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Fuzzy Fractional Evolution Equation with a Non-Dense Domain Operator

Khadija Oufkir, M'hamed Elomari and Hamid Sadiki

ABSTRACT: We aim to establish the existence and uniqueness of the mild fuzzy solution to a fuzzy nonlinear fractional evolution equation. This problem involves a non-dense domain operator and employs the fuzzy Caputo fractional derivative, denoted as ${}^{C}D^{q}_{0+,gH}$ with order $q \in (0,1)$, within the framework of the triangular fuzzy number space Ω. Furthermore, we consider nonlocal conditions and provide an illustrative example to demonstrate the applicability of our findings.

Keywords: Fuzzy fractional derivative, non dense domain, fuzzy mild solution.

Contents

1. Introduction

The study of fractional differential equations has a long and well-established history, dating back over a century. Throughout its development, numerous researchers have explored the existence and uniqueness of solutions for various fractional differential problems, employing a wide range of mathematical techniques. The field underwent a significant expansion with the introduction of fuzzy set theory by Lotfi Zadeh in 1965, which also laid the foundation for the concept of the generalized Hukuhara difference. This advancement enabled the extension of classical fractional derivatives to the fuzzy setting, leading to new approaches for analyzing the existence of mild fuzzy solutions in fractional systems.

One of the earliest and most commonly studied fuzzy derivatives is the Hukuhara derivative, and the systematic investigation of fuzzy fractional differential equations began around 2010 [2]. Since then, researchers have proposed various formulations of fuzzy fractional derivatives, including adaptations based on the Riemann-Liouville derivative within the framework of the Seikkala derivative [5]. Moreover, existence and uniqueness results have been established using conditions similar to the Krasnoselskii-Krein condition and the Nagumo condition [4,3]. Further developments in this area include studies on Caputo-type Hukuhara derivatives, which have gained considerable attention in recent years [15,4]. For a broader discussion on solving fuzzy fractional differential equations, we refer to [10,14].

Building on these foundations, Zhang and O'Regan [20] recently investigated the existence of solutions for a fuzzy fractional evolution equation governed by a closed linear operator A:

$$\begin{cases} {}^{C}_{gH}D^{q}_{0+}x(t) = Ax(t) \oplus f(t, x(t), x_t), & t \in [0, a], \\ x(t) = g(x)(t) \oplus \varphi(t), & t \in [-h, 0], \end{cases}$$

where ${}^{C}_{gH}D^{q}_{0+}$ represents the fuzzy Caputo fractional derivative.

Motivated by this work, we aim to investigate the existence and uniqueness of the mild fuzzy solution for a fuzzy nonlinear fractional evolution equation. Our study considers a non-dense domain operator and employs the fuzzy Caputo fractional derivative within the triangular fuzzy number space Ω, incorporating

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nonlocal conditions. To support our theoretical findings, we provide an illustrative example demonstrating the practical application of our results.

In this article, we specifically examine the existence of solutions for the following fuzzy nonlinear fractional evolution equation in Ω, where the operator $\mathcal{A}$ is non-dense:

$$\begin{cases} {}^{C}D^{q}_{0^+,gH}u(t) = \mathcal{A}u(t) \oplus F(t, u(t), u_t), & t \in [0, T], \\ u(t) = G(u)(t) \oplus H(t), & t \in [-h, 0], \end{cases} \tag{1.1}$$

where ${}^{C}D^{q}_{0^+,gH}$ denotes the fuzzy Caputo fractional derivative of order $q \in (0, 1)$. Here, u is a function mapping into Ω, $\mathcal{A} : D(\mathcal{A}) \subset \Omega \to \Omega$ is a closed linear operator with a non-dense domain, and $F : [0, T] \times \Omega \times C([-h, 0], \Omega) \to \Omega$ is a continuous function. The functions $H \in C([-h, 0], \Omega)$ and G act on $C([-h, T], \Omega)$, while u_t is defined as $u_t(s) = u(t + s)$ for $s \in [-h, 0]$.

The structure of this paper is organized as follows: Section 2 presents the necessary preliminary definitions and background results. In Section 3, we derive the integral formulation of the problem and investigate the existence and uniqueness of solutions. We also analyze the special case where the nonlocal term $G \equiv 0$ to further explore the existence of mild fuzzy solutions. Section 4 provides an illustrative example that validates our theoretical results. Finally, Section 5 concludes with a summary of key findings and future research directions.

2. Preliminaries

In this section, we present the essential results and fundamental concepts to support our study.

Definition 1 *Let $E^1 = \{\mu : \mathbb{R} \to [0, 1]\}$ be the set of fuzzy subsets that satisfy the following conditions:*

1- μ is normal, meaning there exists an $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$.

2- μ is fuzzy convex:

$$\mu(\lambda x + (1 - \lambda)y) \geq \mu(x) \wedge \mu(y), \quad \forall x, y \in \mathbb{R} \text{ and } \lambda \in [0, 1].$$

3- μ is upper semicontinuous.

4- The support of μ, denoted by $\mu^0 = \overline{\{x \in \mathbb{R} \,|\, \mu(x) > 0\}}$, is compact.

Following Zadeh's extension principle, we define the addition of two fuzzy numbers and scalar multiplication as follows.

