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Applications of some fuzzy concepts to Nexuses

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ABSTRACT: The concepts of (fuzzy) $\mathbb{N}^{\infty}$-submoduloid and (fuzzy) $\mathbb{N}^{\infty}$-moduloid homomorphism are introduced, and their properties are explored. We examine the notion of a prime $\mathbb{N}^{\infty}$-submoduloid within nexuses, providing conditions under which a $\mathbb{N}^{\infty}$-submoduloid is prime. Additionally, we study the fractions induced by these structures. A review of relevant fuzzy concepts is presented, accompanied by several examples of nexuses to illustrate these ideas.

Key Words: Nexus, fuzzy subnexus, $\mathbb{N}^{\infty}$-moduloid, prime $\mathbb{N}^{\infty}$-submoduloid, fuzzy $\mathbb{N}^{\infty}$-submoduloid.

Contents

1. Introduction

In 1980, Haristchain [5] introduced a sophisticated database structure, called a *plenix*, to effectively manage the vast and varied data that defines a spatial structure (see also [7]). In 1984, Nooshin [10] defined the concept of a *nexus*, a mathematical object representing the constitution of a plenix. A nexus was axiomatically defined using the concept of the *address set*. In 1988, U. M. Swamy and K. L. N. Swamy [14] defined fuzzy prime ideals of rings and derived several useful results. In 2009, Bolourian [2] introduced the concept of *nexus algebras* as an abstract algebraic structure and studied its properties. The notion of *fuzzy sets* was defined by Zadeh in 1965 [15]. In 2011, Saeidi Rashkolia and Hasankhani [13] introduced the concept of *fuzzy subnexuses*, a generalization of nexuses. In 2012, Afkhami and Hasankhani [1] studied *soft nexuses*. In 2014, Hedaayati and Asadi [6] explored the relationship between *normal*, *maximal*, and *product fuzzy subnexuses*. In 2015, Estaji et al. [4] defined fuzzy subnexuses over a nexus and introduced the concept of prime fuzzy subnexuses. In 2020, Kamrani et al. [8,9] provided a structure of moduloids on a *nexus* and explored the concepts of *submoduloids*, *finitely generated submoduloids*, and *prime submoduloids*. In the same year, Norouzi et al. [12] studied subnexuses on N-structures.

The present paper continues the work in [2], [3], and [13], where the properties of moduloids on a nexus and fuzzy subnexuses were explored. In Section 2, we define fuzzy R-submoduloids of an R-moduloid M and fuzzy $\mathbb{N}^{\infty}$-submoduloids of an $\mathbb{N}^{\infty}$-moduloid N, where N is a nexus. Additionally, we introduce the concept of a prime $\mathbb{N}^{\infty}$-submoduloid related to a fuzzy prime $\mathbb{N}^{\infty}$-submoduloid. In Section 3, we investigate the properties of a prime $\mathbb{N}^{\infty}$-submoduloid and provide conditions under which a $\mathbb{N}^{\infty}$-submoduloid is prime. In Section 4, we review the concept of $\mathbb{N}^{\infty}$-moduloid homomorphisms, extend it to fuzzy $\mathbb{N}^{\infty}$-moduloid homomorphisms, and present results on nexuses. We also discuss the concept of $\mathbb{N}^{\infty}$-moduloid quotients in relation to fuzzy $\mathbb{N}^{\infty}$-moduloids.

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2. Preliminaries

We now review the basic definitions and some elementary aspects that are necessary for this paper.

An address is a sequence from $\mathbb{N}^* := \mathbb{N} \cup 0$ such that if $a_k = 0$, then $a_i = 0$ for all $i \geqslant k$. The sequence consisting entirely of zeros is called the empty address and is denoted by $()$. In other words, every non-empty address is of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, where $a_i \in \mathbb{N}$ and $n \in \mathbb{N}$. This is denoted as $(a_1, a_2, \ldots, a_n)$.

Definition 2.1 ([2]) *A set N of addresses is called a nexus if it satisfies the following conditions:*

(i) If $(a_1, a_2, \ldots, a_{n-1}, a_n) \in N$, then $(a_1, a_2, \ldots, a_{n-1}, t) \in N$ for all $0 \leqslant t \leqslant a_n$.

(ii) If $\{a_i\}_{i=1}^{\infty} \in N$ with $a_i \in \mathbb{N}$, then for all $n \in \mathbb{N}$ and $0 \leqslant t \leqslant a_n$, $(a_1, \ldots, a_n - t) \in N$.

Definition 2.2 ([2]) *Let $a \in N$. The level of a is said to be:*

(i) n, if $a = (a_1, a_2, \ldots, a_n)$, for some $a_k \in \mathbb{N}$,

(ii) ∞, if a is an infinite sequence of N,

(iii) 0, if $a = ()$.

The level of a is denoted by $l(a)$ and the stem of a is $\operatorname{stem} a = a_1$. We define $st(N) = \sup\{i \in \mathbb{N} : (i) \in N\}$.

Definition 2.3 ([2]) *Let $a = \{a_i\}$ and $b = \{b_i\}$, with $i \in \mathbb{N}$, be two addresses. Then, we say $a \leqslant b$ if $l(a) = 0$ or if one of the following cases is satisfied:*

(i) If $l(a) = 1$, that is $a = (a_1)$, for some $a_1 \in \mathbb{N}$, and $a_1 \leqslant b_1$,

(ii) If $1 < l(a) < \infty$, then $l(a) \leqslant l(b)$, $a_{l(a)} \leqslant b_{l(a)}$, and for any $1 \leqslant i < l(a)$, $a_i = b_i$,

(iii) If $l(a) = \infty$, then $a = b$.

Definition 2.4 ([2]) *Let $\emptyset \neq A \subseteq N$. The smallest subnexus of N containing A is called the subnexus of N generated by A and is denoted by $\langle A \rangle$. If $A = \{a_1, a_2, \ldots, a_n\}$, we write $\langle A \rangle$ as $\langle a_1, a_2, \ldots, a_n \rangle$.*

If A consists of a single element a, the subnexus $\langle a \rangle$ is called a cyclic subnexus of N. Clearly, the trivial subnexuses of N are $()$ and N itself.

Remark 2.1 ([3]) *Define $\mathbb{N}^{\infty} = \mathbb{N} \cup \{0, \infty\}$. The maximum and minimum of two elements $a, b \in \mathbb{N}^{\infty}$ are denoted by $a \vee b$ and $a \wedge b$, respectively. The structure $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$ forms a semiring.*

The scalar multiplication

$$\cdot : \mathbb{N}^{\infty} \times N \longrightarrow N$$

is defined on N as follows:

$$r \cdot w = \begin{cases} (a_1, a_2, \ldots, a_r) & \text{if } l(w) > r,\ r > 0, \\ (a_1, a_2, \ldots, a_n) & \text{if } l(w) \leqslant r,\ r > 0, \\ 0 & \text{if } r = 0, \\ w & \text{if } r = \infty. \end{cases}$$

for all $r \in \mathbb{N}^{\infty}$ and $w = (a_1, a_2, \ldots, a_n) \in N$.

Note that the structure $(N, +, 0)$ forms a moduloid over $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$ with scalar multiplication $\cdot$. For simplicity, we refer to N as an $\mathbb{N}^{\infty}$-moduloid.

Definition 2.5 ([3]) *Let S be a non-empty subset of N with $0 \in S$. Then S is called a submoduloid of N if $(S, +, 0)$ forms a moduloid over $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$. The set of all $\mathbb{N}^{\infty}$-submoduloids of N is denoted by $\mathrm{SUB}_M(N)$.*

Definition 2.6 ([3]) *Given $X \subseteq N$, the smallest $\mathbb{N}^\infty$-submoduloid of N containing X is called the $\mathbb{N}^\infty$-submoduloid of N generated by X and is denoted by $\langle X \rangle_m$. If $X = \{a\}$, then the $\mathbb{N}^\infty$-submoduloid of N generated by a is called the cyclic $\mathbb{N}^\infty$-submoduloid, denoted by $\langle a \rangle_m$. Clearly, $\langle a \rangle_m = \{r \cdot a \mid \ r \in \mathbb{N}^\infty\}$.*

Definition 2.7 ([3]) *Let $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ be two addresses of N. Suppose there exists $k \in \mathbb{N}$ such that*

$$(a_1 \vee b_1, a_2 \vee b_2, \ldots, a_k \vee b_k) \in N$$

and

$$(a_1 \vee b_1, a_2 \vee b_2, \ldots, a_{k+1} \vee b_{k+1}) \notin N.$$

Then we define the sum of a and b as

$$a + b = (a_1 \vee b_1, a_2 \vee b_2, \ldots, a_k \vee b_k).$$

If no such k exists, then

$$a + b = (a_1 \vee b_1, a_2 \vee b_2, \ldots).$$

Definition 2.8 ([2]) *Let N be a nexus, and let $a = (a_1, a_2, \ldots, a_k)$ be an address of N. The set*

$$q_a = \{(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n) \in N \mid a_{k+i} \in \mathbb{N}, \ \text{for } i = 1, 2, \ldots, n-k\}$$

is called the panel of a and is denoted by q_a. In other words, if $a = (a_1, a_2, \ldots, a_k)$, then an address b of N belongs to q_a if and only if the first k terms of b match the corresponding terms of a.

Note that the panel q_a does not include a itself. Furthermore, $q_{()}$ contains all addresses of N except for the empty address.

Example 2.1 ([3]) *Consider the nexus $N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,2,1), (2,2,2), (2,3,1), (2,3,2)\}$ with the following diagram:*

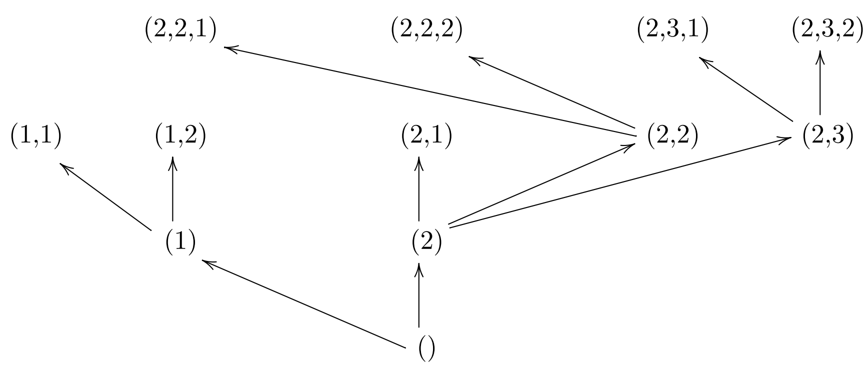

Fig. 1. Diagram of N.

Let $a = (2,2)$ be an address of N. Then $q_a = \{(2,2,1),(2,2,2)\}$ with the following diagram:

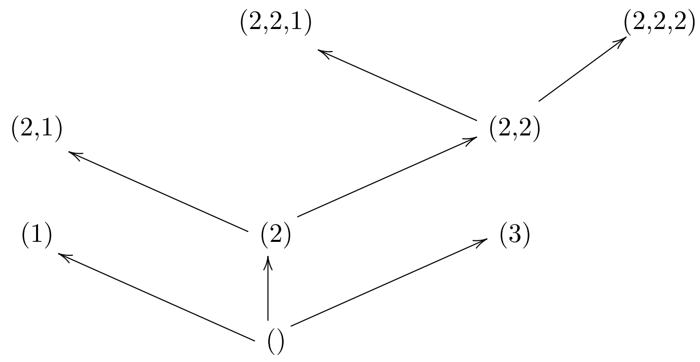

Fig. 2. Diagram of $\mathrm{q_a}$.

Definition 2.9 ([15]) *Let N be a set. A fuzzy subset of N is a mapping*

$$\mu : N \longrightarrow [0,1].$$

If μ and ν are fuzzy subsets of N such that $\nu(x) \leqslant \mu(x)$ for all $x \in N$, we write $\nu \leqslant \mu$ or $\nu \subseteq \mu$ and say that ν is contained in μ or that ν is a fuzzy subset of μ.

Moreovere, for any subsets $A, B \subseteq N$, we write $A \subseteq_\mu B$ if and only if, for every $a \in A$ and $b \in B$,

$$\mu(a) \leqslant \mu(b).$$

Example 2.2 ([13]) *Let N be an arbitrary nexus. Define the fuzzy subset μ of N as follows:*

$$\mu(v) = \begin{cases} \dfrac{1}{a_1 a_2 \dots a_n} & \text{if } l(v) = n, \text{ where } v = (a_1, a_2, \dots, a_n), \\ 1 & \text{if } v = 0, \\ 0 & \text{if } l(v) = \infty. \end{cases}$$

Then $\mu \in FSUB(N)$.

