alpha_n \neq 0, n \geq 0$. Then, its associated form w is said to be quasi-antisymmetric, i.e., $(w)_{2n+2} = 0$ for $n \geq 0$. Equivalently, $x\sigma w = 0$. For more information about these forms, see [25,26].

A form w is called semiclassical when it is regular and there exist two polynomials ϕ and ψ, ϕ monic, $\deg\phi \geq 0$, $\deg\psi \geq 1$, such that w satisfies a Pearson's equation

$$D(\phi w) + \psi w = 0. \tag{2.5}$$

Equivalently, the formal Stieltjes function of w satisfies a nonhomogeneous first order linear differential equation with polynomial coefficients

$$A_0(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z), \tag{2.6}$$

where

$$A_0 = \phi, \quad C_0 = -\phi' - \psi, \quad D_0 = -(w\theta_0\phi)' - (w\theta_0\psi). \tag{2.7}$$

Furthermore, if the polynomials A_0, C_0, and D_0 appearing in (2.7) are coprime, then the class of w is defined by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

If $\{W_n\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS) with respect to a semiclassical form w of class s, then $\{W_n\}_{n\geq 0}$ is referred to as a semiclassical OPS of class s. In particular, when $s = 0$ (so that $\deg\phi \leq 2$ and $\deg\psi = 1$), one obtains, up to an affine change of variables, the four well-known families of classical forms: Hermite, $\mathcal{H}$; Laguerre, $\mathcal{L}(\alpha)$; Jacobi, $\mathcal{J}(\alpha,\beta)$; and Bessel, $\mathcal{B}(\alpha)$ (see [28]). Since Jacobi linear forms $\mathcal{J}(\alpha,\beta)$ will be used in the sequel, we highlight that

$$\phi^{\mathcal{J}}(x) = x^2 - 1, \quad \psi^{\mathcal{J}}(x) = -(\alpha+\beta+2)x + (\alpha-\beta).$$

The semiclassical character of a form is preserved under affine transformations. Specifically, the shifted form

$$\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w, \quad a \in \mathbb{C} - \{0\}, \; b \in \mathbb{C},$$

is also semiclassical and has the same class as w. It satisfies the equation

$$D\left(a^{-\deg\phi}\phi(ax+b)\widehat{w}\right) + a^{1-\deg\phi}\psi(ax+b)\widehat{w} = 0.$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$, where $\widehat{W}_n(x) = a^{-n}W_n(ax+b), \quad n \geq 0$, is orthogonal with respect to $\widehat{w}$. The recurrence coefficients are given by [27]

$$\widehat{\beta}_n = \frac{\beta_n - b}{a}, \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0.$$

The formal Stieltjes function of $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, satisfies [6]

$$S(\widehat{w})(z) = aS(w)(az+b). \tag{2.8}$$

On the other hand, it is easy to check that for any $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$, and $w \in \mathcal{P}'$ we have

$$\Big((h_{a^{-1}} \circ \tau_{-b})w\Big)_n = n!a^{-n} \sum_{\nu+\mu=n} \frac{(-b)^\nu}{\nu!\mu!}(w)_\mu, \quad n \geq 0. \tag{2.9}$$

3. Third degree semiclassical forms

3.1. Third degree form

In this subsection, we briefly review the definitions and list some basic properties of third degree regular forms. Subsequently, we present results concerning strict third degree classical forms (resp. second degree classical forms) that will be needed later in the paper.

Definition 3.1 *The form w is called a third degree regular form (TDRF, in short) if it is regular and if there exist four polynomials, A monic, B, C and D, such that*

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0.$$

Notice that D depends on A, B, C and w. Indeed,

$$D(z) = \big(w^3\theta_0^3 A\big)(z) - \big(w^2\theta_0^2 B\big)(z) + \big(w\theta_0 C\big)(z).$$

Remark 3.1 1. *A regular form w is said to be of second degree if its corresponding Stieltjes function satisfies a quadratic equation with polynomial coefficients B, C, and D (see [29])*

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0,$$

where $B \neq 0, C^2 - 4BD \neq 0, D \neq 0$ since the regularity of w.

2. *If the form w is a TDRF but not a second degree form, it is called a strict third degree regular form (STDRF, for short) [6].*