Definition 2 *Let $\mu, \nu \in E^1$.*

1- The sum of μ and ν is defined by $(\mu \oplus \nu)(z) = \sup_{x+y=z} \mu(x) \wedge \nu(y)$, for all $x, y, z \in \mathbb{R}$.

2- The product of a scalar λ and μ is defined by $(\lambda \odot \mu)(y) = \sup_{y=\lambda x} \mu(x)$, for all $x, y \in \mathbb{R}$.

The α-cut of $\mu \in E^1$ is defined as:

$$[\mu]^\alpha = \{x \in \mathbb{R} \,|\, \mu(x) \geq \alpha\}, \quad \forall \alpha \in (0, 1].$$

We can also express this as $[\mu]^\alpha = [\underline{\mu}^\alpha, \overline{\mu}^\alpha]$.

The fuzzy number zero, $\tilde{0}$, is defined by:

$$\tilde{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

We can apply the concept of the α-cut to perform operations on fuzzy numbers as follows:

Proposition 1 *Let $\mu, \nu \in E^1$ and $\alpha \in [0,1]$. Then:*

1- $[\mu+\nu]^\alpha = [\mu]^\alpha + [\nu]^\alpha$.

2- $[\mu-\nu]^\alpha = [\underline{\mu}^\alpha - \overline{\nu}^\alpha, \overline{\mu}^\alpha - \underline{\nu}^\alpha]$.

3- $[\lambda\mu]^\alpha = \lambda[\mu]^\alpha = \begin{cases} [\lambda\underline{\mu}^\alpha, \lambda\overline{\mu}^\alpha] & \text{if } \lambda \geq 0 \\ [\lambda\overline{\mu}^\alpha, \lambda\underline{\mu}^\alpha] & \text{if } \lambda \leq 0 \end{cases}$

4- $[\mu\nu]^\alpha = \left[\min\{\underline{\mu}^\alpha\underline{\nu}^\alpha, \underline{\mu}^\alpha\overline{\nu}^\alpha, \overline{\mu}^\alpha\underline{\nu}^\alpha, \overline{\mu}^\alpha\overline{\nu}^\alpha\}, \max\{\underline{\mu}^\alpha\underline{\nu}^\alpha, \underline{\mu}^\alpha\overline{\nu}^\alpha, \overline{\mu}^\alpha\underline{\nu}^\alpha, \overline{\mu}^\alpha\overline{\nu}^\alpha\}\right]$.

Definition 3 *The Pompeiu-Hausdorff distance, $D : E^1 \times E^1 \to \mathbb{R}_+ \cup \{0\}$, between the α-cuts of fuzzy sets is defined as follows:*

$$\begin{aligned} D(x,y) &= \sup_{\alpha\in[0,1]} d_H([x]^\alpha, [y]^\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\{|\underline{x}^\alpha - \underline{y}^\alpha|, |\overline{x}^\alpha - \overline{y}^\alpha|\}, \quad x, y \in E^1. \end{aligned}$$

This distance has the following properties:

1- $D(x+z, y+z) = D(x,y)$ *for all* $x, y, z \in E^1$.

2- $D(kx, ky) = |k|D(x,y)$ *for all* $x, y \in E^1$ *and* $k \in \mathbb{R}$.

3- $D(x+s, y+t) \leq D(x,y) + D(s,t)$ *for all* $x, y, s, t \in E^1$.

4- $D(\lambda x, \beta x) = |\lambda - \beta| D(x, \tilde{0})$ *for all* $\lambda, \beta \geq 0$ *and* $x \in E^1$.

We denote by $\mathbb{R}_F$ the set of fuzzy numbers.
Next, we define the Hukuhara difference and derivative.

Definition 4 *i- Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the Hukuhara difference of x and y, and it is denoted by $x \ominus y$.*

ii- The generalized Hukuhara difference of two fuzzy numbers x and y is defined as follows:

$$x \ominus_{gH} y = z \Leftrightarrow \begin{cases} (i)\ x = y + z \\ (ii)\ y = x + (-1)z \end{cases}$$

Definition 5 *Let $x_0 \in (a,b)$ and h be such that $x_0 + h \in (a,b)$. The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,b) \to E^1$ at x_0 is defined as:*

$$\lim_{h\to 0} \left\| \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h} \ominus_{gH} f'_{gH}(x_0) \right\| = 0.$$

If $f'_{gH}(x_0) \in E^1$ exists, we say that f is generalized Hukuhara differentiable (g-differentiable) at x_0, with:

$$f'_{gH}(x_0) = \lim_{h\to 0} \left\| \frac{f(x_0+h) \ominus_{gH} f(x_0)}{h} \right\|.$$

Let Ω be the set of triangular fuzzy numbers in $\mathbb{R}_F$. The space (Ω, D) is a subspace of (E^1, D) and forms a complete metric space. We denote by $C([a,b], \Omega)$ the set of continuous functions from $[a,b]$ to Ω.