Then μ is a fuzzy subset of N. Let $\nu : N \longrightarrow [0,1]$ be such that for every $a \in N$, $\nu(a) = \mu(a)^2$. It is easy to see that $\nu \leqslant \mu$.

Definition 2.10 ([13]) *Let μ be a fuzzy subset of a nexus N. Then μ is called a fuzzy subnexus of N if, for all $a, b \in N$,*

$$a \leqslant b \quad \textit{implies} \quad \mu(b) \leqslant \mu(a).$$

The set of all fuzzy subnexuses of N is denoted by $FSUB(N)$.

Example 2.3 ([13]) Let N be an arbitrary nexus and μ as Example 2.2. It is easy to see that $\mu \in FSUB(N)$.

3. On fuzzy $\mathbb{N}^\infty$-submoduloid of a nexus

In this section, we define several concepts related to fuzzy submoduloids. First, we define a fuzzy R-submoduloid of an R-moduloid M. Next, we define a fuzzy $\mathbb{N}^\infty$-submoduloid of an $\mathbb{N}^\infty$-moduloid N, where N is a nexus. Additionally, we introduce the concept of a prime $\mathbb{N}^\infty$-submoduloid, which is related to a fuzzy prime $\mathbb{N}^\infty$-submoduloid.

Definition 3.1 *Let M be a left R-moduloid and let μ be a fuzzy subset of M. A non-empty set $S \subseteq M$ with $0 \in S$ is called a fuzzy left R-submoduloid of M if S is an R-submoduloid of M and satisfies the following axioms:*

(i) $\mu(ra) \geqslant \mu(a), \quad \forall r \in R, \forall a \in S$,

(ii) $\mu(a+b) \geqslant \min\{\mu(a), \mu(b)\}, \quad \forall a, b \in S$.

The concept of a fuzzy right R-submoduloid is defined similarly. If S is both a fuzzy left R-submoduloid and a fuzzy right R-submoduloid of M, we say that S is a fuzzy R-submoduloid of M.

Example 3.1 Let $R := M_2(\mathbb{N})$ be 2×2 matrices on natural numbers.

(i) If we take $I_1 = \{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{N}\}$, then I_1 is a left R-moduloid, but not a right R-moduloid. We see that $I_1 \neq \emptyset$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I_1$. For every $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$, $B = \begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix} \in I_1$ and $T = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R$, we have $A + B = \begin{bmatrix} a+a' & 0 \\ b+b' & 0 \end{bmatrix} \in I_1$ and $TA = \begin{bmatrix} ax+by & 0 \\ az+bw & 0 \end{bmatrix} \in I_1$. Hence I_1 is a left R-moduloid. Now, let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AT = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \notin I_1$. Thus, I_1 is not a right R-moduloid,

Define $\mu_1 : I_1 \to [0,1]$ by $\mu_1(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}) = 1 - \frac{1}{a \vee b}$ and $\mu_1(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 1$.

Now, let $T \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in R$. Then $\mu(TA) = \mu(\begin{bmatrix} ax+by & 0 \\ az+bw & 0 \end{bmatrix})$.

Without loss of generality, assume $a \geqslant b$.

Case 1. Let $ax + by \geqslant az + bw$. Then $\mu(A) = 1 - \frac{1}{a}$ and $\mu(TA) = 1 - \frac{1}{ax+by}$. Since $ax + by \geqslant a$, we get $1 - \frac{1}{ax+by} \geqslant 1 - \frac{1}{a}$.

Case 2. Let $az + bw \geqslant ax + by$. Then $\mu(A) = 1 - \frac{1}{a}$ and $\mu(TA) = 1 - \frac{1}{az+bw}$. Since $az + bw \geqslant a$, we get $1 - \frac{1}{az+bw} \geqslant 1 - \frac{1}{a}$.

This shows that $\mu(TA) \geqslant \mu(A)$.

Similarly, if $b \geqslant a$, then (i) is valid.

For $T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mu(TA) = \mu(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 1 \geq \mu(A)$.

Without loss of generality, assume $a + a' \geqslant b + b'$. Then $\mu(A + B) = 1 - \frac{1}{a+a'}$. Now, we have:

Case 1. $a' \geqslant b'$. Hence $\mu(B) = 1 - \frac{1}{a'}$. Therefore, $1 - \frac{1}{a+a'} \geqslant 1 - \frac{1}{a}$ and $1 - \frac{1}{a+a'} \geqslant 1 - \frac{1}{a'}$, and so $\mu(A + B) = 1 - \frac{1}{a+a'} \geqslant \min\{1 - \frac{1}{a}, 1 - \frac{1}{a'}\} = \min\{\mu(A), \mu(B)$.

Case 2. $b' \geqslant a'$. Since $a + a' \geqslant b + b' \geqslant b'$, we get $1 - \frac{1}{a+a'} \geqslant 1 - \frac{1}{b'}$, and so $\mu(A + B) = 1 - \frac{1}{a+a'} \geqslant \min\{1 - \frac{1}{a}, 1 - \frac{1}{b'}\} = \min\{\mu(A), \mu(B)$.

By a similar argument if $b + b'' \geqslant a + a'$, then (ii) is valid.

Therefore, I_1 is a left fuzzy R-submoduloid, since I_1 is not a right R-moduloid, does not a right fuzzy R-submoduloid.

(ii) If we take $I_2 = \{\begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} \mid a, c \in \mathbb{N}\}$. Define $\mu_2 : I_2 \to [0,1]$ by $\mu_2(\begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix}) = 1 - \frac{1}{a \vee c}$ and $\mu_2(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 1$. Then I_2 is a right fuzzy R-submoduloid, but not a left fuzzy R-submoduloid.

(iii) If we take $I_3 = \{\begin{bmatrix} a & c \\ b & d \end{bmatrix} \mid a, b, c, d \in 2\mathbb{N}\}$. Define μ_3 by

$\mu_3(\begin{bmatrix} a & c \\ b & d \end{bmatrix}) = 1 - \frac{1}{a \vee b \vee c \vee d}$ and $\mu_3(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 1$. Then I_3 is a left and right fuzzy R-submoduloid, briefly, fuzzy R-submoduloid.

(iv) If we take $I_4 = \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{N}\}$. Define $\mu_4 : I_4 \to [0,1]$ by $\mu_4(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = 1 - \dfrac{1}{a}$ and $\mu_2(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = 1$. Then I_4 is neither a left fuzzy R-submoduloid nor a right fuzzy R-submoduloid.

Let R be a commutative semiring. It is easy to see that for every $a, b \in M$ and every $r, s \in R$, by definition, $ra = ar$. Hence, every fuzzy left R-submoduloid is also a fuzzy right R-submoduloid, and vice versa. Moreover, Let N be a nexus and $R := \mathbb{N}^\infty$. Since $\mathbb{N}^\infty$ is a commutative semiring, every fuzzy left $\mathbb{N}^\infty$-submoduloid is also a fuzzy right $\mathbb{N}^\infty$-submoduloid.

We denoted the set of all fuzzy $\mathbb{N}^\infty$-submoduloid of an $\mathbb{N}^\infty$-moduloid N, by $FSUB_M(N)$.

Example 3.2 Consider the fuzzy subnexuse μ from Example 2.3. If we take $S_a = \langle a \rangle$, then $S_a \in FSUB_M(N)$.

Remark 3.1 *Notice that $\mu(0) \geqslant \mu(a)$ for all $a \in N$, since, by taking $r = 0$ and applying Definition 3.1(i), we obtain*

$$\mu(0) = \mu(0a) \geqslant \mu(a).$$

Let N be a nexus. We define the *stem* of an element $a = (a_1, \ldots, a_n) \in N$ as

$$\text{stem}(a) = a_1.$$

Notice that for all $r \in \mathbb{N}^\infty$ and $a \in N$, if $r \geqslant n$, then $ra = (a_1, \ldots, a_n)$, and if $r < n$, then $ra = (a_1, \ldots, a_r)$. Hence, we have

$$\text{stem}(ra) = \text{stem}(a) = a_1.$$

Example 3.3 Let N be an $\mathbb{N}^\infty$-moduloid and define $\mu : N \longrightarrow [0,1]$ as follows:

$$\mu(a) = \begin{cases} \dfrac{1}{stem(a)} & \text{if } a \neq 0, \\ 1 & \text{if } a = 0. \end{cases}$$

Let $r \in \mathbb{N}^\infty$. If $0 \neq a \in N$, then $\mu(ra) = \mu(a)$, since $stem(ra) = stem(a)$, we get $\dfrac{1}{stem(ra)} = \dfrac{1}{stem(a)}$. If $a = 0$, then $\mu(ra) = \mu(a) = 1$. Now, let $stem(a) = a_1$ and $stem(b) = b_1$, for some $a, b \in N$, and let $a_1 \geqslant b_1$. Thus, $\dfrac{1}{a_1} \leqslant \dfrac{1}{b_1}$. We have

$$\mu(a+b) = \frac{1}{stem(a+b)} = \frac{1}{a_1 \vee b_1} = \frac{1}{a_1} = \min\{\frac{1}{a_1}, \frac{1}{b_1}\} = \min\{\mu(a), \mu(b)\}.$$

Therefore, $\mu \in FSUB_M(N)$.

Example 3.4 Consider Example 2.2 and let $r \in \mathbb{N}^\infty$. If $0 \neq a \in N$, then $\mu(ra) \geqslant \mu(a)$ since $l(ra) \leqslant l(a)$, we get $\dfrac{1}{l(ra)} \geqslant \dfrac{1}{l(a)}$. If $a = 0$, then $\mu(ra) = \mu(a) = 1$.

Now, let $l(a) \leqslant l(b)$, for some $a, b \in N$. Then $l(a+b) \leqslant l(b)$. Thus, $\dfrac{1}{l(a)} \geqslant \dfrac{1}{l(b)}$ and $\dfrac{1}{l(a+b)} \geqslant \dfrac{1}{l(b)}$. Hence

$$\mu(a+b) = \frac{1}{l(a+b)} \geqslant \frac{1}{l(b)} = \min\{\frac{1}{l(a)}, \frac{1}{l(b)}\} = \min\{\mu(a), \mu(b)\}.$$

Therefor, $\mu \in FSUB_M(N)$.

We define $FSUB_M^*(N) := \{\mu \mid \mu \text{ is a fuzzy subnexus of } N \text{ and } \mu(0) = 1\}$. Since the function $1 : N \longrightarrow [0,1]$ is defined by $1(a) = 1$ for all $a \in N$, we have $1 \in FSUB_M^*(N)$. Therefore, $FSUB_M^*(N) \neq \emptyset$.

Proposition 3.1 *Let N be an $\mathbb{N}^\infty$-moduloid. Then $(FSUB^*_M(N), \wedge, 1)$ is a groupoid, where for all $\mu_1, \mu_2 \in FSUB^*_M(N)$ and $a \in N$, we define*

$$(\mu_1 \wedge \mu_2)(a) = \mu_1(a) \wedge \mu_2(a).$$

Proof: For every $\mu \in FSUB^*_M(N)$, we see that $\mu \wedge 1 = 1 \wedge \mu = \mu$. Let $a \leqslant b$. Then $\mu_1(b) \leqslant \mu_1(a)$ and $\mu_2(b) \leqslant \mu_2(a)$. It follows that $\mu_1(b) \wedge \mu_2(b) \leqslant \mu_1(a) \wedge \mu_2(a)$. Hence $\mu_1(b) \wedge \mu_2(b) \leqslant \mu_1(a) \wedge \mu_2(a)$ and the proof is complete. □

Proposition 3.2 *Let N be an $\mathbb{N}^\infty$-moduloid. Then $(FSUB^*_M(N), \wedge, 1)$ is an $\mathbb{N}^\infty$-moduloid such that for every $r \in \mathbb{N}^\infty$, $\mu \in FSUB^*_M(N)$, and $a \in N$, we have*

$$(r\mu)(a) = \mu(ra).$$

Proof: Assume $a = a'$. Then $ra = ra'$, and so $\mu(ra) = \mu(ra')$. Hence $(r\mu)(a) = (r\mu)(a')$. Let $a \leqslant b$. Then $ra \leqslant rb$, and so $\mu(rb) \leqslant \mu(ra)$. Hence $(r\mu)(b) \leqslant (r\mu)(a)$. Therefore, $r\mu \in FSUB^*_M(N)$.