Remark 3.2 *Among the most well-known STDRFs (resp. second degree forms) you get the Jacobi form $\mathcal{V} := \mathcal{J}(-\frac{2}{3}, -\frac{1}{3})$ [6] (resp. the Tchebychev form of first kind $\mathcal{T} := \mathcal{J}\big(-1/2, -1/2\big)$ [3]). Indeed, its formal Stieltjes function is $S(\mathcal{V})(z) = -(z+1)^{-2/3}(z-1)^{-1/3}$ (resp. $S(\mathcal{T})(z) = -(z^2-1)^{-1/2}$) and satisfies the cubic equation*

$$(z+1)^2(z-1)S^3(\mathcal{V})(z) + 1 = 0.$$

(resp. the quadratic equation

$$(z^2-1)S^2(\mathcal{T})(z) - 1 = 0.)$$

Proposition 3.1 *[6] Let w be semiclassical form, satisfying (2.6). Set $A_0(z) = \prod_{i=1}^{m}(z-a_i)^{k_i}$, where $a_1, \ldots, a_m$ are complex numbers and $k_1, \ldots, k_m$ are positive integers such that $k_1 + \cdots + k_m = \deg A_0$. If w is a STDRF, then the rational fraction $\frac{C_0}{A_0}$ has only simple poles and if $\alpha_1, \ldots, \alpha_m$ denote the residues of $\frac{C_0}{A_0}$, then*

$$\frac{C_0(z)}{A_0(z)} = \sum_{i=1}^{m} \frac{\alpha_i}{z-a_i},$$

and

$$3\alpha_1, \ldots, 3\alpha_m \in \mathbb{Z}.$$

Elementary transformations of linear forms, such as k-associated and k-anti-associated perturbations, shifts, multiplication and division by a polynomial, and inversion, among others, preserve the family of third degree linear forms [6,9,24]. In particular, when dealing with the following rational spectral transformations (see [9], [37]), we have:

Lemma 3.1 *[6] Let w and v be two regular forms satisfying $M(x)w = N(x)v$, where $M(x)$ and $N(x)$ are polynomials.*

3.2. Third degree classical forms

As mentioned in the introduction, the classical forms that are STDRFs (resp. second degree forms) are determined in [6] (resp. in [3]). More precisely, only certain Jacobi forms are TDRFs. Indeed,

Theorem 3.1 *[3,6] Among the classical forms, only the Jacobi forms $\mathcal{J}(k+q/3, l-q/3)$, where $k+l \geq -1,\ k,l \in \mathbb{Z},\ q \in \{1,2\}$ (resp. $\mathcal{J}(p-1/2, q-1/2)$, where $p+q \geq 0,\ p,q \in \mathbb{Z}$), are STDRFs (resp. are second degree forms).*

Remark 3.3 *Throughout this paper the following notation will be used:*

$$\begin{aligned}
\mathcal{V}_q^{k,l} &:= \mathcal{J}(k+q/3, l-q/3),\ \text{with } k+l \geq -1,\ k,l \in \mathbb{Z},\ q \in \{1,2\},\\
\mathcal{T}_{p,q} &:= \mathcal{J}(p-1/2, q-1/2),\ \text{with } p+q \geq 0,\ p,q \in \mathbb{Z}.
\end{aligned}$$

The next lemma provides us with fundamental relations to be used in the sequel. More precisely, it emphasizes the fact that the strict third degree classical forms $\mathcal{V}_q^{k,l}$ (resp. second degree classical forms $\mathcal{T}_{p,q}$) are perturbations as Lemma 3.1 of $h_{(-1)^{q-1}}\mathcal{V}$ (resp. $\mathcal{T}$).

Lemma 3.2 *[8,16] Let $q \in \{1,2\}$ and $k,l \in \mathbb{Z}$ with $k+l \geq -1$ (resp. $p,q \in \mathbb{Z}$ with $p+q \geq 0$). The forms $\mathcal{V}_q^{k,l}$ and $\mathcal{V}$ (resp. $\mathcal{T}_{p,q}$ and $\mathcal{T}$) are related by*

$$f_q^{k,l}\mathcal{V}_q^{k,l} = g_q^{k,l} h_{(-1)^{q-1}}\mathcal{V}, \tag{3.1}$$

$$(\textit{resp.} \quad L_{p,q}\mathcal{T}_{p,q} = R_{p,q}\mathcal{T},) \tag{3.2}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ (resp. $L_{p,q}$ and $R_{p,q}$) are polynomials with

$$\begin{aligned}
f_q^{k,l}(x) &:= \Big\langle h_{(-1)^{q-1}}\mathcal{V}, (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}\Big\rangle (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}},\\
g_q^{k,l}(x) &:= \Big\langle \mathcal{V}_q^{k,l}, (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}}\Big\rangle (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}.
\end{aligned}$$

$\Big($*resp.*

$$L_{p,q}(x) := \Big\langle \mathcal{T}, (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}}\Big\rangle (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}}, \tag{3.3}$$

$$R_{p,q}(x) := \Big\langle \mathcal{T}_{p,q}, (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}}\Big\rangle (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}}.\Big) \tag{3.4}$$

In the following remark, we summarize some results concerning the forms $\mathcal{V}_q^{k,l}$ and $\mathcal{T}_{p,q}$ defined above.

