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Hankel Determinant for Bi-Univalent Functions with Bounded Turning Associated with tan Hyperbolic Function

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ABSTRACT: In this article, utilizing the concept of subordination, we have developed two new subclasses of bi-univalent functions that are related to the domain of the hyperbolic tangent function. Our study focuses on determining the upper bound of the second Hankel determinant for particular new subclasses of bi-univalent functions in the open unit disk $\mathbb{U}_0$.

Key Words: Analytic, bi-univalent, Hankel determinant, subordination.

Contents

1. Introduction and definitions

The set of all analytic functions h, satisfying $h(0) = 0$ and $h'(0) = 1$ within the unit disk $\mathbb{U}_0 = \{\vartheta : |\vartheta| < 1\}$, is represented by $\mathcal{A}$. These functions can be expressed through a Taylor series expansion as follows:

$$h(\vartheta) = \vartheta + \sum_{m=2}^{\infty} h_m \vartheta^m, \quad \vartheta \in \mathbb{U}_0. \tag{1.1}$$

The set of functions that are both univalent and analytic within the domain $\mathbb{U}_0$ is represented by $\mathcal{S}$. It is a well-established [5] fact that each function h within the set $\mathcal{S}$ possesses an inverse, denoted as h^{-1}, which satisfies the equation

$$h^{-1}(h(\vartheta)) = \vartheta, \quad \vartheta \in \mathbb{U}_0$$

and

$$h(h^{-1}(u)) = u, \quad \left(|u| < r;\ r \geq \frac{1}{4}\right).$$

The corresponding power series expansion can be expressed as

$$h^{-1}(u) := \eta(u) = u - h_2u^2 + (2h_2^2 - h_3)u^3 - (5h_2^3 - 5h_2h_3 + h_4)u^4 + \cdots. \tag{1.2}$$

A function h within the set $\mathbb{U}_0$ is termed bi-univalent in the domain $\mathbb{U}_0$ if both h and its inverse h^{-1} are univalent in $\mathbb{U}_0$. The class of bi-univalent functions in $\mathbb{U}_0$ is represented by Σ. Examples of functions that fall under the category of Σ include

$$\frac{\vartheta}{1-\vartheta} \quad \textit{and} \quad -\log(1-\vartheta),$$

and others. However, the well-known Koebe function is not included in the class Σ. Lewin [12] explored the class Σ of bi-univalent functions, proving that $|h_2| \leq 1.51$ for every function that belongs to Σ. In recent times, a variety of researchers have introduced and analyzed several noteworthy subclasses of the bi-univalent function class Σ, revealing non-sharp estimates for the initial two Taylor–Maclaurin coefficients

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$|h_2|$ and $|h_3|$ (as noted in [1,7,9,16,17,18,19,20,21]). The challenge of estimating the coefficients $|h_m|$ for $m \geq 4$ remains an unresolved issue.

In their 1976 work, Noonan and Thomas [14] introduced the concept of the ℓ^{th} Hankel determinant for integers $m \geq 1$ and $\ell \geq 1$, which is expressed as:

$$\mathcal{H}_\ell(m) := \begin{vmatrix} h_m & h_{m+1} & \cdots & h_{m+\ell-1} \\ h_{m+1} & h_{m+2} & \cdots & h_{m+\ell} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m+\ell-1} & h_{m+\ell} & \cdots & h_{m+2\ell-2} \end{vmatrix}$$

By specifically altering the parameters ℓ and m, we can formulate

$$\mathcal{H}_2(1) := \begin{vmatrix} h_1 & h_2 \\ h_2 & h_3 \end{vmatrix} = h_3 - h_2^2, \quad as \;\; h_1 = 1$$

and

$$\mathcal{H}_2(2) := \begin{vmatrix} h_2 & h_3 \\ h_3 & h_4 \end{vmatrix} = h_2 h_4 - h_3^2.$$

The Hankel determinants $\mathcal{H}_2(1) = h_3 - h_2^2$ and $\mathcal{H}_2(2) = h_2h_4 - h_3^2$ are recognized as the Fekete–Szegö and second Hankel determinant functionals, respectively. Noor [13] has characterized the growth of $\mathcal{H}_\ell(m)$ as $m \to \infty$ for fixed positive integers ℓ and m, under the constraint of a bounded boundary. Moreover, Fekete and Szegö [6] developed the generalized functional $h_3 - \mu h_2^2$, where μ is defined as a real number. Recently, several researchers have examined related problems in this area (see, for instance, [2,3,4,8,10,15,22]). In their recent work, Lee et al. [11] have defined a precise bound for $|\mathcal{H}_2(2)|$ by generalizing their classes through subordination techniques.