For $t \in [a,b]$, we define $C_t^\Omega := C([-h,t], \Omega)$ as the complete metric space of continuous functions from $[-h,t]$ to Ω, with the metric:

$$D_t(f,g) = \sup_{-h\leq x\leq t} D(f(x), g(x)).$$

Let $\mathcal{P}(\Omega)$ denote the space of bounded linear operators $\mathcal{A} : \Omega \to \Omega$, with norm:

$$\|\mathcal{A}\|_{\mathcal{P}(\Omega)} := \inf\{K : D(\mathcal{A}x, \tilde{0}) \leq KD(x, \tilde{0}) \, , \forall x \in \Omega\}.$$

A linear operator on $\mathcal{P}(\Omega)$ satisfies:

$$(\alpha\mathcal{A}_1 + \beta\mathcal{A}_2)(x) = \alpha \odot (\mathcal{A}_1 x) \oplus \beta \odot (\mathcal{A}_2 x) \quad \forall x \in \Omega.$$

We also have:

$$D(\mathcal{A}x, \tilde{0}) \leq \|\mathcal{A}\|_{\mathcal{P}(\Omega)} D(x, \tilde{0}) \quad \text{and} \quad D(\mathcal{A}x, \mathcal{A}y) \leq \|\mathcal{A}\|_{\mathcal{P}(\Omega)} D(x, y), \quad \forall x, y \in \Omega.$$

Definition 6 *Let $x \in L([0, a], \Omega)$ with $[x(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$, for all $t \in [0, a]$ and $\alpha \in [0, 1]$. The fractional fuzzy Riemann-Liouville integral of order $q \in (0, 1]$ of x is defined as:*

$$I^q_{0^+,gH} x(t) = \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot x(s) \, ds.$$

We then have $\left[I^q_{0^+,gH} x(t)\right]^\alpha = \left[I^q_{0^+,gH} x_1^\alpha(t), I^q_{0^+,gH} x_2^\alpha(t)\right]$, *where*

$$I^q_{0^+,gH} x_1^\alpha(t) = \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot x_1^\alpha(s) \, ds,$$

and

$$I^q_{0^+,gH} x_2^\alpha(t) = \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot x_2^\alpha(s) \, ds.$$

Definition 7 *Let $x \in L([0, a], \Omega)$. The fuzzy Caputo fractional derivative of order $q \in (0, 1]$ of x is defined as:*

$${}^cD^q_{0^+,gH} x(t) = \frac{1}{\gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot x'_{gH}(s) \, ds.$$

We have $\left[{}^cD^q_{0^+,gH} x(t)\right]^\alpha = \left[{}^cD^q_{0^+,gH} x_1^\alpha(t), {}^cD^q_{0^+,gH} x_2^\alpha(t)\right]$, *where*

$${}^cD^q_{0^+,gH} x_1^\alpha(t) = \frac{1}{\gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot (x_1^\alpha)'_{gH}(s) \, ds,$$

and

$${}^cD^q_{0^+,gH} x_2^\alpha(t) = \frac{1}{\gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot (x_2^\alpha)'_{gH}(s) \, ds.$$

Remark 1 *The relationship between the fractional Riemann-Liouville integral and the fractional Caputo derivative is given by:*

$${}^cD^q_{0^+,gH} x(t) = I^{1-q}_{0^+,gH}(x'_{gH}(t)).$$

Definition 8 *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{P}(\Omega)$ be a semigroup of operators on Ω. We say that $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup (or C_0-semigroup) on Ω if it satisfies:*

$$\lim_{t\to 0^+} S(t)x = x, \quad \forall x \in \Omega.$$

Proposition 2 *Suppose that $\{S(t)\}_{t\geq 0} \subset \mathcal{P}(\Omega)$ is a C_0-semigroup on Ω. Then, for every $x \in \Omega$, the mapping $S(\cdot)x : [0, +\infty) \to \Omega$ is continuous.*

3. Main Results

In this section, we first find the integral solution of problem 1.1, and then we investigate the existence and uniqueness of this solution.

Proposition 3 *Suppose that $F:[0,T]\times\Omega\times C([-h,0],\Omega)\to\Omega$ is a continuous function. Then the fuzzy fractional problem 1.1 is equivalent to the following integral solution:*

$$\begin{cases} u(t)=\dfrac{t^{q-1}}{\Gamma(q)}(G(u)(0)\oplus H(0))\oplus I^{q}_{0^+,gH}(\mathcal{A}u(s)\oplus F(s,u(s),u_s))(t), \quad t\in[0,T] \\ u(t)=G(u)(t)\oplus H(t), \quad t\in[-h,0], \end{cases} \tag{3.1}$$

Proof 1 *We have:*

$$\left({}^{C}D^{q}_{0^+,gH}u\right)(t)=\left(D^{q}_{0^+,gH}u\right)(t)\ominus_{gH}\frac{u(0)}{\Gamma(1-q)}\frac{1}{t^q}.$$