Let $r = r'$ and $\mu = \mu'$. Then for every $a \in N$, we have $ra = r'a$, and so $\mu(ra) = \mu(r'a) = \mu'(r'a)$. Thus, $(r\mu)(a) = (r'\mu')(a)$. Therefore, $r\mu = r'\mu'$.

Assume $r \leqslant s$. Then for every $a \in N$, $ra \leqslant sa$, and so $\mu(sa) \leqslant \mu(ra)$. Now, we have:

(i) $(r \vee s)\mu(a) = (s\mu)(a) = \mu(sa) = \mu(ra) \wedge \mu(sa) = (r\mu)(a) \wedge (s\mu)(a)$, $(r \vee s)\mu = r\mu \wedge s\mu$,

(ii) $r(\mu \wedge \nu)(a) = (\mu \wedge \nu)(ra) = \mu(ra) \wedge \nu(ra) = (r\mu)(a) \wedge (r\nu)(a)$,

(iii) $r(s\mu)(a) = r\mu(sa) = \mu(r(sa)) = \mu((r \wedge s)a) = (r \wedge s)\mu(a)$,

(iv) $0\mu(a) = \mu(0a) = \mu(0) = 1$.

On the other hand, $r1(a) = 1(ra) = 1$. Then $0\mu = r1 = 1$.

Hence $(FSUB^*_M(N), \wedge, 1)$ is an $\mathbb{N}^\infty$-moduloid. □

Definition 3.2 *Let μ be a fuzzy subset of N and $0 \leqslant \alpha \leqslant 1$. Define the level set of μ with respect to α as*

$$\mu^\alpha := \mu^{-1}([\alpha, 1]) = \{a \in N \mid \mu(a) \geqslant \alpha\}.$$

Also, the support of μ is defined as

$$supp(\mu) = \{a \in N \mid \mu(a) > 0\}.$$

Example 3.5 Consider Example 3.6. If we take μ in Example 3.4 and let $\alpha := \frac{1}{3}$. Then $\mu^{\frac{1}{3}} = \{(1,2,1), (1,2,2), (3,2,1), (3,2,2)\}$ with the following diagram:

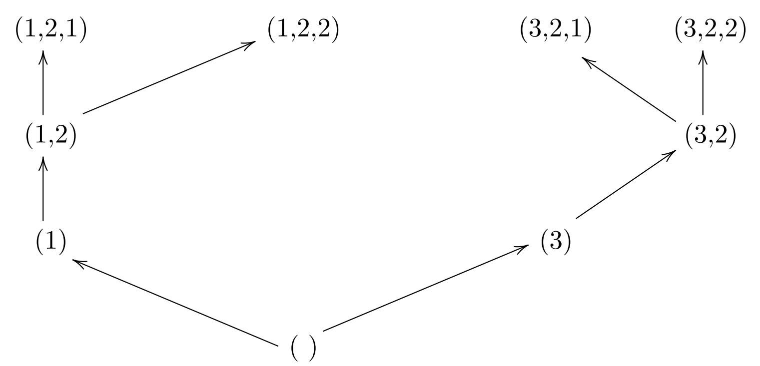

Fig. 5. Diagram of $\mu^{\frac{1}{3}}$.

Also, we see that $supp(\mu) = N$.

Proposition 3.3 *Let μ be a fuzzy subset of N. Then $\mu \in FSUB_M(N)$ if and only if, for every $\alpha \in [0, \mu(0)]$, the level set μ^α is an $\mathbb{N}^\infty$-submoduloid of N.*

Proof: Assume $\mu \in FSUB_M(N)$ and $\alpha \in [0, \mu(0)]$. Clearly, $0 \in \mu^\alpha$. Let $a \in \mu^\alpha$ and $r \in \mathbb{N}^\infty$, then $\mu(ra) \geqslant \mu(a) \geqslant \alpha$, and so $ra \in \mu^\alpha$. Now, let $a, b \in \mu^\alpha$, hence $\mu(a) \geqslant \alpha$, $\mu(b) \geqslant \alpha$, and so $\min\{\mu(a), \mu(b)\} \geqslant \alpha$. This shows that $\mu(a+b) \geqslant \min\{\mu(a), \mu(b)\} \geqslant \alpha$. Thus, $a + b \in \mu^\alpha$. Therefore, μ^α is an $\mathbb{N}^\infty$-submoduloid of N.

Conversely, let for all $\alpha \in [0, \mu(0)]$, μ^α be an $\mathbb{N}^\infty$-submoduloid of N.

Let $\mu(a) = \alpha$, then $a \in \mu^\alpha$. Since μ^α is an $\mathbb{N}^\infty$-submoduloid of N, we get $ra \in \mu^\alpha$, and so $\mu(ra) \geqslant \alpha = \mu(a)$. Let $\mu(a) = \alpha$ and $\mu(b) = \alpha'$, where $a, b \in N$. Suppose $\alpha \leqslant \alpha'$, then $a, b \in \mu^\alpha$ and $\min\{\mu(a), \mu(b)\} = \mu(a) = \alpha$. Since μ^α is an $\mathbb{N}^\infty$-submoduloid of N, we obtain $a + b \in \mu^\alpha$. It follows that $\mu(a + b) \geqslant \alpha = \min\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_M(N)$. □

Proposition 3.4 *Let $A \subseteq N$, where $0 \in A$. Then A is an $\mathbb{N}^\infty$-submoduloid of N if and only if $\chi_A \in FSUB_M(N)$, where*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Proof: Assume $\chi_A \in FSUB_M(N)$, $a \in A$ and $r \in \mathbb{N}^\infty$. Then $\chi_A(a) = 1$, and so $\chi_A(ra) \geqslant \chi_A(a) = 1$. This shows that $\chi_A(ra) = 1$. Thus, $ra \in A$.

Now, let $a, b \in A$. Hence $\mu(a) = \mu(b) = 1$, and so $\min\{\mu(a), \mu(b)\} = 1$. This show sthat $\chi_A(a + b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 1$. It means that $\chi_A(a + b) = 1$, and so $a + b \in A$. Therefore, A is an $\mathbb{N}^\infty$-submoduloid of N.

Conversely, let A be an $\mathbb{N}^\infty$-submoduloid of N. Let $r \in \mathbb{N}^\infty$ and $a, b \in N$. If $a \in A$, then $ra \in A$. Hence $\chi_A(ra) = \chi_A(a) = 1$, and so $\chi_A(ra) \geqslant \chi_A(a)$. If $a \notin A$, then $\chi_A(a) = 0$, and so $\chi_A(ra) \geqslant \chi_A(a) = 0$. Let $a, b \in A$, so $a + b \in A$. Hence $\chi_A(a) = 1$, $\chi_A(b) = 1$, $\chi_A(a + b) = 1 = \min\{\chi_A(a), \chi_A(b)\}$. Therefore, $\chi_A(a + b) = 1 \geqslant 1 = \min\{\chi_A(a), \chi_A(b)\}$. Now, let $a \notin A$, then $\chi_A(a) = 0$, hence $\min\{\chi_A(a), \chi_A(b)\} = 0$. It follows that $\chi_A(a + b) \geqslant 0 = \min\{\chi_A(a), \chi_A(b)\}$. □

For every $a \in N$ and $r \in \mathbb{N}^\infty$, we put

$$\mathbb{N}^\infty[a, \mu] = \{s \in \mathbb{N}^\infty |\ \mu(sa) \neq 0\} \text{ and } N[r, \mu] = \{b \in N |\ \mu(rb) \neq 0\}.$$

Now, let $\alpha \in (0, \mu(0)]$, $r \in \mathbb{N}^\infty$ and $a \in N$, we put

$$\mathbb{N}^\infty_\alpha[a, \mu] = \{s \in \mathbb{N}^\infty |\ \mu(sa) \geqslant \alpha\} \text{ and } N_\alpha[r, \mu] = \{b \in N |\ \mu(rb) \geqslant \alpha\}.$$

Definition 3.3 *Let $\mu \in FSUB_M(N)$. Then μ is said to be prime if for every $a \in N$ with $\mu(a) = 0$, we have either*

$$\mathbb{N}^\infty[a, \mu] = \emptyset$$

or for every $r \in \mathbb{N}^\infty[a, \mu]$, we have

$$N[r, \mu] = N.$$

Definition 3.4 *Let $\mu \in FSUB_M(N)$. Then μ is said to be prime with respect to α, where $\alpha \in (0, \mu(0)]$, if for every $a \in N$ with $\mu(a) < \alpha$, we have either*

$$\mathbb{N}^\infty_\alpha[a, \mu] = \emptyset$$

or for every $r \in \mathbb{N}^\infty_\alpha[a, \mu]$, we have

$$N_\alpha[r, \mu] = N.$$

Definition 3.5 ([3]) *A proper $\mathbb{N}^\infty$-submoduloid P of an $\mathbb{N}^\infty$-moduloid N is called prime if for some $r \in \mathbb{N}^\infty$ and $a \in N$, the condition $ra \in P$ implies that either $a \in P$ or $rN \subseteq P$, where $rN = \{rn |\ n \in N\}$.*

Example 3.6 Consider the nexus $N = \{(), (1), (2), (3), (1,1), (1,2), (3,1), (3,2), (3,3), (1,2,1), (1,2,2), (3,2,1), (3,2,2)\}$ with the following diagram:

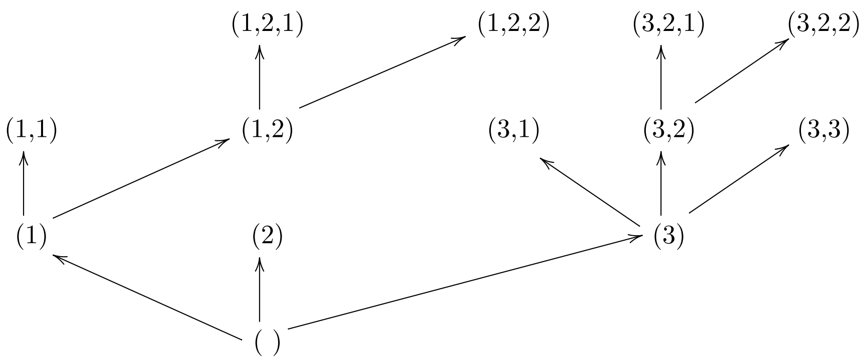

Fig. 3. Diagram of N.

If we take $P := \{(), (1), (2), (3), (1,1), (1,2), (3,1), (3,2), (3,3)\}$ with the following diagram:

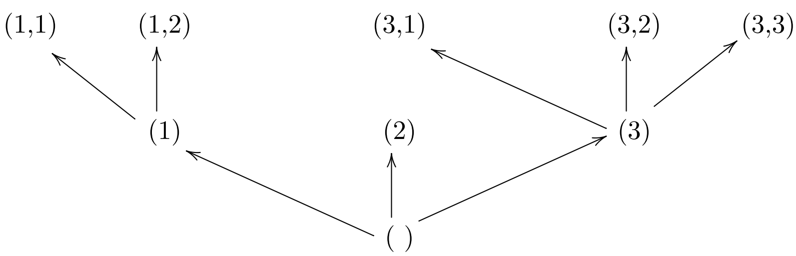

Fig. 4. Diagram of P.

Then P is a prime $\mathbb{N}^\infty$-submoduloid of N.

Proposition 3.5 *Let $\mu \in FSUB_M(N)$. Then $supp(\mu)$ is a prime $\mathbb{N}^\infty$-submoduloid of N if and only if μ is prime.*

Proof: Assume $supp(\mu)$ is a prime $\mathbb{N}^\infty$-submoduloid of N, and let $\mu(a) = 0$. Suppose $\mathbb{N}^\infty[a, \mu] \neq \emptyset$ and $r \in \mathbb{N}^\infty[a, \mu]$. Since $\mu(ra) \neq 0$, $ra \in supp(\mu)$ and $a \notin supp(\mu)$. Then $rN \subseteq supp(\mu)$, and so for every $y \in N$, $ry \in supp(\mu)$. Thus, $\mu(ry) \neq 0$, for every $y \in N$. It follows that $N[r, \mu] = N$. Therefor, μ is prime.