Remark 3.4 *1. From Lemma 3.1, taking into account the expression of the first kind Chebychev form and (3.2), $\mathcal{T}_{p,q}$ is a second degree form since*

$$B_{p,q}(z)S^2\big(\mathcal{T}_{p,q}\big)(z) + C_{p,q}(z)S\big(\mathcal{T}_{p,q}\big)(z) + D_{p,q}(z) = 0,$$

with

$$\begin{aligned}
B_{p,q}(z) &= (z^2-1)f_{p,q}^2(z),\\
C_{p,q}(z) &= 2(z^2-1)f_{p,q}(z)\big((\mathcal{T}_{p,q}\theta_0 f_{p,q})(z) - (\mathcal{T}\theta_0 g_{p,q})(z)\big),\\
D_{p,q}(z) &= (z^2-1)\big((\mathcal{T}_{p,q}\theta_0 f_{p,q})(z) - (\mathcal{T}\theta_0 g_{p,q})(z)\big)^2 - g_{p,q}^2(z).
\end{aligned}$$

2. Using the first order linear differential equation satisfied by the Stieltjes function of the Jacobi form [27], it is a straightforward exercise to prove that $S(\mathcal{T}_{p,q})(z)$ satisfies

$$\Phi(z)S'(\mathcal{T}_{p,q})(z) = C_0^{p,q}(z)S(\mathcal{T}_{p,q})(z) + D_0^{p,q}(z), \tag{3.5}$$

where $\Phi(z), C_0^{p,q}(z)$, and $D_0^{p,q}(z)$ are polynomials given by

$$\Phi(z) = z^2-1, \quad C_0^{p,q}(z) = (p+q-1)z + q - p, \quad D_0^{p,q}(z) = p+q. \tag{3.6}$$

3. Let us recall that the moments of the Jacobi form $\mathcal{T}_{p,q}$, where $p+q \geq 0,\ p,q \in \mathbb{Z}$, are given by [28]

$$\left(\mathcal{T}_{p,q}\right)_n = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(p+q+1)}{\Gamma(\nu+p+q+1)} F_{n,\nu}\big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\big), \quad n \geq 0, \tag{3.7}$$

where

$$F_{n,\nu}\big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\big) = (-1)^{n-\nu}\frac{\Gamma(\nu+p+\frac{1}{2})}{\Gamma(p+\frac{1}{2})} + (-1)^{\nu}\frac{\Gamma(\nu+q+\frac{1}{2})}{\Gamma(q+\frac{1}{2})}, \tag{3.8}$$

and Γ is the gamma function [28].

Remark 3.5 *In the sequel, we denote $\widehat{\mathcal{T}}_{p,q} := (h_{2^{-1}} \circ \tau_1)\mathcal{T}_{p,q}$, with $p+q \geq 0,\ p,q \in \mathbb{Z}$.*

In the sequel, we need the following lemma:

Corollary 3.1 *One has*

$$S(\widehat{\mathcal{T}})(z^2) = z^{-1} S(\mathcal{T})(z). \tag{3.9}$$

Proof: From Remarks 3.3 and 3.5 we get $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_{0,0} = (h_{2^{-1}} \circ \tau_1)\mathcal{T}$. As a consequence,

$$\begin{aligned} S(\widehat{\mathcal{T}})(z^2) &\overset{by\ (2.8)}{=} 2S(\mathcal{T})(2z^2-1) \\ &= 2\left(4z^4-4z^2\right)^{-\frac{1}{2}} \\ &= -z^{-1}\left(z^2-1\right)^{-\frac{1}{2}} \\ &= z^{-1} S(\mathcal{T})(z). \end{aligned}$$

□

4. Third degree quasi-antisymmetric semiclassical forms of class one

In this section, we provide several characterizations of the quasi-antisymmetric semiclassical forms of class one that are of second degree, highlighting their connection with the forms $\mathcal{T}_{0,q}$, their corresponding Stieltjes function, and their moments.

In [34], all the quasi-antisymmetric semiclassical forms of class one are determined. Only two solutions appear, up to affine transformation. Indeed,

Theorem 4.1 *[34] For a semiclassical linear functional w of class $s=1$ fulfilling (2.5) such that the corresponding MOPS satisfies (1.1), we get the following.*

(a) First family: If $\phi(x) = x$, we have

$$(xw)' + 2x^2 w = 0, \tag{4.1}$$

with

$$\alpha_{2n} = -\lambda\sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}, \quad \alpha_{2n+1} = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\lambda\Gamma(n+1)}, \quad n \geq 0.$$

(b) Second family: If $\phi(x) = x^3 - x$, we have

$$\begin{cases} \left(\left(x^3-x\right)w\right)' - 2(\beta+1)x^2 w = 0, \\ |\beta| + |(2\beta+1)\lambda+1| \neq 0, \end{cases} \tag{4.2}$$

with, for $n \geq 0$,

$$\alpha_{2n} = \begin{cases} -\dfrac{2\lambda\sqrt{\pi}\Gamma\left(\beta+\frac{3}{2}\right)\Gamma(n+1)\Gamma(n+\beta+1)}{(4n+2\beta+1)\Gamma(\beta+1)\Gamma\left(n+\beta+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}, & \text{if } \beta \neq -\frac{1}{2} \\ \lambda, & \text{if } \beta = -\frac{1}{2} \end{cases}$$

and

$$\alpha_{2n+1} = \begin{cases} -\dfrac{2\Gamma(\beta+1)\Gamma\left(n+\beta+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\lambda\sqrt{\pi}(4n+2\beta+3)\Gamma\left(\beta+\frac{3}{2}\right)\Gamma(n+1)\Gamma(n+\beta+1)}, & \text{if } \beta \neq -\frac{1}{2} \\ \frac{1}{4\lambda}, & \text{if } \beta = -\frac{1}{2}. \end{cases}$$

In the sequel, we denote by $\mathcal{F}(\beta)$ the linear functional w that satisfies (4.2).