For $t\in[0,T]$, we have:

$${}^{C}D^{q}_{0^+,gH}u(t)=\mathcal{A}u(t)\oplus F(t,u(t),u_t).$$

Then,

$$\begin{aligned}\left(I^{q}_{0^+,gH}\left({}^{C}D^{q}_{0^+,gH}u\right)(s)\right)(t)&=I^{q}_{0^+,gH}\left(\left(D^{q}_{0^+,gH}u\right)(s)\ominus_{gH}\frac{u(0)}{\Gamma(1-q)}\frac{1}{s^q}\right)(t)\\&=I^{q}_{0^+,gH}\left(\mathcal{A}u(s)\oplus F(s,u(s),u_s)\right)(t),\end{aligned}$$

and we have:

$$\begin{aligned}I^{q}_{0^+,gH}\left(\left(D^{q}_{0^+,gH}u\right)(s)\ominus_{gH}\frac{u(0)}{\Gamma(1-q)}\frac{1}{s^q}\right)(t)&=I^{q}_{0^+,gH}\left(\left(D^{q}_{0^+,gH}u\right)(s)\right)(t)\\&\quad\ominus_{gH}I^{q}_{0^+,gH}\left(\frac{u(0)}{\Gamma(1-q)}\frac{1}{s^q}\right)(t)\\&=u(t)\ominus_{gH}\frac{t^{q-1}}{\Gamma(q)}u(0)\\&=u(t)\ominus_{gH}\frac{t^{q-1}}{\Gamma(q)}\left(G(u)(0)\oplus H(0)\right).\end{aligned}$$

Then,

$$u(t)\ominus_{gH}\frac{t^{q-1}}{\Gamma(q)}\left(G(u)(0)\oplus H(0)\right)=I^{q}_{0^+,gH}\left(\mathcal{A}u(s)\oplus F(s,u(s),u_s)\right)(t).$$

So, for $t\in[0,T]$:

$$u(t)=\frac{t^{q-1}}{\Gamma(q)}(G(u)(0)\oplus H(0))\oplus I^{q}_{0^+,gH}(\mathcal{A}u(s)\oplus F(s,u(s),u_s))(t).$$

Let $\Omega'=\overline{D(\mathcal{A})}$ and $\mathcal{A}'$ a part of $\mathcal{A}$ in Ω' defined by:

$$D(\mathcal{A}')=\{x\in D(\mathcal{A}),\ \mathcal{A}x\in\overline{D(\mathcal{A})},\ \mathcal{A}'x=\mathcal{A}x\}.$$

The new operator $\mathcal{A}'$ generates a C_0-semigroup $\{S(t)\}_{t\geq0}$ on Ω'.

We are now working on the following auxiliary fuzzy fractional problem:

$$\begin{cases} {}^{C}D^{q}_{0^+,gH}u(t)=\mathcal{A}'u(t)\oplus F(t,u(t),u_t) \quad , \quad t\in[0,T] \\ u(t)=G(u)(t)\oplus H(t) \quad , \quad t\in[-h,0], \end{cases} \tag{3.2}$$

and the integral solution of the problem 3.2 remains of the following form:

$$\begin{cases} u(t)=\dfrac{t^{q-1}}{\Gamma(q)}(G(u)(0)\oplus H(0))\oplus I^{q}_{0^+,gH}(\mathcal{A}'u(s) \\ \qquad\oplus F(s,u(s),u_s))(t),\ t\in[0,T] \\ u(t)=G(u)(t)\oplus H(t),\ t\in[-h,0], \end{cases} \tag{3.3}$$

Definition 9 *The Wright-type function Λ_q is defined by:*

$$\Lambda_q(\xi) = \sum_{n=0}^{\infty} \frac{(-\xi)^{n-1}}{(n-1)!\Gamma(1-qn)} \ , \ q \in (0,1) \ , \ \xi \in \mathbb{C},$$

and satisfies:

1- $\Lambda_q(\xi) \geq 0$.

2- $\int_0^\infty \Lambda_q(s)ds = 1$.

3- $\int_0^\infty s^\alpha \Lambda_q(s)ds = \dfrac{\Gamma(1+\alpha)}{\Gamma(1+q\alpha)}$, $\alpha \in [0,1]$.

Lemma 1 *Suppose that the function F has values in Ω'. Then the integral solution 3.3 becomes:*

$$\begin{cases} u(t) = \mathcal{M}_q(t) \odot (G(u)(0) \oplus H(0)) \\ \qquad \oplus \int_0^t (t-s)^{q-1} \odot \mathcal{N}_q(t-s) F(s,u(s),u_s)ds, \ t \in [0,T] \\ u(t) = G(u)(t) \oplus H(t), \ t \in [-h,0], \end{cases} \tag{3.4}$$

with $\mathcal{M}_q(t) = \int_0^\infty \Lambda_q(s) \odot S(t^q s)ds$ and $\mathcal{N}_q(t) = q \odot \int_0^\infty s\Lambda_q(s) \odot S(t^q s)ds$.

Proof 2 *See lemma 4.1 in [17].*

Before advancing in these results, we need the following hypotheses:

(H_1) The family $\{S(t)\}_{t\geq 0}$ is uniformly bounded; there exists $K > 0$ such that

$$K = \sup_{t\in[0,T]} \|S(t)\|_{\mathcal{P}(\Omega)} \geq 1.$$

(H_2) There are two positive constants M_1, M_2 such that

$$D(F(\xi,x_1,y_1), F(\xi,x_2,y_2)) \leq M_1 D(x_1,x_2) + M_2 D_0(y_1,y_2)$$

for all $x_1, x_2 \in \Omega$, $y_1, y_2 \in C_0^\Omega$, and $\xi \in [0,T]$.