In the context of two analytic functions h and f defined on $\mathbb{U}_0$, we say that h is subordinate to f, represented as $g \prec h$, if there exists an analytic function χ in $\mathbb{U}_0$ satisfying $\chi(0) = 0$ and $|\chi(\vartheta)| < 1$, such that $h(\vartheta)$ can be expressed as $h(\vartheta) = f(\chi(\vartheta))$ for all $\vartheta \in \mathbb{U}_0$. If f is univalent in $\mathbb{U}_0$, this relationship is equivalent to the conditions $h(0) = f(0)$ and $h(\mathbb{U}_0)$ being a subset of $f(\mathbb{U}_0)$.

This study assumes that the function Ψ is analytic and possesses a positive real part in the unit disk $\mathbb{U}_0$. We establish that $\Psi(0) = 1$ and $\Psi'(0) > 0$, with the additional property that $\Psi(\mathbb{U}_0)$ is symmetric about the real axis. Such a function can be represented by a power series expansion of the form

$$\Psi(\vartheta) = 1 + \Psi_1\vartheta + \Psi_2\vartheta^2 + \Psi_3\vartheta^3 + \cdots, \quad \Psi_1 > 0.$$

Definition 1.1 *The function $h \in \Sigma$, as described in (1.1), qualifies as part of the class $\mathcal{H}_\Sigma(\Pi)$ provided it fulfills the stipulated conditions*

$$h'(\vartheta) \prec \Pi(\vartheta) := 1 + \tanh(z)$$

and

$$\eta'(u) \prec \Pi(u) := 1 + \tanh(u).$$

Here, η signifies the analytic continuation of the inverse of h across the area $\mathbb{U}_0$.

Definition 1.2 *The function $h \in \Sigma$, as described in (1.1), qualifies as part of the class $\mathcal{R}_\Sigma(\Pi)$ provided it fulfills the stipulated conditions*

$$\frac{h(\vartheta)}{\vartheta} \prec \Pi(\vartheta) := 1 + \tanh(z)$$

and

$$\frac{\eta(u)}{u} \prec \Pi(u) := 1 + \tanh(u).$$

Here, η signifies the analytic continuation of the inverse of h across the area $\mathbb{U}_0$.

To delineate our essential findings, we must utilize the following lemmas.

Lemma 1.1 *[5] Let t represent an analytic function in the unit disk $\mathbb{U}_0$, satisfying the condition $t(0) = 0$ and $|t(\vartheta)| < 1$ for all $\vartheta \in \mathbb{U}_0$. This function can be characterized by its power series expansion*

$$t(\vartheta) = t_1\vartheta + t_2\vartheta^2 + t_3\vartheta^3 + \cdots, \quad \vartheta \in \mathbb{U}_0.$$

Then, $|t_m| \leq 1$ for all $m \in \mathbb{N}$. Furthermore, $|t_m| = 1$ for some m $m \in \mathbb{N}$ if and only if $t(\vartheta) = e^{ir}\vartheta^m$, $r \in \mathbb{R}$.

Lemma 1.2 *[8] Given that $t(\vartheta) = t_1\vartheta + t_2\vartheta^2 + t_3\vartheta^3 + \cdots$, for ϑ within the domain $\mathbb{U}_0$ and t is classified as a Schwarz function, then it follows that*

$$t_2 = \xi(1 - t_1^2)$$

and

$$t_3 = (1 - t_1^2)(1 - |\xi|^2)c - t_1(1 - t_1^2)\xi^2.$$

For selected values of ξ and c, with the condition that $|\xi| \leq 1$ and $|c| \leq 1$.

In this article, utilizing the concept of subordination, we have developed two new subclasses of bi-univalent functions that are related to the domain of the hyperbolic tangent function. Our study focuses on determining the upper bound of the second Hankel determinant for particular new subclasses of bi-univalent functions in the open unit disk $\mathbb{U}_0$.