Conversely, let μ be prime. Let $ra \in supp(\mu)$ and $a \notin supp(\mu)$. Then $\mu(a) = 0$ and $\mu(ra) \neq 0$. Hence $N[r, \mu] = N$, and so $\mu(ry) \neq 0$, for every $y \in N$. Thus, $rN \subseteq supp(\mu)$. This shows that $supp(\mu)$ is a prime $\mathbb{N}^\infty$-submoduloid of N. $\square$

Example 3.7 Consider Example 3.6, and let μ be in Example 3.3. Define $\nu : N \longrightarrow [0,1]$ such that for every $a \in N$, $\nu(a) = \mu(a)$, if $l(a) \leqslant 2$ and $\nu(a) = 0$, if $l(a) > 2$. For every $a \in N$, such that $\nu(a) = 0$, we have $0, 1, 2 \in \mathbb{N}^\infty[a, \nu]$. On the other side, we see that $N[0, \nu] = N[1, \nu] = N[2, \nu] = N$. Hence by Proposition 3.5, $supp(\nu)$ is a prime $\mathbb{N}^\infty$-submoduloid of N. Since $supp(\nu) = P$, P is a prime $\mathbb{N}^\infty$-submoduloid of N.

Proposition 3.6 *Let $\mu \in FSUB_M(N)$ and $\alpha \in (0, \mu(0)]$. Then μ^α is a prime $\mathbb{N}^\infty$-submoduloid of N if and only if μ is prime with respect to α.*

Proposition 3.7 *Let $\mu \in FSUB_M(N)$ and $a = (a_1, \dots, a_k) \in N$. Then $q_a \subseteq_\mu \{a\} \subseteq_\mu \langle a\rangle_m$.*

Proof: Assume $b \in q_a$. Then there exists $k \in \mathbb{N}^\infty$ such that $b = (a_1, \dots, a_k, a_{k+1}, \dots, a_n)$, and so $a = kb$. Hence $\mu(a) = \mu(kb) \geqslant \mu(b)$. Thus, $q_a \subseteq_\mu \{a\}$. For every $b \in \langle a\rangle_m$, there exists $r \in \mathbb{N}^\infty$ such that $b = ra$. Then $\mu(b) = \mu(ra) \geqslant \mu(a)$. This shows that $\{a\} \subseteq_\mu \langle a\rangle_m$. Therefor, $q_a \subseteq_\mu \{a\} \subseteq_\mu \langle a\rangle_m$. □

Proposition 3.8 *Let $N = \langle a\rangle_m$ be a cyclic $\mathbb{N}^\infty$-moduloid and μ be a fuzzy subset of N. Then $\mu \in FSUB_M(N)$ if and only if, for every $b, b' \in N$, with $b \leqslant b'$, it holds that $\mu(b') \leqslant \mu(b)$.*

Proof: Let $\mu \in FSUB_M(N)$ and $b, b' \in N$ such that $b \leqslant b'$ for some $b, b' \in N = \langle a\rangle_m$. Hence there exist $r, r' \in \mathbb{N}^\infty$ such that $b = ra$, $b' = r'a$. Thus, $r \leqslant r'$. Let $a = \{a_i\}$. Then $ra = (a_1, \dots, a_r)$, $r'a = (a_1, \dots, a_r, \dots, a_{r'})$, $ra = r(r'a) = rb'$, $b = rb'$. It means that $\mu(b) = \mu(rb') \geqslant \mu(b')$.

Conversely, let $r \in \mathbb{N}^\infty$ and $b \in N$. Since $rb \leqslant b$, we obtain $\mu(rb) \geqslant \mu(b)$. Let $b, b' \in N$ and $b \leqslant b'$. Hence $b + b' = b'$, and so $\mu(b) \geqslant \mu(b')$. Then $\mu(b + b') = \mu(b') = min\{\mu(b), \mu(b')\}$. □

The following corollary is a direct consequence of Proposition 3.8.

Corollary 3.1 *Let $N = \langle a\rangle$ be a cyclic nexus and $\mu : N \longrightarrow [0, 1]$ be a fuzzy subnexus of N. Then $\mu \in FSUB_M(N)$.*

Example 3.8 Let $N = \langle a\rangle$ be a cyclic nexus and μ be the fuzzy subnexus that defined in Example 2.3. By Corollary 3.1, μ is a fuzzy $\mathbb{N}^\infty$-submoduloid of N.

4. On prime $\mathbb{N}^\infty$-submoduloids of $FSUB^*_M(N)$

In this section, we introduce the concept of a prime $\mathbb{N}^\infty$-submoduloid and explore its properties. We provide conditions under which a $\mathbb{N}^\infty$-submoduloid is prime.

Definition 4.1 *We say that $\emptyset \neq I \subseteq \mathbb{N}^\infty$ is an ideal of $\mathbb{N}^\infty$ if I satisfies the following axioms:*

(i) $s \vee t \in I$ for all $s, t \in I$,

(ii) $r \wedge s \in I$ for all $s \in I$ and $r \in \mathbb{N}^\infty$.

Hence, if I is an ideal of $\mathbb{N}^\infty$ and $s \in I$, then for every $r \leqslant s$, $r \in I$. Thus, for every proper ideal I of $\mathbb{N}^\infty$, there exists $n \in \mathbb{N}^\infty$ such that $I = \{0, 1, 2, \dots, n\} := I_n$.

Let N be an $\mathbb{N}^\infty$-moduloid and let P be a proper $\mathbb{N}^\infty$-submoduloid of N. Then, the set $(P : N)$, defined by

$$(P : N) = \{r \in \mathbb{N}^\infty |\ rN \subseteq P\},$$

is a proper ideal of $\mathbb{N}^\infty$.

Example 4.1 Consider the nexus $N = \{(), (1), (2), (3), (1, 1), (1, 2), (1, 3), (1, 1, 1), (1, 1, 2), (1, 1, 3)\}$ with the following diagram:

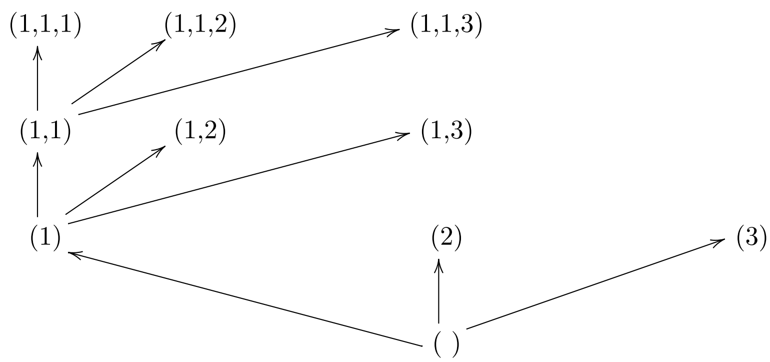

Fig. 6. Diagram of N.

If we take $P := \{(), (1), (2), (3), (1,1), (1,2), (1,3)\}$ with the following diagram.

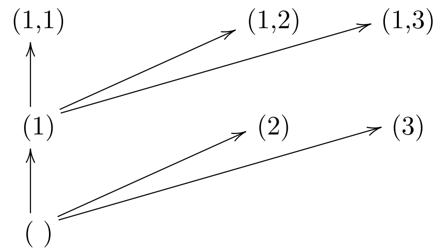

Fig. 7. Diagram of P.

Then we see that $(P : N) = I_2 = \{0, 1, 2\}$.

Lets define

$$Spec(FSUB^*_M(N)) = \{P|\ P \text{ is a prime } \mathbb{N}^\infty\text{-submoduloid of } FSUB^*_M(N)\}.$$

Proposition 4.1 *Let P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. Then $P \in Spec(FSUB^*_M(N))$ if and only if there exists $n \in \mathbb{N}^\infty$ such that $nFSUB^*_M(N) \subseteq P$ and for every $r > n$, if $r\mu \in P$, then $\mu \in P$.*

Proof: Assume $P \in Spec(FSUB^*_M(N))$, there exists $n \in \mathbb{N}^\infty$ such that $(P : FSUB^*_M(N)) = I_n$. Let $r\mu \in P$ and $r > n$, then $r \notin I_n$. Hence $\mu \in P$. Conversely, let there exists $n \in \mathbb{N}^\infty$ such that $nFSUB^*_M(N) \subseteq P$. Then $n \in (P : FSUB^*_M(N))$. Let $(P : FSUB^*_M(N)) = I_m$. Then $n \in I_m$, and so $n \leqslant m$. If $m \neq n$, since $P \neq N$, there exists $\mu \in N \setminus P$. So, $\mu \notin P$ and $m\mu \in P$, which is a contradiction. Therefor, $m = n$. Hence $(P : FSUB^*_M(N)) = I_n$. If $r\mu \in P$ and $r \notin I_n$, then $r > n$, and so $\mu \in P$. Therefore, $P \in Spec(FSUB^*_M(N))$. □

Proposition 4.2 *Let P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$, and assume that $\bigcup_{\mu \in P} Im(\mu) \neq [0, 1]$. Then $(P : FSUB^*_M(N)) = \{0\}$.*

Proof: Let $\delta \in [0, 1] \setminus \bigcup_{\mu \in P} Im\mu$ and $\nu : N \longrightarrow [0, 1]$ be such that $\nu(x) = \delta$, for every $0 \neq x \in N$.

If $0 \neq r \in (P : FSUB^*_M(N))$, then $r\nu \in P$. There exists $\mu \in P$ such that $r\nu = \mu$, so $r\nu(x) = \nu(rx) = \delta = \mu(x)$. Hence $\delta \in \bigcup_{\mu \in P} Im\mu$, which is a contradiction. Therefore, $r = 0$, and so $(P : FSUB^*_M(N)) = \{0\}$. □

The following example shows that the converse of Proposition 4.2, may not true.

Example 4.2 Consider the nexus $N = \{(), (1), (2), \ldots\}$ with the following diagram: for $n \in \mathbb{N}$

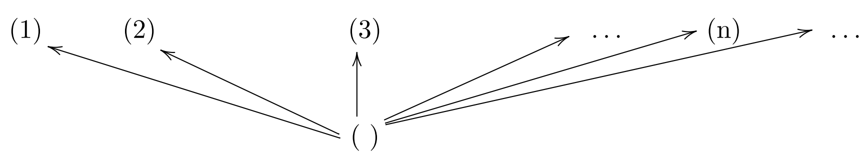

Fig. 8. Diagram of N.

Let $\delta \in [0,1]$, for every $0 \neq a \in N$, put $\mu(a) := \delta$ $\mu = \delta$ and $P := \{\delta | \ \delta \in [0,1]\}$. Then $\bigcup_{\mu \in P} Im\mu = [0,1]$. On the other hand, if we define $\nu : N \longrightarrow [0,1]$ by $\nu(i) = \dfrac{1}{i}$ for every $(i) \in N$ and $\nu(()) = 1$. Thus, $\nu \in FSUB^*_M(N)$. If $0 \neq r \in (P : FSUB^*_M(N))$, then $r\nu \in P$. Since $r\nu = \nu$, $\nu \in P$, which is a contradiction. Therefore, $(P : FSUB^*_M(N)) = \{0\}$.

Corollary 4.1 *Let P be a countable proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. Then*

$$(P : FSUB^*_M(N)) = \{0\}.$$

Proof: Since P is a countable set, we get $\bigcup_{\mu \in P} Im\mu$ is a countable set. Then there exists $\delta \in [0,1] \setminus \bigcup_{\mu \in P} Im\mu$. Now, by Proposition 4.2, the proof is complete. □

Proposition 4.3 *Let P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$ and $(P : FSUB^*_M(N)) \neq \{0\}$. Then*

(i) $\bigcup_{\mu \in P} Im(\mu) = [0,1]$,

(ii) $\{\delta \mid \delta \in [0,1]\} \subseteq P$.

Proof: (i) Using Proposition 4.2, the proof is obvious.

(ii) Assume $0 \neq r \in (P| \ FSUB^*_M(N))$. Hence for every $\delta \in [0,1]$, $r\delta \in P$. This shows that $r\delta = \delta$, and so $\{\delta | \delta \in [0,1]\} \subseteq P$. □

Proposition 4.4 *Let $rise(N) = n$ and P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$, with $(P : FSUB^*_M(N)) = I_k$. Then $n > k$.*

Proof: Since $rise(N) = n$, for every $a \in N$, we have $na = a$. Hence for every $\mu \in FSUB^*_M(N)$, we get $n\mu = \mu$. Then $nFSUB^*_M(N) = FSUB^*_M(N) \nsubseteq P$. Therefore, $n \notin (P : FSUB^*_M(N))$, and so $n \notin I_k$. Hence $n > k$. □

Proposition 4.5 *Let $rise(N) = n$, P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$, and $(P : FSUB^*_M(N)) = I_{n-1}$. Then P is a prime $\mathbb{N}^\infty$-submoduloid.*

Proof: Assume $r \in \mathbb{N}^\infty$, $\nu \in FSUB^*_M(N)$ and $r\nu \in P$. Let $r \notin (P : FSUB^*_M(N))$. Then $n \leqslant r$. Since $rise(N) = n$, we have $r\nu = \nu$. This shows that $\nu \in P$. Therefore, P is a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. □

Example 4.3 Consider Example 5.2 and let $P = \{r\mu | \mu \in FSUB^*_M(N), 0 \leqslant r \leqslant 2\}$. Then P is a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$ and $(P : FSUB^*_M(N)) = I_2$. Then by Propositions 4.4 and 4.5, P is a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.