(H_3) There is a positive constant L_G such that:

$$D_0(G(u),G(v)) \leq L_G D_T(u,v), \quad \forall u,v \in C_T^\Omega.$$

Proposition 4 *Suppose the hypothesis (H_1) is verified. Then:*

(i) The two operators $\{\mathcal{M}_q(t)\}_{t\geq 0}$ and $\{\mathcal{N}_q(t)\}_{t\geq 0}$ are bounded such that, for all $u \in \Omega$:

$$D(\mathcal{M}_q(t)u, \tilde{0}) \leq MD(u,\tilde{0}),$$

$$D(\mathcal{N}_q(t)u, \tilde{0}) \leq \frac{M}{\Gamma(q)} D(u,\tilde{0}).$$

(ii) The two operators $\{\mathcal{M}_q(t)\}_{t\geq 0}$ and $\{\mathcal{N}_q(t)\}_{t\geq 0}$ are strongly continuous.

Let $\theta_\varepsilon = \varepsilon(\varepsilon I - \mathcal{A})^{-1}$ with: $\lim_{\varepsilon\to+\infty} \theta_\varepsilon u = u, \ \forall u \in \Omega'$.

Remark 2 *1- The integral solution 3.4 can be written in the following form:*

$$\begin{cases} u(t) = \mathcal{M}_q(t) \odot (G(u)(0) \oplus H(0)) \\ \qquad \oplus \lim_{\varepsilon\to+\infty} \int_0^t (t-s)^{q-1} \odot \mathcal{N}_q(t-s)\theta_\varepsilon F(s,u(s),u_s)ds, \quad t \in [0,T] \\ u(t) = G(u)(t) \oplus H(t), \quad t \in [-h,0], \end{cases} \tag{3.5}$$

2- The operator $\mathcal{A}$ generates an operator $\{\mathcal{S}_q(t)\}_{t\geq 0}$ such that:

$$\mathcal{S}_q(t) = \lim_{\varepsilon\to+\infty}\int_0^t (t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon u(s)ds.$$

Theorem 5 *Suppose that the hypotheses (H_1)-(H_3) are verified and that the function H has values in C_0^Ω. Then the fuzzy fractional problem 1.1 admits a unique mild fuzzy solution in C_T^Ω if the following condition holds:*

$$K\left(L_G+\frac{T^q(M_1+M_2)}{\Gamma(q+1)}\right)<1.$$

Proof 3 *Let $\mathcal{R}: C_T^\Omega\to C_T^\Omega$ be an operator defined by:*

$$(\mathcal{R}u)(t)=\begin{cases}\mathcal{M}_q(t)\odot(G(u)(0)\oplus H(0))\oplus\lim\limits_{\varepsilon\to+\infty}\displaystyle\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,u(s),u_s)ds, & t\in[0,T]\\ G(u)(t)\oplus H(t), \quad t\in[-h,0], & \end{cases} \tag{3.6}$$

It is clear that the operator $\mathcal{R}$ is well defined.

The mild fuzzy solution of the problem 1.1 is the fixed point of operator $\mathcal{R}$.

Now, we show that operator $\mathcal{R}$ admits a unique fixed point in C_T^Ω.

For $t\in[-h,0]$, we have:

$$(\mathcal{R}u)(t)=G(u)(t)\oplus H(t).$$

Let $u,v\in C_T^\Omega$, we have:

$$\begin{aligned}D\left((\mathcal{R}u)(t),(\mathcal{R}v)(t)\right)&=D\left(G(u)(t)\oplus H(t),G(v)(t)\oplus H(t)\right)\\&=D\left(G(u)(t),G(v)(t)\right)\\&=L_GD(u(t),v(t))\\&=L_GD_T(u,v).\end{aligned}$$

And for $t\in[0,T]$, we have:

$$(\mathcal{R}u)(t)=\mathcal{M}_q(t)\odot(G(u)(0)\oplus H(0))\oplus\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,u(s),u_s)ds.$$

Let $u,v\in C_T^\Omega$, then:

$$\begin{aligned}D\left((\mathcal{R}u)(t),(\mathcal{R}v)(t)\right)&\leq KD\left(G(u)(0),G(v)(0)\right)\\&+\frac{K}{\Gamma(q)}\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\theta_\varepsilon D(F(s,u(s),u_s),F(s,v(s),v_s))ds\\&\leq KD_0\left(G(u),G(v)\right)+\frac{K}{\Gamma(q)}\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\theta_\varepsilon\left(M_1D(u(s),v(s))\right.\\&\left.+M_2D(u_s,v_s)\right)ds\\&\leq KL_GD_T(u,v)+\frac{K(M_1+M_2)D_T(u,v)}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds\\&\leq K\left(L_G+\frac{T^q(M_1+M_2)}{\Gamma(q+1)}\right)D_T(u,v).\end{aligned}$$

Then,

$$\begin{aligned}D_T(\mathcal{R}u,\mathcal{R}v)&\leq\sup_{-h\leq t\leq T}D((\mathcal{R}u)(t),(\mathcal{R}v)(t))\\&\leq K\left(L_G+\frac{T^q(M_1+M_2)}{\Gamma(q+1)}\right)D_T(u,v)\\&\leq\lambda D_T(u,v).\end{aligned}$$

Where $0 < \lambda = K\left(L_G + \frac{T^q(M_1+M_2)}{\Gamma(q+1)}\right) < 1.$

Then, according to Banach's theorem, the operator $\mathcal{R}$ *admits a unique fixed point in* C_T^Ω.