Remark 4.1 *Moreover, the linear form $\sigma\left(x\mathcal{F}(\beta)\right)$ is classical fulfilling*

$$\left(h_2 \circ \tau_{-1/2}\right)\sigma\left(x\mathcal{F}(\beta)\right) = \mathcal{J}\left(-\frac{1}{2}, \beta\right). \tag{4.3}$$

The following proposition plays an important role in proving our results.

Proposition 4.1 *Let w be a regular quasi-antisymmetric form. The following statements are equivalent*

(a) w is a second degree regular form (resp. STDRF).

(b) $(w)_1 v = \sigma(xw)$ is a second degree regular form (resp. STDRF).

Proof: The formal Stieltjes function $S(w)$ is a rational spectral transformation of the formal Stieltjes function $S(v)$ of given by

$$zS(w)(z) = \lambda z S(v)(z^2) - 1. \tag{4.4}$$

with $\lambda = (w)_1$. Thus, the result is a direct consequence of Proposition 2 from [17]. □

With the above results, we can now formulate and establish the main results of this section.

Proposition 4.2 *The forms arising in Theorem 4.1 are not STDRFs.*

Proof: *First family:* From (4.1), the formal Stieltjes function of the form w in the first family in Theorem 4.1 fulfils (2.6) with

$$A_0(z) = z,$$
$$C_0(z) = -2z^2 - 1.$$

Therefore

$$\frac{C_0(z)}{A_0(z)} = \frac{-\frac{1}{4}}{z - i\frac{\sqrt{2}}{2}} + \frac{-\frac{1}{4}}{z + i\frac{\sqrt{2}}{2}}.$$

By applying Proposition 3.1, it can be concluded that this form is not a STDRF.

Second family: From (4.2), the formal Stieltjes function of the form w in the second family of Theorem 4.1 satisfies the equation (2.6) with

$$A_0(z) = z(z^2 - 1),$$
$$C_0(z) = 2(q-1)z^2 + 1.$$

Then

$$\frac{C_0(z)}{A_0(z)} = \frac{q - \frac{1}{2}}{z - 1} + \frac{q - \frac{1}{2}}{z + 1} - \frac{1}{z}.$$

By applying Proposition 3.1, it follows that this form cannot be a STDRF.

□

In [36] all the second degree quasi-antisymmetric semiclassical forms of class one are determined.

Theorem 4.2 *[36] Among the quasi-antisymmetric semiclassical forms of class $s = 1$, only the forms $\mathcal{F}\left(q - \frac{1}{2}\right)$ are second degree forms, provided $q \in \mathbb{N}$.*

The main result of this work offers a characterization of the second degree quasi-antisymmetric semiclassical forms of class one through their formal Stieltjes function, which is explicitly provided. Consequently, the moments are derived.

Theorem 4.3 *Let w be a regular and normalized form. The following statements are equivalent.*

(a) w is a second degree quasi-antisymmetric semiclassical form of class one.

(b) (The connection between the regular forms via the operator σ). There exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$\sigma(xw) = (h_{2^{-1}} \circ \tau_1)\mathcal{T}_{0,q}.$$

(c) (The connection between the regular forms). There exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$L_{0,q}(2x^2-1)xw = \lambda R_{0,q}(2x^2-1)\mathcal{T}, \tag{4.5}$$

and

$$\begin{aligned}\big(w(L_{0,q}(2x^2-1))\big)(z) = &-2\lambda z\Big(\big(\mathcal{T}\theta_0 R_{0,q}\big) - \big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)\Big)(2z^2-1)\\ &+ L_{0,q}(2z^2-1) + \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z),\end{aligned} \tag{4.6}$$

where $L_{0,q}$ and $R_{0,q}$ are polynomials defined by (3.3) and (3.4), respectively.