2. Main Results

Theorem 2.1 *For a function h in the space $\mathcal{H}_\Sigma(\Pi)$ as described in (1.1), the following inequality is valid:*

$$|h_2h_4 - h_3^2| \leq \frac{127}{144}.$$

Proof: If h belongs to the space $\mathcal{H}_\Sigma(\Pi)$, then according to Definition 1.1, there are two Schwarz functions, $\phi(\vartheta)$ and $\chi(u)$, such that

$$h'(\vartheta) = 1 + \tanh(\phi(\vartheta)) \tag{2.1}$$

and

$$\eta'(u) = 1 + \tanh(\chi(u)) \tag{2.2}$$

with

$$\phi(\vartheta) = y_1\vartheta + y_2\vartheta^2 + y_3\vartheta^3 + \cdots$$

and

$$\chi(u) = q_1u + q_2u^2 + q_3u^3 + \cdots.$$

Employing the subordination property, we can conclude

$$1 + \tanh(\phi(\vartheta)) = 1 + y_1\vartheta + y_2\vartheta^2 + \left(y_3 - \frac{y_1^3}{3}\right)\vartheta^3 + \cdots \tag{2.3}$$

and

$$1 + \tanh(\chi(u)) = 1 + q_1u + q_2u^2 + \left(q_3 - \frac{q_1^3}{3}\right)u^3 + \cdots. \tag{2.4}$$

Upon reviewing equations (2.1) and (2.3), we ascertain that, respectively,

$$2h_2 = y_1, \tag{2.5}$$

$$3h_3 = y_2 \tag{2.6}$$

and

$$4h_4 = y_3 - \frac{1}{3}y_1^3. \tag{2.7}$$

Similarly, upon reviewing equations (2.2) and (2.4), we ascertain that, respectively,

$$-2h_2 = q_1, \tag{2.8}$$

$$3(2h_2^2 - h_3) = q_2 \tag{2.9}$$

and

$$4(5h_2h_3 - h_4 - 5h_2^3) = q_3 - \frac{1}{3}q_1^3. \tag{2.10}$$

By analyzing (2.5) and (2.8), we derive that

$$y_1 + q_1 = 0 \tag{2.11}$$

and

$$h_2 = \frac{y_1}{2}. \tag{2.12}$$

Using (2.12) in combination with (2.6) and (2.9), we determine that

$$h_3 = \frac{y_1^2}{4} + \frac{1}{6}(y_2 - q_2). \tag{2.13}$$

Again, using (2.12) and (2.13) in combination with (2.7) and (2.10), we determine that

$$h_4 = -\frac{1}{12}y_1^3 + \frac{5}{24}(y_2 - q_2) + \frac{1}{8}(y_3 - q_3). \tag{2.14}$$

Therefore, by consulting equations (2.12), (2.13), and (2.14), we determine that

$$|h_2h_4 - h_3^3| = \left| -\frac{5}{48}y_1^4 + \frac{1}{48}y_1^2(y_2 - q_2) + \frac{1}{16}y_1(y_3 - q_3) - \frac{1}{36}(y_2 - q_2)^2 \right|. \tag{2.15}$$

Based on Lemma 1.2, it follows that

$$y_2 = \xi(1 - y_1^2) \quad and \quad q_2 = z(1 - q_1^2)$$

Hence, based on equation (2.11), it follows that

$$y_2 - q_2 = (1 - y_1^2)(\xi - z). \tag{2.16}$$

Moreover,

$$y_3 = (1 - y_1^2)(1 - |\xi|^2)c - y_1(1 - y_1^2)\xi^2$$

and

$$q_3 = (1 - q_1^2)(1 - |z|^2)s - q_1(1 - q_1^2)z^2.$$

Hence, based on equation (2.11), it follows that

$$y_3 - q_3 = (1 - y_1^2)[(1 - |\xi|^2)c - (1 - |z|^2)s] - y_1(1 - y_1^2)(\xi^2 + z^2), \tag{2.17}$$

for some ξ, z, c and s, with $|\xi| \leq 1$, $|z| \leq 1$, $|c| \leq 1$ and $|s| \leq 1$. By applying equations (2.16) and (2.17) to equation (2.15), we have

$$|h_2h_4 - h_3^2| \leq \left| \begin{array}{l} -\frac{5}{48}y_1^4 + \frac{(\xi - z)}{48}y_1^2(1 - y_1^2) - \frac{y_1^2}{16}(1 - y_1^2)(\xi^2 + z^2) - \frac{1}{36}(1 - y_1^2)^2(\xi - z)^2 \\ +\frac{y_1}{16}(1 - y_2^2)[(1 - |\xi|^2)c - (1 - |z|^2)s]. \end{array} \right|$$