Hence, the problem 1.1 admits a unique mild fuzzy solution in C_T^Ω.

Now, if we assume the term $G \equiv 0$, we get a new form of the fuzzy fractional problem 1.1:

$$\begin{cases} {}^C D^q_{0^+,gH} u(t) = \mathcal{A}u(t) \oplus F(t, u(t), u_t) \quad , \quad t \in [0,T] \\ u(t) = H(t) \quad , \quad t \in [-h, 0], \end{cases} \tag{3.7}$$

Thus, we will use a new method to show the existence of the mild fuzzy solution of problem 3.7. For this, we need the following proposition.

Theorem 6 *Suppose the operator* $\mathcal{A} : D(\mathcal{A}) \subset \Omega \to \Omega$ *generates a* C_0*-semigroup* $\{\mathcal{S}_q(t)\}_{t\geq 0} \subset \mathcal{P}(\Omega)$ *and* $G \equiv 0$*, and suppose both hypotheses* (H_1) *and* (H_2) *hold. Then, the fuzzy fractional problem 3.7 admits a unique mild fuzzy solution.*

Proof 4 *Let* $\mathcal{Z} : C_T^\Omega \to C_T^\Omega$ *be an operator defined by:*

$$(\mathcal{Z}u)(t) = \begin{cases} \mathcal{M}_q(t) \odot H(0) \\ \qquad \oplus \lim\limits_{\varepsilon \to +\infty} \int_0^t (t-s)^{q-1} \odot \mathcal{N}_q(t-s)\theta_\varepsilon F(s, u(s), u_s)ds, \quad t \in [0,T] \\ H(t) \quad , \quad t \in [-h, 0], \end{cases} \tag{3.8}$$

It is clear that the operator $\mathcal{Z}$ *is well defined. The mild fuzzy solution of the problem 3.7 is the fixed point of operator* $\mathcal{Z}$.

Now, we show that operator $\mathcal{Z}$ *admits a unique fixed point in* C_T^Ω.

For $t \in [0,T]$*, we have:*

$$(\mathcal{Z}u)(t) = \mathcal{M}_q(t) \odot H(0) \oplus \lim_{\varepsilon \to +\infty} \int_0^t (t-s)^{q-1} \odot \mathcal{N}_q(t-s)\theta_\varepsilon F(s, u(s), u_s)ds.$$

Let $u, v \in C_T^\Omega$*, and using the two hypotheses* (H_1) *and* (H_2)*, we obtain:*

$$\begin{aligned} D\left((\mathcal{Z}u)(t), (\mathcal{Z}v)(t)\right) &\leq \frac{K}{\Gamma(q)} \lim_{\varepsilon \to +\infty} \int_0^t (t-s)^{q-1}\theta_\varepsilon D(F(s,u(s),u_s), F(s,v(s),v_s))ds \\ &\leq \frac{K}{\Gamma(q)} \lim_{\varepsilon \to +\infty} \int_0^t (t-s)^{q-1}\theta_\varepsilon \left(M_1 D(u(s),v(s)) + M_2 D_0(u_s,v_s)\right)ds \\ &\leq \frac{K}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(M_1 D_T(u,v) + M_2 D_T(u,v)\right)ds \\ &\leq \frac{K}{\Gamma(q)} (M_1+M_2) D_T(u,v) \int_0^t (t-s)^{q-1} ds \\ &\leq \frac{t^q K}{\Gamma(q+1)} (M_1+M_2) D_T(u,v). \end{aligned}$$

In the same way, if we apply the operator $\mathcal{Z}$ twice, we obtain:

$$\begin{aligned}
D\left((\mathcal{Z}^2u)(t),(\mathcal{Z}^2v)(t)\right) &= D\left((\mathcal{Z}(\mathcal{Z}u))(t),(\mathcal{Z}(\mathcal{Z}v))(t)\right)\\
&\leq D(\mathcal{M}_q(t)\odot(\mathcal{Z}u)(0),\mathcal{M}_q(t)\odot(\mathcal{Z}v)(0))\\
&+D\left(\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}u)(s),(\mathcal{Z}u)_s)ds,\right.\\
&\left.\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}v)(s),(\mathcal{Z}v)_s)ds\right)\\
&\leq D(\mathcal{M}_q(t)\odot H(0),\mathcal{M}_q(t)\odot H(0))+\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\\
&D\left(\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}u)(s),(\mathcal{Z}u)_s),\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}v)(s),(\mathcal{Z}v)_s)\right)ds\\
&\leq\frac{K}{\Gamma(q)}\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\theta_\varepsilon D\left(F(s,(\mathcal{Z}u)(s),(\mathcal{Z}u)_s),F(s,(\mathcal{Z}v)(s),(\mathcal{Z}v)_s)\right)ds\\
&\leq\frac{K}{\Gamma(q)}\int_0^t(t-s)^{q-1}\left(M_1D\left((\mathcal{Z}u)(s),(\mathcal{Z}v)(s)\right)+M_2D_0\left((\mathcal{Z}u)_s,(\mathcal{Z}v)_s\right)\right)ds\\
&\leq\frac{K}{\Gamma(q)}\left(\frac{K}{\Gamma(q+1)}(M_1+M_2)D_T(u,v)\right)\left(M_1\int_0^t(t-s)^{q-1}s^qds+M_2\int_0^t(t-s)^{q-1}s^qds\right)\\
&\leq\frac{K^2}{\Gamma(q)\Gamma(q+1)}(M_1+M_2)^2D_T(u,v)\int_0^t(t-s)^{q-1}s^qds\\
&\leq\frac{t^{2q}K^2}{\Gamma(2q+1)\delta(1+q,q)}(M_1+M_2)^2D_T(u,v),
\end{aligned}$$