By Propositions 4.4 and 4.5 we have:

Corollary 4.2 *Let $rise(N) = 1$. Then every proper $\mathbb{N}^\infty$-submoduloid P of $FSUB^*_M(N)$ is prime.*

Example 4.4 Consider Example 4.2. By Corollary 4.2, P is a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.

Definition 4.2 *Let $r \in \mathbb{N}^\infty$. Define a relation R_r on $FSUB^*_M(N)$ such that for every $\mu, \nu \in FSUB^*_M(N)$, we say $\mu R_r \nu$ if and only if $r\mu = r\nu$.*

Clearly, R_r is an equivalence relation on $FSUB^*_M(N)$, and for every $\mu \in FSUB^*_M(N)$, we define the equivalence class $[\mu]_r = \{\nu \in FSUB^*_M(N) \mid \mu R_r \nu\}$.

Theorem 4.1 *Let P be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$, $(P : FSUB^*_M(N)) = I_k$, and suppose that $rise(N) \geqslant k+2$. If there exist $r \in \mathbb{N}^\infty$, $x \in N$, $\mu_0 \in P$, and $\delta \in [0,1]$ such that:*

- $k < r < l(x)$,
- $\delta \leqslant \inf\{\mu_0(a) \mid \ a < x, l(a) \leqslant r\}$, *and*
- $\delta \notin \bigcup_{\mu \in ([\mu_0]_r \cap P)} Im\, \mu$.

Then P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.

Proof: Define $\nu : N \longrightarrow [0,1]$ with

$$\nu(a) = \begin{cases} \delta & \text{if } a = x, \\ \mu_0(a) & \text{if } l(a) \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following cases:
Case 1. Let $l(a), l(b) \leqslant r$. Then $\mu_0(b) \leqslant \mu_0(a)$, and so $\nu(b) \leqslant \nu(a)$.
Case 2. Let $l(a) \leqslant r < l(b)$ and $b = x$. Then $\nu(b) = \nu(x) = \delta < \mu_0(a) = \nu(a)$.
Case 3. Let $l(a) \leqslant r < l(b)$ and $b \neq x$. Then $\nu(b) = 0 < \nu(a)$.
Case 4. Let $l(a), l(b) > r$. Then $\nu(b) = \nu(a) = 0$.
Therefore, $\nu(a) \leqslant \nu(b)$, and so ν is a subnexus of N. Now, we will show that $r\nu \in P$, but $r \notin (P : F^*)$ and $\nu \notin P$. Let $a \in N$. Then $l(ra) \leqslant r$. Hence $r\nu(a) = \nu(ra) = \mu_0(ra) = r\mu_0(a)$, and so $r\nu \in P$. Since $r > k$, $r \notin (P : FSUB^*_M(N))$. If $\nu \in P$, there exist $\mu \in P$ such that $\nu = \mu$. If $\mu \in [\mu_0]_r$, we have $\delta = \nu(x) = \mu(x)$, and so $\delta \in \bigcup\limits_{\mu \in [\mu_0]_r} Im\mu$, which is a contradiction.

If $\mu \notin [\mu_0]_r$, then $r\mu \neq r\mu_0$. Hence there exist $a \in N$ such that $r\mu(a) \neq r\mu_0(a)$. So, $\mu(ra) \neq \mu_0(ra)$. Since $l(ra) \leqslant r$, we have $\nu(ra) = \mu_0(ra) \neq \mu(ra)$, which is a contradiction. Thus, P is not prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. □

Example 4.5 Consider the nexus $N = \{(), (1), (2), (3), (1,1), (1,2), (2,1), (2,2), (2,3), (3,1)\}$ with the following diagram:

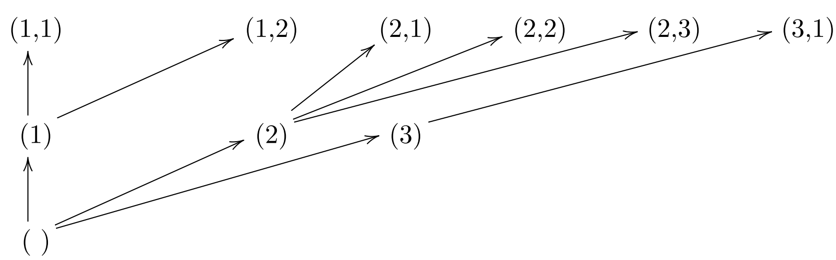

Fig. 9. Diagram of N.

We see that $rise(N) = 2$. Consider μ_1 in Example 2.3 and μ_2 in Example 4.2, and let $P := \langle \mu_1, \mu_2 \rangle_m$. By Corollary 4.1, we get $(P : FSUB^*_M(N)) = 0$. Take $\mu_0 := \mu_2$, $r := 1$, $x := (1,1)$ and $\delta := \dfrac{1}{7}$. Then by Theorem 4.1, P is not a prime $\mathbb{N}^\infty$-submodulod of $FSUB^*_M(N)$.

Proposition 4.6 *Let $rise(N) \geqslant 2$ and let P be a countable proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. If there exists $\mu_0 \in P$ such that $\alpha = \inf\{\mu_0(a) \mid a \in N\} \neq 0$, then P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.*

Proof: Using Corollary 4.1, $(P : FSUB^*_M(N)) = 0$. There exist $r \in \mathbb{N}^\infty$ and $x \in N$ such that $1 \leqslant r < l(x)$. Since P is countable, $\bigcup\limits_{\mu \in [\mu_0]_r \cap P} Im\mu$ is countable. Hence there exist $\delta \in [0,1]$ such that $\delta \notin \bigcup\limits_{\mu \in [\mu_0]_r \cap P} Im\mu$ and $\delta \leqslant \alpha$. Now, by Theorem 4.1, the proof is complete. □

The following example demonstrates that in Proposition 4.6, the condition of countability is necessary, and it cannot be removed.

Example 4.6 Let $N = \{(), (1), (2), (3), (1,1), (1,2)\}$ be a nexus with the following diagram:

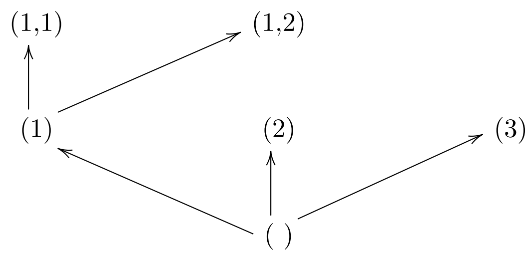

Fig. 10. Diagram of N.

and let $P := \{\delta |\ \delta \in [0,1]\} \cup \{\mu |\ \mu \in FSUB^*_M(N)\} \cup \{1\}$. We can see that P is uncountable $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$ and $(P : FSUB^*_M(N)) = I_1$. Now, since $rise(N) = 2$, by Proposition 4.5, P is a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.

Corollary 4.3 *Let $rise(N) \geqslant 2$ and let P be a countable proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$ such that there exists $\mu_0 \in P$ with $0 \notin Im(\mu_0)$. Then P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.*

Proof: There exist $r \in \mathbb{N}^\infty$ and $x \in N$ such that $1 \leqslant r < l(x)$. Since P is countable, $\bigcup\limits_{\mu \in P} Im\mu$ is countable. On the other hand, since N is finite and $0 \in Im\mu_0$, we get $0 \neq \alpha = min\{\mu(a) |\ a \in N\}$. Now, by Proposition 4.6, P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$. □

Corollary 4.4 *Let $rise(N) \geqslant 2$ and let $\mu \in FSUB^*_M(N)$ such that there exists $x \in N$ with $l(x) \geqslant 2$ and $\mu(x) \neq 0$. Then $\langle \mu \rangle_m$ is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$.*

Proof: Let $P = \langle \mu \rangle_m$. By Corollary 4.1, we get $(P : FSUB^*_M(N)) = \{0\}$. Let $r = 1$. Since $\mu(x) \neq 0$, for every $a \leqslant x$, we have $\mu(x) \leqslant \mu(a)$, and so $\mu(a) \neq 0$. Hence $0 \neq \alpha = inf\{\mu(a) |\ a < x, l(a) \leqslant r\}$. Therefor, there exist $\delta \leqslant \alpha$ such that $\delta \notin \bigcup\limits_{\nu \in ([\mu]_r \cap P)} Im\nu$. Now, by Theorem 4.1, the proof is complete. □

Example 4.7 Let N be an $\mathbb{N}^\infty$-moduloid with $rise(N) \geqslant 2$. Consider μ in Example 2.3 and ν in Example 4.2. Then there exist $x \in N$ such that $\nu(x) \neq 0$ and $\mu(x) \neq 0$. Hence by Corollary 4.4, $\langle \mu \rangle_m$ and $\langle \nu \rangle_m$ are not prime.

Proposition 4.7 *Let $N = \langle a \rangle$ be a cyclic nexus and Q be a proper $\mathbb{N}^\infty$-submoduloid of $FSUB^*_M(N)$ such that for every $\mu \in Q$, $0 \in Im\,\mu$. Then $Q = \{0\}$.*

Proof: Assume $\mu \in Q$. Since $0 \in Im\mu$, there exist $b \in n$ such that $\mu(b) = 0$. Since $b \leqslant a$, $\mu(a) \leqslant \mu(b) = 0$, and so $\mu(a) = 0$. Now, let $x \in N$ be arbitrary, there exist $r \in \mathbb{N}^\infty$ such that $x = ra$. Hence $r\mu \in Q$, and so $r\mu(a) = 0$. Thus, $\mu(x) = \mu(ra) = r\mu(a) = 0$. Therefor, $Q = \{0\}$. □

Corollary 4.5 *Let $N = \langle a \rangle$ be a cyclic nexus with $l(a) \geqslant 2$, and let $P \neq \{0\}$ be a countable proper $\mathbb{N}^\infty$-submoduloid of $FSUB_M^*(N)$. Then P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB_M^*(N)$.*

Proof: Since $P \neq \{0\}$, by Proposition 4.7, there exist $\mu \in P$ such that $0 \notin Im\mu$. Hence by Corollary 4.3, P is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB_M^*(N)$. $\square$

Example 4.8 Consider the cyclic nexus $N = \langle (1, 2, 3) \rangle$ with the following diagram:

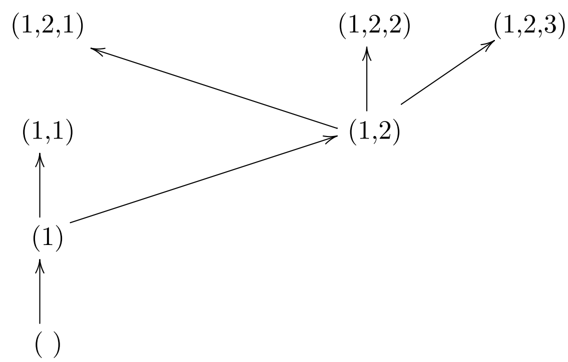

Fig. 11. Diagram of cyclic nexus N.

By Corollary 4.5, for every $0 \neq \mu \in FSUB_M^*(N)$, we get $\langle \mu \rangle_m$ is not a prime $\mathbb{N}^\infty$-submoduloid of $FSUB_M^*(N)$.

5. On fuzzy $\mathbb{N}^\infty$-moduloid homomorphisms

We extend the concept of $\mathbb{N}^\infty$-moduloid homomorphisms to fuzzy $\mathbb{N}^\infty$-moduloid homomorphisms and explore several related properties within the framework of nexuses. Additionally, we examine the relationship between $\mathbb{N}^\infty$-moduloid quotients and their fuzzy counterparts, providing further insights into their structural properties.

Definition 5.1 *Let N and M be two $\mathbb{N}^\infty$-moduloids, and let $f : N \to M$ be a mapping. We say that f is an $\mathbb{N}^\infty$-moduloid homomorphism if it satisfies the following conditions:*

(i) $f(a + b) = f(a) + f(b)$, *for all* $a, b \in N$,

(ii) $f(ra) = rf(a)$, *for all* $r \in \mathbb{N}^\infty$ *and* $a \in N$.