Fixed $y = y_1$, as $|y_1| \leq 1$, then $y \in [0, 1]$, and by using $|z| \leq 1$ and $|c| \leq 1$, so we have

$$|h_2h_4 - h_3^2| \leq \left| \begin{array}{l} -\frac{5}{48}y^4 + \frac{y(1 + y^2)}{8} + \frac{y^2}{48}(1 + y^2)(|\xi| + |z|) \\ \frac{y(y - 1)}{16}(1 + y^2)(|\xi|^2 + |z|^2) + \frac{(1 + y^2)^2}{36}(|\xi| + |z|)^2. \end{array} \right|$$

Considering $\gamma = |\xi| \le 1$ and $\beta = |z| \le 1$, we get

$$|h_2h_4 - h_3^2| \le [L_1 + L_2(\gamma+\beta) + L_3(\gamma^2+\beta^2) + L_4(\gamma+\beta)^2] = \Upsilon(\gamma,\beta),$$

where

$$\begin{aligned}
L_1 =& L_1(y) = \frac{5}{24}y^4 + \frac{y}{8}(1+y^2) \ge 0,\\
L_2 =& L_2(y) = \frac{1}{48}y^2(1+y^2) \ge 0,\\
L_3 =& L_3(y) = \frac{y}{16}(y-1)(1+y^2) \le 0,\\
L_4 =& L_4(y) = \frac{1}{36}(1+y^2)^2 \ge 0.
\end{aligned}$$

Our next step is to identify the maximum value of the function $\Upsilon(\gamma,\beta)$ within the closed square $[0,1]\times[0,1]$ for $y \in [0,1]$. This analysis requires us to examine the maximum of $\Upsilon(\gamma,\beta)$ for $y \in (0,1)$, as well as for $y = 0$ and $y = 1$, while considering the expression $\Upsilon_{\gamma\gamma}\Upsilon_{\beta\beta} - \Upsilon^2_{\gamma\beta}$.

To begin with, if we assign $y = 0$, then we have

$$\Upsilon(\gamma,\beta) := \frac{1}{36}(\gamma+\beta)^2,$$

and it is straightforward to see that

$$\max\{\Upsilon(\gamma,\beta) : (\gamma,\beta) \in [0,1]\times[0,1]\} = \Upsilon(1,1) = \frac{1}{9}.$$

In the next step, by choosing $y = 1$, we can conclude that

$$\Upsilon(\gamma,\beta) := \frac{11}{24} + \frac{1}{24}(\gamma+\beta) + \frac{1}{9}(\gamma+\beta)^2,$$

and it is straightforward to see that

$$\max\{\Upsilon(\gamma,\beta) : (\gamma,\beta) \in [0,1]\times[0,1]\} = \Upsilon(1,1) = \frac{213}{216}.$$

Finally, let us consider the case $y \in (0,1)$. Given that $L_3 + 2L_4 > 0$ and $L_3 < 0$, we can deduce that

$$\Upsilon_{\gamma\gamma}\Upsilon_{\beta\beta} - \Upsilon^2_{\gamma\beta} < 0.$$

Therefore, the function Υ cannot possess a local maximum within the interior of the square defined by $[0,1]\times[0,1]$.
For the case where $\gamma = 0$ and $0 \le \beta \le 1$ (and likewise for $\beta = 0$ and $0 \le \gamma \le 1$), we get

$$G(\beta) := \Upsilon(0,\beta) = L_1 + L_2\beta + (L_3+L_4)\beta^2.$$

(i) Assuming $L_3 + L_4 \ge 0$, it is clear that $G'(\beta) = 2(L_3+L_4)\beta + L_2 > 0$, for $0 < \beta < 1$ and for any constant $y \in (0,1)$. Thus, $G(\beta)$ is recognized as an increasing function. Therefore, for a fixed value of y within the interval $(0,1)$, the maximum of $G(\beta)$ is achieved at $\beta = 1$, where