where $\delta(1+q,q)=\int_0^t(t-s)^{q-1}s^qds$ is the Beta function.

For any $x,y\in C_T^{\Omega}$ and $t\in[0,T]$, we have:

$$D\left((\mathcal{Z}^iu)(t),(\mathcal{Z}^iv)(t)\right)\leq\frac{t^{iq}K^i}{\Gamma(iq+1)}(M_1+M_2)^iD_T(u,v),$$

with i an integer greater than or equal to 3.

According to this inequality and the two hypotheses (H_1) and (H_2), we obtain:

$$\begin{aligned}
D\left((\mathcal{Z}^{i+1}u)(t),(\mathcal{Z}^{i+1}v)(t)\right) &= D\left((\mathcal{Z}(\mathcal{Z}^iu))(t),(\mathcal{Z}(\mathcal{Z}^iv))(t)\right)\\
&\leq D(\mathcal{M}_q(t)\odot(\mathcal{Z}^iu)(0),\mathcal{M}_q(t)\odot(\mathcal{Z}^iv)(0))\\
&+D\left(\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}^iu)(s),(\mathcal{Z}^iu)_s)ds,\right.\\
&\left.\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\odot\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}^iv)(s),(\mathcal{Z}^iv)_s)ds\right)\\
&\leq D(\mathcal{M}_q(t)\odot H(0),\mathcal{M}_q(t)\odot H(0))+\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\\
&D\left(\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}^iu)(s),(\mathcal{Z}^iu)_s),\mathcal{N}_q(t-s)\theta_\varepsilon F(s,(\mathcal{Z}^iv)(s),(\mathcal{Z}^iv)_s)\right)ds\\
&\leq\frac{K}{\Gamma(q)}\lim_{\varepsilon\to+\infty}\int_0^t(t-s)^{q-1}\theta_\varepsilon D\left(F(s,(\mathcal{Z}^iu)(s),(\mathcal{Z}^iu)_s),F(s,(\mathcal{Z}^iv)(s),(\mathcal{Z}^iv)_s)\right)ds\\
&\leq\frac{K}{\Gamma(q)}\int_0^t(t-s)^{q-1}\left(M_1D\left((\mathcal{Z}^iu)(s),(\mathcal{Z}^iv)(s)\right)+M_2D_0\left((\mathcal{Z}^iu)_s,(\mathcal{Z}^iv)_s\right)\right)ds\\
&\leq\frac{K^{i+1}}{\Gamma(q)\Gamma(iq+1)}(M_1+M_2)^{i+1}D_T(u,v)\int_0^t(t-s)^{q-1}s^qds\\
&\leq\frac{t^{(i+1)q}K^{i+1}}{\Gamma((i+1)q+1)\delta(1+q,q)}(M_1+M_2)^{i+1}D_T(u,v)\\
&\leq\frac{t^{(i+1)q}K^{i+1}}{\Gamma((i+1)q+1)}(M_1+M_2)^{i+1}D_T(u,v),
\end{aligned}$$

Then, according to the method of mathematical induction:

$$D\left((\mathcal{Z}^n u)(t), (\mathcal{Z}^n v)(t)\right) \leq \frac{t^{nq} K^n}{\Gamma(nq+1)} (M_1 + M_2)^n D_T(u, v),$$

for all n a positive integer, $u, v \in C_T^\Omega$ and $t \in [0, T]$.

For $t \in [-h, 0]$, we have:

$$(\mathcal{Z}u)(t) = H(t).$$

Then, for all $u, v \in C_T^\Omega$:

$$D\left((\mathcal{Z}^n u)(t), (\mathcal{Z}^n v)(t)\right) = 0.$$

Consequently, according to the two inequalities, for all n a positive integer and $u, v \in C_T^\Omega$, we have:

$$D_T\left(\mathcal{Z}^n u, \mathcal{Z}^n v\right) \leq \frac{T^{nq} K^n}{\Gamma(nq+1)} (M_1 + M_2)^n D_T(u, v).$$

According to the last inequality and the following proposition: Let n be a positive integer and $\alpha \in (0,1)$. The well-known Stirling's formula is:

$$\Gamma(n\alpha + 1) = \sqrt{2\pi n\alpha} \left(\frac{n\alpha}{e}\right) e^{\frac{\beta}{12n\alpha}} \quad , \quad \beta \in (0, 1).$$