(d) (The connection between the Stieltjes functions). There exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$zS(w)(z) = \lambda z S\big(\widehat{\mathcal{T}}_{0,q}\big)(z^2) - 1. \tag{4.7}$$

(e) (The connection between the regular forms via the operator $\boldsymbol{\alpha}$). There exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$xw = \lambda x \boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big).$$

(f) (The moments). There exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$(w)_{2n+2} = 0, \tag{4.8}$$

$$(w)_{2n+1} = \lambda n! 2^{-n} \sum_{\nu+\mu=n} \frac{1}{\nu!\mu!} \sum_{i=0}^{\mu} \binom{\mu}{i} 2^{i-1} \frac{\Gamma(q+1)}{\Gamma(i+q+1)} F_{\mu,i}\big(-\tfrac{1}{2}, q-\tfrac{1}{2}\big), \quad n \geq 0, \tag{4.9}$$

where $F_{\mu,i}\big(-\frac{1}{2}, q-\frac{1}{2}\big)$ is defined by (3.8).

Proof: $(a) \Rightarrow (b)$ Let w be a second degree quasi-antisymmetric semiclassical form of class one. According to Theorem 4.2, there exist $q \in \mathbb{N}$ and $\lambda \in \mathbb{C} - \{0\}$ such that

$$w = \mathcal{F}\left(q - \frac{1}{2}\right).$$

From (2.1), (4.3) becomes

$$\sigma(xw) = (h_{2^{-1}} \circ \tau_1)\mathcal{T}_{0,q}. \tag{4.10}$$

$(b) \Rightarrow (c)$ According to (4.4), (2.8) and (4.10), we get

$$zS(w)(z) + 1 = \lambda 2zS(\mathcal{T}_{0,q})(2z^2-1).$$

By multiplying both sides of the last equation by $L_{0,q}(2z^2-1)$, and using (2.3), we deduce that

$$\begin{aligned}&L_{0,q}(2z^2-1)zS(w)(z) + L_{0,q}(2z^2-1)\\ &\quad= 2\lambda zS\big(L_{0,q}\mathcal{T}_{0,q}\big)(2z^2-1) - 2\lambda z\big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)(2z^2-1)\\ &\quad\overset{by\ (3.2)}{=} 2\lambda zS\big(R_{0,q}\mathcal{T}\big)(2z^2-1) - 2\lambda z\big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)(2z^2-1)\\ &\quad\overset{by\ (2.3)}{=} 2\lambda z R_{0,q}(2z^2-1)S(\mathcal{T})(2z^2-1) + 2\lambda z\Big(\big(\mathcal{T}\theta_0 R_{0,q}\big) - \big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)\Big)(2z^2-1)\\ &\quad\overset{by\ (3.9)}{=} \lambda R_{0,q}(2z^2-1)S(\mathcal{T})(z) + 2\lambda z\Big(\big(\mathcal{T}\theta_0 R_{0,q}\big) - \big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)\Big)(2z^2-1).\end{aligned}$$

Using (2.3), the above relation becomes

$$S\big(xL_{0,q}(2x^2-1)w\big)(z) = S\big(\lambda R_{0,q}(2x^2-1)\mathcal{T}\big)(z) + P(z),$$

with

$$\begin{aligned}P(z) =& 2\lambda z\Big(\big(\mathcal{T}\theta_0 R_{0,q}\big) - \big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)\Big)(2z^2-1) - L_{0,q}(2z^2-1)\\ &+ \big(w(L_{0,q}(2x^2-1))\big)(z) - \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z),\end{aligned}$$

or equivalently,

$$S\big(xL_{0,q}(2x^2-1)w - \lambda R_{0,q}(2x^2-1)\mathcal{T}\big)(z) = P(z) \in \mathcal{P}.$$

Thus, taking into Remark 2.1 we get

$$L_{0,q}(2x^2-1)xw - \lambda R_{0,q}(2x^2-1)\mathcal{T} = 0 \quad \text{in } \ \mathcal{P}',$$

and

$$P(z) = 0.$$

Thus the result follows.
$(c) \Rightarrow (d)$ Applying the operator S to (4.5) and using (2.3), we obtain

$$\begin{aligned}L_{0,q}(2z^2-1)zS(w)(z) =& \lambda R_{0,q}(2z^2-1)S\big(\mathcal{T}\big)(z) - \big(w(L_{0,q}(2x^2-1))\big)(z)\\ &+ \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z).\end{aligned}$$

Thus, from (3.9)

$$\begin{aligned}&L_{0,q}(2z^2-1)zS(w)(z)\\ &= 2\lambda z R_{0,q}(2z^2-1)S\big(\mathcal{T}\big)(2z^2-1) - \big(w(L_{0,q}(2x^2-1))\big)(z)\\ &+ \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z)\\ &\overset{by\ (2.3)-(3.2)}{=} 2\lambda z S\big(L_{0,q}\mathcal{T}_{0,q}\big)(2z^2-1) - \big(w(L_{0,q}(2x^2-1))\big)(z)\\ &+ \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z) - 2\lambda z\big(\mathcal{T}\theta_0 R_{0,q}\big)(2z^2-1)\\ &\overset{by\ (2.3)}{=} 2\lambda z L_{0,q}(2z^2-1)S\big(\mathcal{T}_{0,q}\big)(2z^2-1) - \big(w(L_{0,q}(2x^2-1))\big)(z)\\ &+ \lambda\big(\mathcal{T}\theta_0(R_{0,q}(2x^2-1))\big)(z) - 2\lambda z\big(\mathcal{T}\theta_0 R_{0,q}\big)(2z^2-1) + 2\lambda z\big(\mathcal{T}_{0,q}\theta_0 L_{0,q}\big)(2z^2-1).\end{aligned}$$