Proposition 5.1 *Let $f : N \longrightarrow N'$ be a homomorphism of nexuses, and suppose that for every $r \in \mathbb{N}^\infty$ and $a \in N$, we have $f(ra) = rf(a)$. Then, $\Gamma_f : FSUB_M^*(N') \longrightarrow FSUB_M^*(N)$, defined by $\Gamma_f(\mu) = \mu \circ f$, is an $\mathbb{N}^\infty$-homomorphism of moduloids.*

Proof: Since $(\mu \circ f)(a) = \mu(f(a))$, $\mu \circ f$ is well defined. Let $a, b \in N$ and $a \leqslant b$, then $f(a) \leqslant f(b)$, and so $\mu(f(b)) \leqslant \mu(f(a))$. Hence $(\mu \circ f)(b) \leqslant (\mu \circ f)(a)$, and so $\mu \circ f$ is a fuzzy subnexus of N.

$\Gamma_f(\mu_1 \wedge \mu_2)(a) = (\mu_1 \wedge \mu_2)(f(a)) = \mu_1(f(a)) \wedge \mu_2(f(a)) = \Gamma_f(\mu_1)(a) \wedge \Gamma_f(\mu_2)(a)$. Then $\Gamma_f(\mu_1 \wedge \mu_2) = \Gamma_f(\mu_1) \wedge \Gamma_f(\mu_2)$.

$\Gamma_f(r\mu)(a) = (r\mu)(f(a)) = \mu(rf(a)) = \mu(f(ra)) = (\mu \circ f)(ra) = r(\mu \circ f)(a) = (r(\Gamma_f(\mu))(a)$. Then $\Gamma_f(r\mu) = r\Gamma_f(\mu)$. Hence the proof is complete. $\square$

Example 5.1 Let N be an $\mathbb{N}^\infty$-moduloid and $\mu : N \longrightarrow [0, 1]$ be such that for every $a \in N$, $\mu(a) = \dfrac{1}{l(a)}$ if $a \neq 0$ and $\mu(a) = 1$ if $a = 0$. Then $\mu \in FSUB_(N)$.

Example 5.2 Consider two nexuses $N = \{(), (1), (2), (2,1), (2,2)\}$ and $N' = \{(), (1), (2), (1,1), (1,2)\}$ with the following diagrams:

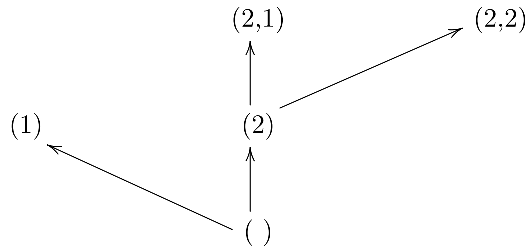

Fig. 12. Diagram of N.

and

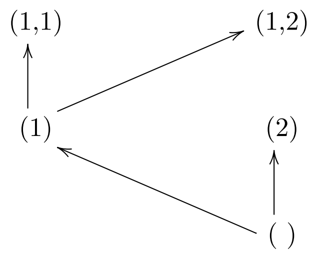

Fig. 13. Diagram of N′.

Define a map $f : N \longrightarrow N'$ by $f(()) = 0$, $f((1)) = f((2)) = (1)$, $f((2,1)) = f((2,2)) = (1,1)$. Then f is a homomorphism of nexuses, and for every $r \in \mathbb{N}^\infty$ and $a \in N$, $f(ra) = rf(a)$. Let μ be in Example 3.4. Hence $\mu \in FSUB(N')$. Now, let $\nu : N \longrightarrow [0,1]$ be such that $\nu(()) = \nu((1)) = \nu((2)) = 1$ and $\nu((2,1)) = \nu((2,2)) = \dfrac{1}{2}$. By Proposition 5.1, we have $\Gamma_f : FSUB^*_M(N') \longrightarrow FSUB^*_M(N)$ is an $\mathbb{N}^\infty$-homomorphism of moduloids. Since $\Gamma_f(\mu) = \nu$, we get ν is a fuzzy subnexuses of N.

Proposition 5.2 *Let N and M be two $\mathbb{N}^\infty$-moduloids, and let $f : N \longrightarrow M$ be an $\mathbb{N}^\infty$-moduloid homomorphism. Let $\mu \in FSUB_M(N)$. Then $f(\mu)(y) \in FSUB_M(M)$, where*

$$f(\mu)(y) = \begin{cases} 0 & \text{if } y \notin Im(f), \\ \sup\{\mu(a) \mid \ f(a) = y\} & \text{if } y \in Im(f). \end{cases}$$

Proof: Assume $y \in M$ and $r \in \mathbb{N}^\infty$. If $y \notin Imf$, $f(\mu)(y) = 0$, hence $f(\mu)(ry) \geqslant 0 = f(\mu)(y)$. Let $y \in M$ and $y \in Imf$. Since f is an $\mathbb{N}^\infty$-moduloid homomorphism, every $a \in f^{-1}(y)$ implies $ra \in f^{-1}(ry)$, on the other hand, since μ is a fuzzy $\mathbb{N}^\infty$-submoduloid of N, for every $a \in f^{-1}(y)$, $\mu(ra) \geqslant \mu(a)$. Then we have

$$\begin{array}{rcl} f(\mu)(ry) & = & sup\{\mu(b) | \ f(b) = ry\} \\ & \geqslant & sup\{\mu(ra) | \ f(a) = y\} \\ & \geqslant & sup\{\mu(a) | \ f(a) = y\} \\ & = & f(\mu)(y). \end{array}$$

Now, let $y, y' \in M$ and $y \notin Imf$ or $y' \notin Imf$. Then $f(\mu)(y) = 0$ or $f(\mu)(y') = 0$, and so $\min\{f(\mu)(y), f(\mu)(y')\} = 0$. Hence $f(\mu)(y + y') \geqslant 0 = \min\{f(\mu)(y), f(\mu)(y')\}$. Now, let $y, y' \in Imf$. By the proof of [12, Prop. 1.1.10], we have $f(\mu)(y + y') \geqslant \min\{f(\mu)(y), f(\mu)(y')\}$. $\square$

Proposition 5.3 *Let $f : N \longrightarrow M$ be an $\mathbb{N}^{\infty}$-moduloid homomorphism, and let $\mu \in FSUB_M(N)$. Then the following hold:*

(i) $f(\mu)(0) = \mu(0)$,

(ii) For every $\alpha \in (0, \mu(0)]$, *we have* $\mu^{\alpha} \subseteq f^{-1}(f(\mu)^{\alpha})$,

(iii) $supp(\mu) \subseteq f^{-1}(supp(f(\mu)))$,

(iv) If f is bijective, then for every $b \in M$, $f(\mu)(b) = \mu(f^{-1}(b))$.

Proof: (i) We have $f(\mu)(0) = sup\{\mu(a)|\ a \in Ker(f)\}$, where $Ker(f) = \{a \in N|\ f(a) = 0\}$. Since $0 \in Ker(f)$ and for every $a \in N$, $\mu(0) \geqslant \mu(a)$, we have $sup\{\mu(a)|\ a \in Ker(f)\} = \mu(0)$. Therefore, $f(\mu)(0) = \mu(0)$.

(ii) Assume $a \in \mu^{\alpha}$. Then $\mu(a) \geqslant \alpha$. It follows that $f(\mu)(f(a)) \geqslant \mu(a) \geqslant \alpha$. Hence $f(a) \in f(\mu)^{\alpha}$. Thus, $a \in f^{-1}(f(\mu)^{\alpha})$. Therefore, $\mu^{\alpha} \subseteq f^{-1}(f(\mu)^{\alpha})$.

The proofs (iii)-(iv) are straightforward. $\square$

Example 5.3 Consider Example 5.2 and define $f : N \longrightarrow M$ by $f(()) = 0$, $f((1)) = f((2)) = (1)$, $f((2,1)) = f((2,2)) = (1,1)$. Then f is an $\mathbb{N}^{\infty}$-moduloid homomorphism and $Imf = \{(), (1), (1,1)\}$ with the following diagram:

$$\begin{tikzcd}
(1,1) \\
(1) \arrow[u] \\
(\ ) \arrow[u]
\end{tikzcd}$$

Fig. 14. Diagram of Imf.

Now, define $\lambda : M \to M$ by $\lambda((2)) = \lambda((1,2)) = 0$, $\lambda(()) = 1$, $\lambda((1)) = 1$, $\lambda((1,1)) = \dfrac{1}{2}$. Let μ be in Example 3.4. Since $\lambda = f(\mu)$, by Proposition 5.2, we get $\lambda \in FSUB_M(M)$.

Proposition 5.4 *Let $f : N \longrightarrow M$ be an $\mathbb{N}^{\infty}$-moduloid homomorphism. Then for every $a \in N$, $f(\mu)(f(a)) = \mu(a)$ if and only if, $f^{-1}(f(a)) \subseteq_{\mu} \{a\}$.*

Proof: We have $f(\mu)(f(a)) = \mu(a)$ if and only if, for every $b \in f^{-1}(f(a))$, $\mu(b) \leqslant \mu(a)$ if and only if, $f^{-1}(f(a)) \subseteq_{\mu} \{a\}$. $\square$

Proposition 5.5 *Let $\mu \in FSUB_M(N)$. Then for every $a \in N$, the map $(a + \mu) : N \longrightarrow [0, 1]$ defined by*

$$(a + \mu)(x) = \frac{\mu(a) + \mu(x)}{2},$$

is a fuzzy $\mathbb{N}^{\infty}$-submoduloid of N.

Proof: Clearly $a + \mu$ is well define. Let $x \in N$. Then $\mu(rx) \geqslant \mu(x)$, hence $\mu(rx) + \mu(a) \geqslant \mu(x) + \mu(a)$. Therefore, $(a + \mu)(rx) \geqslant (a + \mu)(x)$. Let $x, x' \in N$. Then $\mu(x + x') \geqslant \min\{\mu(x), \mu(x')\}$. Without loss of generality, let $\mu(x) \leqslant \mu(x')$. Then $\mu(x) + \mu(a) \leqslant \mu(x') + \mu(a)$, and so $\min\{\dfrac{\mu(x) + \mu(a)}{2}, \dfrac{\mu(x') + \mu(a)}{2}\} = \dfrac{\mu(x) + \mu(a)}{2}$. It is clear $\mu(x + x') \geqslant \mu(x)$, hence $\mu(x + x') + \mu(a) \geqslant \mu(x) + \mu(a)$. Therefore, $(a + \mu)(x + x') \geqslant \min\{(a + \mu)(x), (a + \mu)(x')\}$. $\square$

Proposition 5.6 *Let $\mu \in FSUB_M(N)$, and let $N_\mu = \{a+\mu \mid a \in N\}$. Then $(N_\mu, *)$ is a groupoid with the zero element $0+\mu$, such that*

$$(a+\mu)*(b+\mu) = \begin{cases} a+\mu & \text{if } \mu(a) \leqslant \mu(b), \\ b+\mu & \text{if } \mu(b) < \mu(a). \end{cases}$$

Proof: Let $a, a', b, b' \in N$ such that $a+\mu = a'+\mu$, $b+\mu = b'+\mu$. Hence for every $x \in N$, we have $\mu(a)+\mu(x) = \mu(a')+\mu(x)$ and $\mu(b)+\mu(x) = \mu(b')+\mu(x)$. So, $\mu(a) = \mu(a')$ and $\mu(b) = \mu(b')$.

If $\mu(a) < \mu(b)$, then $\mu(a') < \mu(b')$ and hence $(a+\mu)*(b+\mu) = a+\mu = a'+\mu = (a'+\mu)*(b'+\mu)$.

If $\mu(a) > \mu(b)$, then $\mu(a') > \mu(b')$, and so $(a+\mu)*(b+\mu) = b+\mu = b'+\mu = (a'+\mu)*(b'+\mu)$.

If $\mu(a) = \mu(a') = \mu(b) = \mu(b')$, then for every $x \in N$, $\mu(a)+\mu(x) = \mu(b)+\mu(x)$, and so $a+\mu = b+\mu$. Hence $(a+\mu)*(b+\mu) = a+\mu = b+\mu = b'+\mu = a'+\mu = (a'+\mu)*(b'+\mu)$. Therefore, $(N_\mu, *)$ is a groupoid. Now, since for every $a \in N$, $\mu(0) \geqslant \mu(a)$, $(a+\mu)*(0+\mu) = (0+\mu)*(a+\mu) = a+\mu$. □

Proposition 5.7 *N_μ is an $\mathbb{N}^\infty$-moduloid such that for every $r \in \mathbb{N}^\infty$ and for every $a+\mu \in N_\mu$, we have $r(a+\mu) = a+\mu$, if $r \neq 0$ and $r(a+\mu) = 0+\mu$, if $r = 0$.*

Proof: We will prove that for every $r, s \in \mathbb{N}^\infty$ and for every $a, b \in N$:

(i) $(r \vee s)(a+\mu) = r(a+\mu) * s(a+\mu)$,

(ii) $r(a+\mu*b+\mu) = r(a+\mu)*r(b+\mu)$,

(iii) $r(s(a+\mu)) = (r \wedge s)(a+\mu)$,

(iv) $0(a+\mu) = r(0+\mu) = 0+\mu$.