$$\max\{G(\beta) : \beta \in [0,1]\} = G(1) = L_1 + L_2 + L_3 + L_4.$$

(ii) Assuming $L_3 + L_4 < 0$, we examine the critical point $\beta = -\dfrac{L_2}{2(L_3+L_4)} = \dfrac{L_2}{2c}$ for a fixed value of $y \in (0,1)$, where $c = -(L_3+L_4) > 0$, and consider the following two cases:
Case 1. For $\beta = \dfrac{L_2}{2c} > 1$, it can be concluded that $c < \dfrac{L_2}{2} \le L_2$, which implies that $L_2 + L_3 + L_4 \ge 0$. Consequently,

$$G(0) = L_1 \le L_1 + L_2 + L_3 + L_4 = G(1).$$

Case 2. For $\beta = \dfrac{L_2}{2c} \leq 1$, as $\dfrac{L_2}{2} \geq 0$, we have $\dfrac{L_2^2}{4c} \leq \dfrac{L_2}{2} \leq L_2$. Furthermore, it follows that

$$G(1) = L_1 + L_2 + L_3 + L_4 \leq L_1 + L_2,$$

which implies

$$G(0) = L_1 \leq L_1 + \frac{L_2^2}{4c} = G\left(\frac{L_2}{2c}\right) \leq L_1 + L_2.$$

Upon examining cases (i) and (ii) with $\gamma = 0$, and for $0 \leq \beta \leq 1$ while keeping y fixed within the range $(0,1)$, it can be deduced that $G(\beta)$ attains its maximum when $L_3 + L_4 \geq 0$. Consequently, we find that the maximum of $G(\beta)$ for $\beta \in [0,1]$ is expressed as

$$\max\{G(\beta) : \beta \in [0,1]\} = G(1) = L_1 + L_2 + L_3 + L_4.$$

For the case where $\gamma = 1$ and $0 \leq \beta \leq 1$ (and likewise for $\beta = 1$ and $0 \leq \gamma \leq 1$), we get

$$M(\beta) := \Upsilon(1,\beta) = (L_3 + L_4)\beta^2 + (L_2 + 2T_4)\beta + L_1 + L_2 + L_3 + L_4.$$

(iii) Assuming $L_3 + L_4 \geq 0$, it is clear that $M'(\beta) = 2(L_3 + L_4)\beta + L_2 + 2T_4 > 0$, for $0 < \beta < 1$ and for any constant $y \in (0,1)$. Thus, $M(\beta)$ is recognized as an increasing function. Therefore, for a fixed value of y within the interval $(0,1)$, the maximum of $M(\beta)$ is achieved at $\beta = 1$, where

$$\max\{M(\beta) : \beta \in [0,1]\} = M(1) = L_1 + 2L_2 + 2L_3 + 4L_4.$$

(iv) Assuming $L_3 + L_4 < 0$, we examine the critical point $\beta = -\dfrac{(L_2 + 2L_4)}{2(L_3 + L_4)} = \dfrac{(L_2 + 2L_4)}{2c}$ for a fixed value of $y \in (0,1)$, where $c = -(L_3 + L_4) > 0$, and consider the following two cases:

Case 1. For $\beta = \dfrac{L_2 + 2L_4}{2c} > 1$, it can be concluded that $c < \dfrac{L_2 + 2L_4}{2} \leq L_2 + 2L_4$, which implies that $L_2 + L_3 + 3L_4 \geq 0$. Consequently,

$$M(0) = L_1 + L_2 + L_3 + L_4 \leq L_1 + 2L_2 + 2L_3 + 4L_4 = M(1).$$

Case 2. For $\beta = \dfrac{L_2 + 2L_4}{2c} \leq 1$, as $\dfrac{L_2 + 2L_4}{2} \geq 0$, Furthermore, it follows that

$$\frac{(L_2 + 2L_4)^2}{4c^2} \leq \frac{L_2 + 2L_4}{2} \leq L_2 + 2L_4,$$

which implies

$$\begin{aligned} M(0) =& L_1 + L_2 + L_3 + L_4 \\ \leq& L_1 + L_2 + L_3 + L_4 + \frac{(L_2 + 2L_4)^2}{4c^2} \\ =& M\left(\frac{L_2 + 2L_4}{2c}\right) \leq L_1 + 2L_2 + L_3 + 3L_4. \end{aligned}$$