We have:

$$\lim_{n \to \infty} \frac{T^{nq} K^n}{\sqrt{2\pi nq} \left(\frac{nq}{e}\right) e^{\frac{\beta}{12nq}}} (M_1 + M_2)^n = 0.$$

Then, according to this limit, there is a positive integer m such that:

$$\frac{T^{mq} K^m}{\sqrt{2\pi mq} \left(\frac{mq}{e}\right) e^{\frac{\beta}{12mq}}} (M_1 + M_2)^m < 1.$$

Then the operator $\mathcal{Z}^m$ is a contraction, so it admits a unique fixed point $\mu \in C_T^\Omega$, that is to say: $\mathcal{Z}^m \mu = \mu$.

Let us show that μ is a fixed point of the operator $\mathcal{Z}$.

We have: $\mathcal{Z}^m \mu = \mu$.

Then, $\mathcal{Z}^{m+1} \mu = \mathcal{Z}^m(\mathcal{Z}\mu) = \mathcal{Z}\mu$.

So, $\mathcal{Z}\mu$ is a fixed point of the operator $\mathcal{Z}^m$ and since the fixed point is unique, then: $\mathcal{Z}\mu = \mu$.

Hence, $\mu \in C_T^\Omega$ is the unique fixed point of operator $\mathcal{Z}$.

Thus, μ is the unique mild fuzzy solution of the fuzzy fractional problem 3.7.

4. An Illustrative Example

In this section, we propose a sample problem to test the validity of our results regarding the existence and uniqueness of the mild solution.

We consider the following example problem coincides with the fuzzy fractional problem 1.1:

$$\begin{cases} {}^C D^q_{0^+, gH} u(t) = \mathcal{A}u(t) \oplus F(t, u(t), u_t) \quad , \quad t \in [0, T] \\ u(t) = G(u)(t) \oplus H(t) \quad , \quad t \in [-h, 0], \end{cases} \tag{4.1}$$

with, $q = \dfrac{1}{3}$, $T = 1$ and the constant $h \in (0, +\infty)$

the function $u \in C_T^\Omega$ and $\mathcal{A} : D(\mathcal{A}) \subset \Omega \to \Omega$ a non-dense closed linear operator.

The function F is defined by the formula :

$$F(t, u(t), u_t) = \frac{1}{4} sin(t) \odot u(t) \oplus \int_{-h}^{0} \varphi(s) u(s) ds \oplus t e^{\frac{-4}{5} t} \mathcal{B},$$

where, $\mathcal{B} = (0, 1, 2) \in \Omega$ is a triangular fuzzy number, and $\varphi : [-h, 0] \to \mathbb{R}^+$ is an integrable function. The function $G(u)(t) = \frac{2}{7} cos(t) \odot u(t)$.
Suppose that hypothesis (H_1) holds, with : $K = 1$. For the hypothesis (H_2), we have for all $u, v \in C_T^\Omega$:

$$D(F(t, u(t), u_t), F(t, v(t), v_t)) \leq \frac{1}{4} D(u, v) + \int_{-h}^{0} \varphi(s) ds D_0(u, v),$$

then, $M_1 = \frac{1}{4}$ and suppose that $M_2 = \int_{-h}^{0} \varphi(s) ds = \frac{1}{5}$.
For the hypothesis (H_3), we have:

$$D(G(u)(t), G(v)(t)) \leq \frac{2}{7} D(u, v) \ , \ \forall u, v \in C_T^\Omega,$$

then, $L_G = \frac{2}{7}$.
So, we have:

$$K \left(L_G + \frac{T^q (M_1 + M_2)}{\Gamma(q + 1)} \right) \simeq 0,7913 < 1.$$

Hence, according to theorem 5 the problem example admits a unique milde fuzzy solution in C_T^Ω.
Now, if we assume that $G \equiv 0$, then according to theorem 6 the problem example admits a unique mild fuzzy solution in C_T^Ω.

5. Conclusion

In this article, the focus has been on the study of the fuzzy nonlinear fractional evolution equation. The approach began with deriving the integral solution, followed by a rigorous proof of the existence and uniqueness of the mild fuzzy solution. Finally, our work concluded with an application example to demonstrate practical implications of our findings.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Khadija Oufkir,
Applied Mathematics and Scientific Computing Laboratory,
Sultan Moulay Slimane University (SMSU), 23000, Beni Mellal
Morocco.
E-mail address: `oufkirkhadijabzou@gmail.com ; https://orcid.org/0000-0002-5465-7118`

and

M'hamed Elomari,
Applied Mathematics and Scientific Computing Laboratory,
Sultan Moulay Slimane University (SMSU), 23000, Beni Mellal
Morocco.
E-mail address: `m.elomari@usms.ma ; https://orcid.org/0000-0002-4256-1434`

and

Hamid Sadiki,
Applied Mathematics and Scientific Computing Laboratory,
Sultan Moulay Slimane University (SMSU), 23000, Beni Mellal
Morocco.
E-mail address: `h.sadiki@usms.ma ; https://orcid.org/0009-0009-9900-8486`