Without loss of generality, let $\mu(a) \leqslant \mu(b)$. Consider the following cases:
Case 1. Let $r \neq 0$, $s \neq 0$, then $r \vee s \neq 0$ and $r \wedge s \neq 0$.

(i) $(r \vee s)(a+\mu) = a+\mu = (a+\mu)*(a+\mu) = r(a+\mu)*s(a+\mu)$,

(ii) $r(a+\mu*b+\mu) = r(a+\mu) = a+\mu = (a+\mu)*(b+\mu) = r(a+\mu)*r(b+\mu)$,

(iii) $r(s(a+\mu)) = r(a+\mu) = a+\mu = (r \wedge s)(a+\mu)$.

Case 2. Let $r = 0$, $s \neq 0$, then $r \vee s = s \neq 0$ and $r \wedge s = 0$.

(i) $(r \vee s)(a+\mu) = s(a+\mu) = a+\mu = (0+\mu)*(a+\mu) = r(a+\mu)*s(a+\mu)$

(ii) $r(a+\mu*b+\mu) = r(a+\mu) = 0+\mu = (0+\mu)*(0+\mu) = r(a+\mu)*r(b+\mu)$,

(iii) $r(s(a+\mu)) = r(a+\mu) = 0+\mu = (r \wedge s)(a+\mu)$.

Case 3. Let $r = 0, s = 0$, then $r \vee s = r \wedge s = 0$. Then (i)-(iii) are valid.
The proof of (iv) is clear. □

Theorem 5.1 *Let $\mu, \nu \in FSUB_M(N)$, where $\nu \subseteq \mu$, and suppose that for every $a, a' \in N$,*

$$\nu(a) = \nu(a') \implies \mu(a) = \mu(a'), \quad \text{and} \quad \nu(a) < \nu(a') \implies \mu(a) < \mu(a').$$

Then the map

$$\frac{\mu}{\nu} : N_\nu \longrightarrow [0,1], \quad \frac{\mu}{\nu}(a+\nu) = \mu(a),$$

is a fuzzy $\mathbb{N}^\infty$-submoduloid of $(N_\nu, *)$.

Proof: Assume $a+\nu = a'+\nu$. Then $\nu(a)=\nu(a')$, and so $\mu(a)=\mu(a')$. Hence $\frac{\mu}{\nu}(a+\nu)=\frac{\mu}{\nu}(a'+\nu)$. This shows that $\frac{\mu}{\nu}$ is well defined. Let $0 \neq r \in \mathbb{N}^\infty$ and for every $a \in N$, $\frac{\mu}{\nu}(r(a+\nu))=\frac{\mu}{\nu}(a+\nu)=\mu(a)=\frac{\mu}{\nu}(a+\nu)$. Then $\frac{\mu}{\nu}(r(a+\nu)) \geqslant \frac{\mu}{\nu}(a+\nu)$. If $r=0$, since for every $a \in N$, $\mu(0) \geqslant \mu(a)$, we get $\frac{\mu}{\nu}(r(a+\nu))=\frac{\mu}{\nu}(0+\nu)=\mu(0)\geqslant \mu(a)=\frac{\mu}{\nu}(a+\nu)$. Without loss of generality, let $\nu(a) \leqslant \nu(b)$. $\frac{\mu}{\nu}((a+\nu)*(b+\nu))=\frac{\mu}{\nu}(a+\nu)=\mu(a)=\min\{\mu(a),\mu(b)\}=\min\{\frac{\mu}{\nu}(a+\nu),\frac{\mu}{\nu}(b+\nu)\}$. Then $\frac{\mu}{\nu}((a+\nu)*(b+\nu)) \geqslant \min\{\frac{\mu}{\nu}(a+\nu),\frac{\mu}{\nu}(b+\nu)\}$. Therefore, $\frac{\mu}{\nu}$ is a fuzzy $\mathbb{N}^\infty$-submoduloid of N_ν. □

Definition 5.2 *Let $\mu \in FSUB_M(N)$. We say that μ satisfies property $(\star)$ if μ satisfies the following condition:*

If $\mu(a)=\mu(a')$ and $\mu(b)=\mu(b')$ for some $a,a',b,b' \in N$, then $\mu(a+b)=\mu(a'+b')$, and for every $r \in \mathbb{N}^\infty$, we have $\mu(ra)=\mu(ra')$.

Example 5.4 Consider μ in Example 3.4, and the nexus $N=\{(),(1),(1,1),(1,2),(1,2,1),(1,2,2),(1,2,3)\}$ with the following diagram:

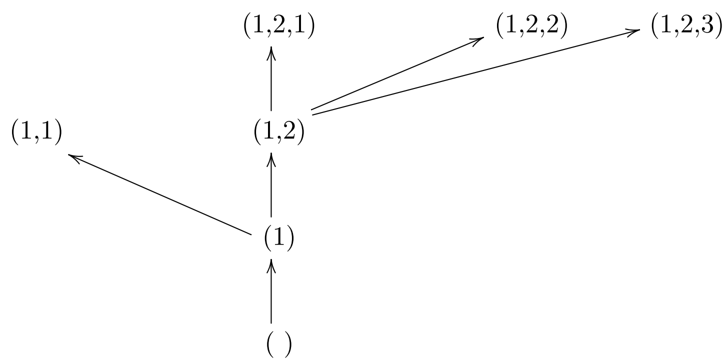

Fig. 15. Diagram of N.

Then μ satisfy in property $(\star)$.

Now, we define another fuzzy $\mathbb{N}^\infty$-submoduloid of N_μ, if μ satisfy in property $(\star)$.

Proposition 5.8 *Let $N_\mu=\{a+\mu \mid a \in N\}$. Then $(N_\mu,+)$ is a groupoid with the zero element $0+\mu$, such that for every $a,b \in N$,*

$$(a+\mu)+(b+\mu)=(a+b)+\mu.$$

Proof: Let $a+\mu=a'+\mu$ and $b+\mu=b'+\mu$. Then $\mu(a)=\mu(a')$ and $\mu(b)=\mu(b')$, hence $\mu(a+b)=\mu(a'+b')$. Thus, for every $x \in N$, $\mu(a+b)+\mu(x)=\mu(a'+b')+\mu(x)$. It means that $(a+b)+\mu=(a'+b')+\mu$. Then $+$ is well defined. For every $a \in N$, clearly $(a+\mu)+(0+\mu)=(0+\mu)+(a+\mu)=(a+0)+\mu=a+\mu$. Therefore, $(N_\mu,+)$ is a groupoid. □

Proposition 5.9 *N_μ is an $\mathbb{N}^\infty$-moduloid such that for every $r \in \mathbb{N}^\infty$ and for every $a+\mu \in N_\mu$, we have*

$$r(a+\mu)=ra+\mu.$$

Proof: It is easy to see that $ra+\mu \in N_\mu$. Let $r,r' \in \mathbb{N}^\infty$ and $a,a' \in N$ such that $a+\mu=a'+\mu$ and $r=r'$, $\mu(ra)=\mu(ra')=\mu(r'a')$, $ra+\mu=r'a'+\mu$. For every $r,s \in \mathbb{N}^\infty$ and for every $a,b \in N$, we have:

(i) $(r \vee s)(a+\mu)=(r\vee s)a+\mu=(ra+sa)+\mu=ra+\mu+sa+\mu=r(a+\mu)+s(a+\mu)$,

(ii) $r(a+\mu+b+\mu) = r((a+b)+\mu)) = r(a+b)+\mu = ra+rb+\mu = ra+\mu+rb+\mu = r(a+\mu)+r(b+\mu)$,

(iii) $r(s(a+\mu)) = r(sa+\mu) = r(sa)+\mu = (r \wedge s)a + \mu = (r \wedge s)(a+\mu)$,

(iv) $0(a+\mu) = 0a+\mu = 0+\mu$,

(v) $r(0+\mu) = r0+\mu = 0+\mu$.

Hence the proof is complete. □

Theorem 5.2 *Let $\mu, \nu \in FSUB_M(N)$ where $\nu \subseteq \mu$ and ν satisfy in property $(\star)$ and for every $a, a' \in N$, $\nu(a) = \nu(a')$, implies $\mu(a) = \mu(a')$. Then $\frac{\mu}{\nu} : N_\nu \longrightarrow [0,1]$ with $\frac{\mu}{\nu}(a+\nu) = \mu(a)$ is a fuzzy $\mathbb{N}^\infty$-submoduloid of $(N_\nu, +)$.*

Proof: Assume $a+\nu = a'+\nu$, $\nu(a) = \nu(a')$. Then $\mu(a) = \mu(a')$. Thus, $\frac{\mu}{\nu}(a+\nu) = \frac{\mu}{\nu}(a'+\nu)$. So, $\frac{\mu}{\nu}$ is well defined. For every $r \in \mathbb{N}^\infty$ and for every $a \in N$, $\frac{\mu}{\nu}(r(a+\nu)) = \frac{\mu}{\nu}(ra+\nu) = \mu(ra) \geqslant \mu(a) = \frac{\mu}{\nu}(a+\nu)$. Also, for every $a, b \in N$, $\frac{\mu}{\nu}(a+\nu+b+\nu) = \frac{\mu}{\nu}(a+b+\nu) = \mu(a+b) \geqslant \min\{\mu(a), \mu(b)\} = \min\{\frac{\mu}{\nu}(a+\nu), \frac{\mu}{\nu}(b+\nu)\}$. Therefore, $\frac{\mu}{\nu}$ is a fuzzy $\mathbb{N}^\infty$-submoduliod of $(N_\nu, +)$. □

Example 5.5 Let μ be in Example 2.3 and N be the cyclic nexus in Example 4.8.

If we take $\nu(x) := \mu(x)^2$, for all $x \in N$, then μ and ν satisfy in the Theorem 5.2. Therefore, $\frac{\mu}{\nu}$ is a fuzzy $\mathbb{N}^\infty$-submoduloid of $N_\nu = \{a_i | \ 0 \leq i \leq 7\}$, where $a_0 := () + \nu$, $a_1 := (1) + \nu$, $a_2 := (1,2) + \nu$, $a_3 := (1,2,2) + \nu$, $a_4 := (1,2,3) + \nu$, $a_5 := (1,2,3,1) + \nu$, $a_6 := (1,2,3,2) + \nu$, $a_7 := (1,2,3,3) + \nu$ is a nexus with the following diagram:

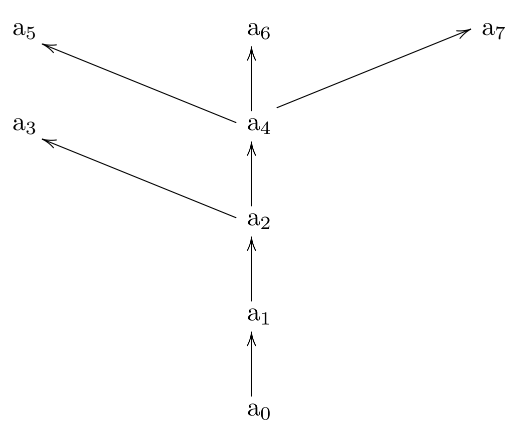

Fig. 16. Diagram of N_ν.

Let R be a semiring. A subset $S \subseteq R$ is called a *multiplicatively closed subset* (briefly, m.c.s.) of R, if $0 \notin S$ and for all $s, s' \in S$, we have $ss' \in S$.

We observe that the subset $S \subseteq \mathbb{N}^\infty$ is an m.c.s. of $\mathbb{N}^\infty$. Let N be an $\mathbb{N}^\infty$-moduloid and $S \subseteq \mathbb{N}^\infty$ be an m.c.s. For every $a, a' \in N$ and for every $s, s' \in S$, we define the relation $(a, s) \sim (a', s')$ if there exists $s'' \in S$ such that

$$(s'' \wedge s')a = (s'' \wedge s)a'.$$

The relation $\sim$ is an equivalence relation on $S \times N$.