Therefore, by using (4.6), the last equation becomes

$$L_{0,q}(2z^2-1)zS(w)(z) = 2\lambda z L_{0,q}(2z^2-1)S\big(\mathcal{T}_{0,q}\big)(2z^2-1) - L_{0,q}(2z^2-1).$$

As a consequence,

$$zS(w)(z) = \lambda z S\big(\widehat{\mathcal{T}}_{0,q}\big)(z^2) - 1.$$

The statement (d) holds.
$(d) \Rightarrow (e)$ First, observe that

$$S\left(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\right)(z) = -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_n}{z^{n+1}} \overset{by\ (2.2)}{=} -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_{2n+1}}{z^{2n+2}} \overset{by\ (2.2)}{=} -\sum_{n\geq 0}\frac{\big(\widehat{\mathcal{T}}_{0,q}\big)_n}{z^{2(n+1)}} = S\big(\widehat{\mathcal{T}}_{0,q}\big)(z^2). \tag{4.11}$$

Together with (4.7) we have

$$zS(w)(z) = \lambda z S\left(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\right)(z) - 1.$$

Therefore, using (2.3), the last equation becomes

$$S(zw)(z) = \lambda S\left(x\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\right)(z) - 1.$$

By applying Remark 2.1, the desired relation follows.
$(e) \Rightarrow (f)$

$$(w)_{2n+2} = (xw)_{2n+1} = \Big(\lambda x\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_{2n+1} = \lambda\Big(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_{2n+2} \overset{by\ (2.2)}{=} 0, \quad n \geq 0. \tag{4.12}$$

$$(w)_{2n+1} = (xw)_{2n} = \Big(\lambda x\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_{2n} = \lambda\Big(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)_{2n+1} \overset{by\ (2.2)}{=} \lambda\big(\widehat{\mathcal{T}}_{0,q}\big)_n, \quad n \geq 0. \tag{4.13}$$

Using (2.9) and taking into account (3.7)-(3.8) (4.8)-(4.9) follow in a straightforward way.
$(f) \Rightarrow (a)$ By hypothesis we have

$$(w)_0 = 1, \quad (w)_{2n+2} = 0, \quad (w)_{2n+1} = \lambda\big(\widehat{\mathcal{T}}_{0,q}\big)_n, \quad n \geq 0.$$

It remains to show that

$$S(w)(z) = -\sum_{n\geq 0}\frac{(w)_{2n}}{z^{2n+1}} - \sum_{n\geq 0}\frac{(w)_{2n+1}}{z^{2n+2}} = -\frac{1}{z} - \lambda\sum_{n\geq 0}\frac{\big(\widehat{\mathcal{T}}_{0,q}\big)_n}{z^{2(n+1)}} = -\frac{1}{z} + \lambda S\big(\widehat{\mathcal{T}}_{0,q}\big)(z^2).$$

Then,

$$zS(w)(z) = \lambda zS\Big(\boldsymbol{\alpha}\big(\widehat{\mathcal{T}}_{0,q}\big)\Big)(z) - 1. \tag{4.14}$$

Moreover, the fact that the affine transformation of a second degree form is also a second degree form implies that $\widehat{\mathcal{T}}_{0,q}$ is a second degree form such that

$$\widehat{B}_{0,q}(z)S^2\big(\widehat{\mathcal{T}}_{0,q}\big)(z) + \widehat{C}_{0,q}(z)S\big(\widehat{\mathcal{T}}_{0,q}\big)(z) + \widehat{D}_{0,q}(z) = 0, \tag{4.15}$$

with

$$\widehat{B}_{0,q}(z) = (-2)^{-t}B_{0,q}(2z-1), \quad \widehat{C}_{0,q}(z) = (-2)^{1-t}C_{0,q}(2z-1), \quad \widehat{D}_{0,q}(z) = (-2)^{2-t}D_{0,q}(2z-1).$$

Making $z \leftarrow z^2$ in (4.15), multiplying this equation by z^2 and taking into account (4.14) we get

$$\widehat{B}_{0,q}(z^2)S^2(w)(z) + z\widehat{C}_{0,q}(z^2)S(w)(z) + z^2\widehat{D}_{0,q}(z^2) = 0.$$

As a result, w is a second degree form.
To conclude the proof, we must show that the class of w is one. Using (2.8), the relation (4.14) transforms into

$$zS(w)(z) = 2\lambda zS(\mathcal{T}_{0,q})(2z^2-1) - 1. \tag{4.16}$$

By taking the formal derivatives of the last equation, we obtain

$$zS'(w)(z) + S(w)(z) = 8\lambda z^2 S'(\mathcal{T}_{0,q})(2z^2-1) + 2\lambda S(\mathcal{T}_{0,q})(2z^2-1).$$