Upon examining cases (iii) and (iv) with $\gamma = 1$, and for $0 \leq \beta \leq 1$ while keeping y fixed within the range $(0,1)$, it can be deduced that $M(\beta)$ attains its maximum when $L_3 + L_4 \geq 0$. Consequently, we find that the maximum of $M(\beta)$ for $\beta \in [0,1]$ is expressed as

$$\max\{M(\beta) : \beta \in [0,1]\} = M(1) = L_1 + 2L_2 + 2L_3 + 4L_4.$$

Since it is established that $G(1) \leq M(1)$ for $y \in [0,1]$, we can assert that the maximum of $\Upsilon(\gamma,\beta)$, where (γ,β) falls within the range $[0,1] \times [0,1]$, is $\Upsilon(1,1)$. Therefore, the maximum of Υ in the closed square

$[0,1]\times[0,1]$ occurs at the coordinates $\gamma=1$ and $\beta=1$.
Consider the function H defined as $H:[0,1]\to\mathbb{R}$, where

$$H(y):=\max\{\Upsilon(\gamma,\beta):(\gamma,\beta)\in[0,1]\times[0,1]\}=\Upsilon(1,1)$$
$$=L_1+2L_2+2L_3+4L_4.$$

Upon substituting L_1, L_2, L_3, and L_4 into the function H outlined above,

$$H(y)=\frac{55}{144}y^4+\frac{7}{18}y^2+\frac{1}{9}.$$

By assigning $y^2=p$, we analyze the maximum of $H(p)$ within the interval $(0,1)$. After performing a simple calculation, we obtain the following

$$H'(y^2):=H'(p)=\frac{55}{72}p+\frac{7}{18}.$$

Tt is clear that $H'(p)>0$, for $0<p<1$. Thus, $H(p)$ is recognized as an increasing function. Therefore, for a fixed value of p within the interval $(0,1)$, the maximum of $H(p)$ is achieved at $p=1$, where

$$\max\{H(p):p\in[0,1]\}=H(1)=\frac{127}{144}.$$

Therefore, we conclude

$$|h_2h_4-h_3^2|\le\frac{127}{144}.$$

This finalizes the proof of the Theorem 2.1. □

Theorem 2.2 *For a function h in the space $\mathcal{R}_\Sigma(\Pi)$ as described in (1.1), the following inequality is valid:*

$$|h_2h_4-h_3^2|\le\frac{43}{6}.$$

Proof: If h belongs to the space $\mathcal{R}_\Sigma(\Pi)$, then according to Definition 1.2, there are two Schwarz functions, $\phi(\vartheta)$ and $\chi(u)$, such that

$$\frac{h(\vartheta)}{\vartheta}=1+\tanh(\phi(\vartheta)) \tag{2.18}$$

and

$$\frac{\eta(u)}{u}=1+\tanh(\chi(u)). \tag{2.19}$$

Upon reviewing equations (2.18) and (2.3), we ascertain that, respectively,

$$h_2=y_1, \tag{2.20}$$

$$h_3=y_2 \tag{2.21}$$

and

$$h_4=y_3-\frac{1}{3}y_1^3. \tag{2.22}$$

Similarly, upon reviewing equations (2.19) and (2.4), we ascertain that, respectively,

$$-h_2=q_1, \tag{2.23}$$

$$2h_2^2-h_3=q_2 \tag{2.24}$$

and

$$5h_2h_3-h_4-5h_2^2=q_3-\frac{1}{3}q_1^3. \tag{2.25}$$

By analyzing (2.20) and (2.23), we derive that

$$y_1 + q_1 = 0. \tag{2.26}$$

Using (2.20) in combination with (2.21) and (2.24), we determine that

$$h_3 = y_1^2 + \frac{1}{2}(y_2 - q_2). \tag{2.27}$$

Again, using (2.20) and (2.27) in combination with (2.22) and (2.25), we determine that

$$h_4 = -\frac{1}{3}y_1^3 + \frac{5}{4}y_1(y_2 - q_2) + \frac{1}{2}(y_3 - q_3). \tag{2.28}$$