Define

$$\frac{a}{s} := \{(a', s') \mid \ a' \in N, s' \in S, (a, s) \sim (a', s')\}$$

as the equivalence class of (a,s), and let

$$S^{-1}N := \left\{\frac{a}{s} \mid \ a \in N, s \in S\right\}.$$

Example 5.6 Let μ be in Example 3.4, $S := 2\mathbb{N}$ and consider the nexus

$$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,2,1), (2,2,2), (2,3,1), (2,3,2)\}$$

with the following diagram:

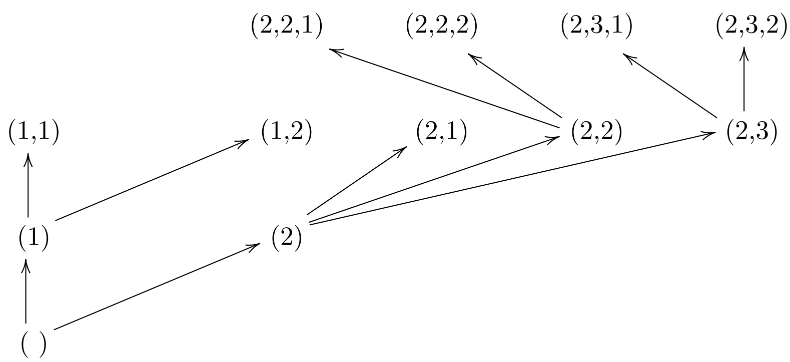

Fig. 17. Diagram of N.

Since $S = 2\mathbb{N}$, we get $2 = \bigwedge_{n \in S} n$ and so $S^{-1}N = \{\alpha_i | \ 0 \leq i \leq 7\}$, where $\alpha_0 := \frac{()}{2}$, $\alpha_1 := \frac{(1)}{2}$, $\alpha_2 := \frac{(2)}{2}$, $\alpha_3 := \frac{(1,1)}{2}$, $\alpha_4 := \frac{(1,2)}{2}$, $\alpha_5 := \frac{(2,1)}{2}$, $\alpha_6 := \frac{(2,2,2)}{2}$ and $\alpha_7 := \frac{(2,3,2)}{2}$ with the following diagram:

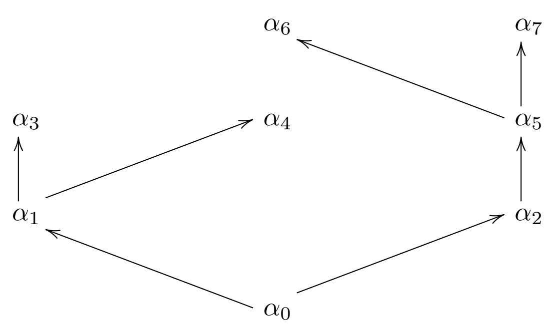

Fig. 18. Diagram of $\mathrm{S}^{-1}\mathrm{N}$.

For every $\frac{a}{s}, \frac{b}{t} \in S^{-1}N$ and $r \in \mathbb{N}^{\infty}$, we define "$+$" and "$.$" on $S^{-1}N$, by $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{s \wedge t}$ and respectively, $r\frac{a}{s} = \frac{ra}{s}$.

Proposition 5.10 *$(S^{-1}N, +)$ is an $\mathbb{N}^{\infty}$-moduloid, where $u = \bigwedge_{s \in S} s$.*

Proof: Assume $\frac{a}{s} = \frac{a'}{s'}$ and $\frac{b}{t} = \frac{b'}{t'}$. Then there exist $k, l \in S$ such that $(k \wedge s')a = (k \wedge s)a'$ and

$(l \wedge t')b = (l \wedge t)b'$. Put $r := k \wedge l$. We have:

$$\begin{aligned}
(r \wedge s' \wedge t')(ta + sb) &= (k \wedge l \wedge s' \wedge t')(ta + sb) \\
&= (k \wedge l \wedge s' \wedge t')ta + (k \wedge l \wedge s' \wedge t')sb \\
&= (k \wedge l \wedge s' \wedge t' \wedge t)a + (k \wedge l \wedge s' \wedge t' \wedge s)b \\
&= (l \wedge t' \wedge t) \wedge (k \wedge s')a + (k \wedge s' \wedge s) \wedge (l \wedge t')b \\
&= (l \wedge t' \wedge t) \wedge (k \wedge s)a' + (k \wedge s' \wedge s) \wedge (l \wedge t)b' \\
&= (l \wedge k \wedge t \wedge s)(t'a') + (k \wedge l \wedge t \wedge s)(s'b') \\
&= (r \wedge s \wedge t)(t'a' + t'b').
\end{aligned}$$

Also, we can see that $\frac{a}{s} + \frac{0}{u} = \frac{0}{u} + \frac{a}{s} = \frac{s0 + ua}{s \wedge u} = \frac{ua}{s \wedge u} = \frac{a}{s}$. Hence $(S^{-1}N, +)$ is a groupoid. It is easy to see that the multiplication $r\frac{a}{s} = \frac{ra}{s}$ is well defined. Now, we have:

(i) $(r \vee t)\frac{a}{s} = \frac{(r \vee t)a}{s} = \frac{ra + ta}{s} = \frac{s(ra + ta)}{s \wedge s} = \frac{ra}{s} + \frac{ta}{s} = r\frac{a}{s} + t\frac{a}{s}$,

(ii) $r(\frac{a}{s} + \frac{b}{t}) = r(\frac{ta + sb}{s \wedge t}) = \frac{r(ta + sb)}{s \wedge t} = \frac{t(ra) + s(rb)}{s \wedge t} = \frac{ra}{s} + \frac{rb}{t} = r\frac{a}{s} + r\frac{b}{t}$,

(iii) $r(t\frac{a}{s}) = r(\frac{ta}{s}) = \frac{r(ta)}{s} = \frac{(r \wedge t)a}{s} = (r \wedge t)\frac{a}{s}$,

(iv) $0(\frac{a}{s}) = r\frac{0}{u} = \frac{0}{u}$.

Therefore, $S^{-1}N$ is an $\mathbb{N}^\infty$-moduloid. □

Let $\mu \in FSUB_M(N)$ and $u := \bigwedge_{s \in S} s$. Define $\mu_s : S^{-1}N \longrightarrow [0, 1]$ by $\mu_s(\frac{a}{s}) = sup\{\mu(b) | \ \frac{b}{u} = \frac{a}{u}\}$.

Theorem 5.3 *Let $\mu \in FSUB_M(N)$. Then $\mu_s \in FSUB_M(S^{-1}N)$.*

Proof: Define $\Phi : N \longrightarrow S^{-1}N$ by $\Phi(a) = \frac{a}{u}$, where $u = \bigwedge_{s \in S} s$. If $a = a'$, then $\frac{a}{u} = \frac{a'}{u}$. This shows that Φ is well defined. For every $a, b \in N$ and $r \in \mathbb{N}^\infty$, we have $\Phi(a + b) = \frac{a + b}{u} = \frac{u(a + b)}{u \wedge u} = \frac{a}{u} + \frac{b}{u} = \Phi(a) + \Phi(b)$ and $\Phi(ra) = \frac{ra}{u} = r\frac{a}{u} = r\Phi(a)$. Hence Φ is a homomorphism moduloid. Since $\mu_s = \Phi(\mu)$, by Proposition 5.2, we get $\mu_s \in FSUB_M(S^{-1}N)$. □

Example 5.7 Let S be the set of all odd numbers of $\mathbb{N}^\infty$, μ be in Example 2.3 and consider the nexus $N = \{(), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2), (3, 3), (1, 2, 1), (1, 2, 2), (3, 2, 1), (3, 2, 2)\}$ with the following diagram:

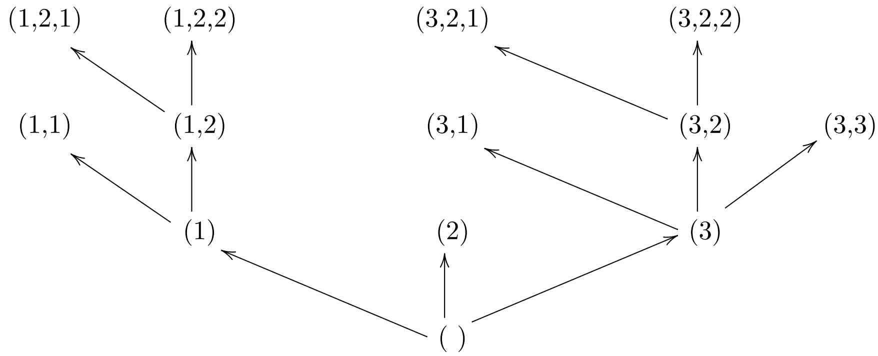

Fig. 19. Diagram of N.

Also, since $S = \{n \in \mathbb{N}^{\infty} |\ n \text{ is odd}\}$, we get $1 = \bigwedge_{n \in S} n$ and so $S^{-1}N = \{\alpha_i |\ 0 \leq i \leq 10\}$, where $\alpha_0 := \frac{()}{1}$, $\alpha_1 := \frac{(1)}{1}$, $\alpha_2 := \frac{(2)}{1}$, $\alpha_3 := \frac{(3)}{1}$, $\alpha_4 := \frac{(1,1)}{1}$, $\alpha_5 := \frac{(1,2)}{1}$, $\alpha_6 := \frac{(3,1)}{1}$, $\alpha_7 := \frac{(3,2)}{1}$, $\alpha_8 := \frac{(3,3)}{1}$, $\alpha_9 := \frac{(3,2,1)}{1}$ and $\alpha_{10} := \frac{(3,2,2)}{1}$ with the following diagram:

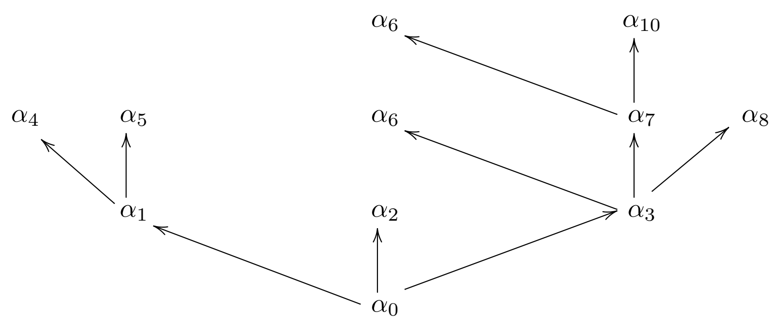

Fig. 20. Diagram of $\mathrm{S}^{-1}\mathrm{N}$.

Define $\Phi : S^{-1}N \longrightarrow [0,1]$ by, for every $s \in S$, $\Phi(\frac{()}{s}) = \Phi(\frac{(1)}{s}) = \Phi(\frac{(1,1)}{s}) = \Phi(\frac{(1,2)}{s}) = \Phi(\frac{(1,2,1)}{s}) = \Phi(\frac{(1,2,2)}{s}) = 1$, $\Phi(\frac{(2)}{s}) = \frac{1}{2}$ and $\Phi(\frac{(3)}{s}) = \Phi(\frac{(3,1)}{s}) = \Phi(\frac{(3,2)}{s}) = \Phi(\frac{(3,3)}{s}) = \Phi(\frac{(3,2,1)}{s}) = \Phi(\frac{(3,2,2)}{s}) = \frac{1}{3}$.
Then $\Phi = \mu_s$. Using Theorem 5.3, we get $\Phi \in FSUB_M(S^{-1}N)$.

6. Conclusions

We have generalized the concept of a $\mathbb{N}^{\infty}$-moduloid to the framework of fuzzy $\mathbb{N}^{\infty}$-submoduloids, providing a more nuanced structure for analysis. In this context, we explore the characterizations of prime fuzzy $\mathbb{N}^{\infty}$-submoduloids of nexuses and derive the necessary conditions for a fuzzy $\mathbb{N}^{\infty}$-submoduloid to be prime. Furthermore, we investigate the fractions induced by these submoduloids, highlighting their importance in understanding the interactions within fuzzy structures. We then establish a significant connection between fuzzy subnexuses and fuzzy $\mathbb{N}^{\infty}$-submoduloids, facilitating a deeper understanding of the underlying algebraic properties. Building on this, we define the notion of a $\mathbb{N}^{\infty}$-moduloid homomorphism, extending the classical homomorphism concept to fuzzy structures and providing the necessary framework for their analysis. Finally, by utilizing the concept of multiplicatively closed subsets, we extend the related fuzzy subsets, offering new insights into their behavior and applications within the broader context of fuzzy algebraic structures. This extension allows for a more flexible approach to understanding the relationships between fuzzy elements, leading to potential new applications in various areas of algebra and logic.

Conflict of interest

The authors declare that they have no conflict of interest.

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