From the above two equations, we derive

$$S'(\mathcal{T}_{0,q})(2z^2-1) = \frac{z^2S'(w)(z) - 1}{8\lambda z^3}. \tag{4.17}$$

In (3.5) the change of variable $z \leftarrow 2z^2-1$ yields

$$\Phi(2z^2-1)S'(\mathcal{T}_{0,q})(2z^2-1) = C_0^{0,q}(2z^2-1)S(\mathcal{T}_{0,q})(2z^2-1) + D_0^{0,q}(2z^2-1). \tag{4.18}$$

Substituting (4.16) and (4.17) into (4.18), and multiplying both sides of the resulting equation by $8\lambda z^3$, one obtains

$$\phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{4.19}$$

where the polynomials ϕ_w, C_w and D_w are

$$\begin{aligned}
\phi_w(z) &= z^2\Phi(2z^2-1),\\
C_w(z) &= 4z^3C_0^{0,q}(2z^2-1),\\
D_w(z) &= \Phi(2z^2-1)+4z^2C_0^{0,q}(2z^2-1)+8\lambda z^3D_0^{0,q}(2z^2-1).
\end{aligned}$$

Therefore, from (3.6) $S(w)(z)$ fulfils (4.19) with

$$\begin{aligned}
\phi_w(z) &= 4z^4(z^2-1),\\
C_w(z) &= 4z^3\big[(q-1)(2z^2-1)+q\big],\\
D_w(z) &= 4z^2(z^2-1)+4z^2\big[(q-1)(2z^2-1)+q\big]+8\lambda qz^3.
\end{aligned} \tag{4.20}$$

Therefore, the polynomials ϕ_w, C_w, and D_w given by (4.20) have $4z^3$ as a common factor. Dividing these polynomials by $4z^3$, we obtain

$$\begin{aligned}
\phi_w(z) &= z(z^2-1),\\
C_w(z) &= 2(q-1)z^2+1,\\
D_w(z) &= (2q-1)z+2\lambda q.
\end{aligned}$$

Now, considering that $C_w(0)\neq 0$ and $C_w(\pm 1)=2q-1\neq 0$, we conclude that ϕ_w, C_w, and D_w are coprime. Consequently, since $\deg D_w=1$ and $\deg C_w=2$, the class of w is one, which completes the proof of the theorem. □

5. Acknowledgement

The author is grateful to the referee for his helpful remarks and suggestions.

References

1. Alaya, A., Bouras, B., Marcellán, F., *A non-symmetric second degree semi-classical form of class one*. Integral Transforms Spec. Funct. **23**, 149–159, (2012).
2. Beghdadi, D., *Second degree semi-classical forms of class $s=1$. The symmetric case*. Appl. Numer. Math. **34**, 1–11, (2000).
3. Beghdadi, D., Maroni, P., *Second degree classical forms*. Indag. Math. (N.S.) **8**, 439–452, (1997).
4. Belmehdi, S., *On semi-classical linear functionals of class $s=1$. Classification and integral representations*. Indag. Math. **3**, 253–275, (1992).
5. Belmehdi, S., Ronveaux, A., *Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials*. J. Approx. Theory **76** no. 3, 351-368, (1994).
6. Ben Salah, I., *Third degree classical forms*. Appl. Numer. Math. **44**, 433–447, (2003).
7. Ben Salah, I., Khalfallah, M., *third degree semiclassical forms of class one arising from cubic decomposition*. Integral Transforms Spec. Funct. **31** no. 9, 720–743, (2020).
8. Ben Salah, I., Khalfallah, M., *A description via second degree character of a family of quasi-symmetric forms*. Period. Math. Hung. **85** no. 1, 81–108, (2022).
9. Ben Salah, I., Marcellán, F., Khalfallah, M., *Stability of Third Degree Linear Functionals and Rational Spectral Transformations*. Mediterr. J. Math. **19**, no. 4, Article 155, 23 pages, (2022).
10. Ben Salah, I., Marcellán, F., Khalfallah, M., *Second degree linear forms and semiclassical forms of class one. A case study*. Filomat **36** no. 3, 781–800, (2022).
11. Ben Salah, I., Maroni, P., *The connection between self-associated two-dimensional vector functionals and third degree forms*. Adv. Comput. Math. **13** no. 1, 51–77, (2000).
12. Bouras, B., Alaya, A., *A large family of semi-classical polynomials of class one*. Integral Transforms Spec. Funct. **18**, 913–931, (2007).
13. Chihara, T.S., *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York, (1978).
14. Khalfallah, M., *A canonical Christoffel transformation of the strict third degree classical linear forms*. Korean J. Math. **32**(4), 733–744 (2024).

15. Khalfallah, M., *A Description of Second Degree Semiclassical Forms of Class Two Arising Via Cubic Decomposition*. Mediterr. J. Math. **19** no. 1, Paper No. 30, 22 pp., (2022).

16. Khalfallah, M., *A description via third degree character of quasi-symmetric semiclassical forms of class two*. Period. Math. Hung. **86**, 58–75, (2023).

17. Khalfallah, M., *A new generator of third degree linear forms*. Issues of Analysis, **13**(2), 63–83, (2024).

18. Khalfallah M., *Characterization theorem for strict third degree semiclassical forms of class one obtained via cubic decomposition*. Z. Anal. Anwend. **42**, no. 1/2, 17–36, (2023).

19. Khalfallah M., *Characterization theorem for third degree symmetric semiclassical forms of class two*. Filomat **39**, no. 7, 2101–2119, (2025).

20. Khalfallah, M., *Several characterizations of third degree semiclassical linear forms of class two appearing via cubic decomposition*. Integral Transforms Spec. Funct. **35**(3), 206–233, (2024).

21. Marcellán, F., Khalfallah, M., *On the characterizations of third degree semiclassical forms via polynomial mappings*. Integral Transforms Spec. Funct. **34**, no. 1, 65–87, (2023).

22. Marcellán, F., Khalfallah, M., *Third Degree Linear Forms and Semiclassical Forms of Class One—A Case Study*. In: Castillo, K., Durán, A.J. (eds) Orthogonal Polynomials and Special Functions. Coimbra Mathematical Texts, vol 3. Springer, Cham, (2024).

23. García-Ardila, J. C., Marcellán, F., Marriaga, M., *From standard orthogonal polynomials to Sobolev orthogonal polynomials: The role of semiclassical linear functionals*. In *Orthogonal Polynomials. 2nd AIMS-Volkswagen Stiftung Workshop, Douala, Cameroon, 5-12 October, 2018,* M. Foupouagnigni, W. Koepf Editors. Series Tutorials, Schools, and Workshops in the Mathematical Sciences (TSWMS), Birkhäuser, Cham, 245-292, (2020).

24. Marcellán, F., Prianes, E., *Orthogonal polynomials and linear functionals of second degree*, In *Proceedings 3 rd International Conference on Approximation and Optimization.* J. Guddat et al. Editors. Aport. Mat., Serie Comun. **24**, 149–162, (1998).

25. Maroni, P., *Sur la décomposition quadratique d'une suite de polynˆomes orthogonaux II*. Port. Math. **50**(3), 305–329, (1993).

26. Maroni, P., *Sur la suite de polynômes orthogonaux associ´ee 'a la forme* $u = \delta_c + \lambda(x-c)L$ *(French)*. Period. Math. Hunger. **21**(3), 223–248, (1990).

27. Maroni, P., *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques (French)*. In *Orthogonal Polynomials and their Applications (Erice, 1990)*, IMACS Ann. Comput. Appl. Math. Vol. **9**, C. Brezinski, L. Gori, A. Ronveaux Editors, Baltzer, Basel, 95–130, (1991).

28. Maroni, P., *Fonctions eulériennes. Polynômes orthogonaux classiques*, Techniques de l' Ingénieur. Paris, (1994).

29. Maroni, P., *An introduction to second degree forms*, Adv. Comput. Math. **3**, 59–88, (1995).

30. Maroni, P., Tounsi, M.I., *Quadratic decomposition of symmetric semi-classical polynomials sequences of even class: an example from the case* $s = 2$. J. Differ. Equ. Appl. **18**, 1519–1530, (2012).

31. Maroni, P., Mejri, M., *Some semiclassical orthogonal polynomials of class one*. Eurasian Math.J. **2**, 108–128, (2011).

32. Sghaier, M., *A family of second degree semi-classical forms of class* $s = 1$. Ramanujan J. **26**, 55–67, (2011).

33. Sghaier, M., *A family of symmetric second degree semiclassical forms of class* $s = 2$. Arab J. Math. **1**, 363–375, (2012).

34. Sghaier, M., Zaatra, M., Khlifi, A., *Laguerre-Freud equations associated with the D-Laguerre-Hahn forms of class one*. Adv. Pure Appl. Math. **10**, no. 4, 395–411, (2019).

35. Shohat, J.A., *A differential equation for orthogonal polynomials*. Duke Math. J. **5**, 401–407 (1939).

36. Zaatra, M., *Second degree semiclassical linear functionals of class one. The quasi-antisymmetric case*. Methods of Functional Analysis and Topology, **29**, no. 3, 134–144, (2023).

37. Zhedanov, A., *Rational spectral transformations and orthogonal polynomials*. J. Comput. Appl. Math. **85**, no. 1, 67–86, (1997).

Mohamed Khalfallah,

Department of Mathematics,

Faculty of Sciences at Monastir, University of Monastir, 5019 Monastir,

Preparatory Engineering Institute, University of Monastir, 5019 Monastir,

Tunisia

E-mail address: khalfallah.mohamed91@gmail.com