Therefore, by consulting equations (2.20), (2.27), and (2.28), we determine that

$$|h_2h_4 - h_3^3| = \left| -\frac{4}{3}y_1^2 + \frac{y_1^2(y_2 - q_2)}{4} + \frac{y_1(y_3 - q_3)}{2} - \frac{(y_2 - q_2)^2}{4} \right|. \tag{2.29}$$

By applying equations (2.16) and (2.17) to equation (2.29), we have

$$|h_2h_4 - h_3^2| \leq \left| \begin{array}{l} -\frac{4}{3}y_1^4 + \frac{(\xi - z)}{4}y_1^2(1 - y_1^2) - \frac{y_1^2}{2}(1 - y_1^2)(\xi^2 + z^2) - \frac{1}{4}(1 - y_1^2)^2(\xi - z)^2 \\ +\frac{y_1}{2}(1 - y_2^2)[(1 - |\xi|^2)c - (1 - |z|^2)s]. \end{array} \right|$$

Fixed $y = y_1$, as $|y_1| \leq 1$, then $y \in [0, 1]$, and by using $|z| \leq 1$ and $|c| \leq 1$, so we have

$$|h_2h_4 - h_3^2| \leq \left| \begin{array}{l} -\frac{4}{3}y^4 + y(1 + y^2) + \frac{y^2}{4}(1 + y^2)(|\xi| + |z|) \\ \frac{y(y-1)}{2}(1 + y^2)(|\xi|^2 + |z|^2) + \frac{(1 + y^2)^2}{4}(|\xi| + |z|)^2. \end{array} \right|$$

Considering $\gamma = |\xi| \leq 1$ and $\beta = |z| \leq 1$, we get

$$|h_2h_4 - h_3^2| \leq [W_1 + W_2(\gamma + \beta) + W_3(\gamma^2 + \beta^2) + W_4(\gamma + \beta)^2] = J(\gamma, \beta),$$

where

$$\begin{aligned} W_1 =& W_1(y) = \frac{4}{3}y^4 + y(1 + y^2) \geq 0, \\ W_2 =& W_2(y) = \frac{1}{4}y^2(1 + y^2) \geq 0, \\ W_3 =& W_3(y) = \frac{y}{2}(y - 1)(1 + y^2) \leq 0, \\ W_4 =& W_4(y) = \frac{1}{4}(1 + y^2)^2 \geq 0. \end{aligned}$$

In order to ascertain the maximum value of $J(\gamma, \beta)$, we employ a method similar to that used for $J(\gamma, \beta)$, in Theorem 2.1. Clearly, the maximum of $J(\gamma, \beta)$ is located at $\gamma = 1$ and $\beta = 1$ within the closed square $[0, 1] \times [0, 1]$ for $c \in (0, 2)$. Consider the function I defined as $I : [0, 1] \to \mathbb{R}$, where

$$\begin{aligned} I(y) :=& \max\{M(\gamma, \beta) : (\gamma, \beta) \in [0, 1] \times [0, 1]\} = M(1, 1) \\ =& W_1 + 2W_2 + 2W_3 + 4W_4. \end{aligned}$$

Upon substituting W_1, W_2, W_3, and W_4 into the function I outlined above,

$$I(y) = \frac{23}{6}y^4 + \frac{7}{3}y^2 + 1.$$

By assigning $y^2 = p$, we analyze the maximum of $I(p)$ within the interval $(0, 1)$. After performing a simple calculation, we obtain the following

$$I'(y^2) := I'(p) = \frac{23}{3}p + \frac{7}{3}.$$

It is clear that $I'(p) > 0$, for $0 < p < 1$. Thus, $I(p)$ is recognized as an increasing function. Therefore, for a fixed value of p within the interval $(0, 1)$, the maximum of $I(p)$ is achieved at $p = 1$, where

$$\max\{I(p) : p \in [0, 1]\} = I(1) = \frac{43}{6}.$$

Therefore, we conclude

$$|h_2h_4 - h_3^2| \leq \frac{43}{6}.$$

This finalizes the proof of the Theorem 2.2. □

3. Concluding remarks and observations

This article focuses on determining the upper bound of the second Hankel determinant for selected subclasses of bi-univalent functions within the open unit disc, using tan hyperbolic function for our investigation.

Furthermore, the study considered in this article can be extended by taking the generalized tan hyperbolic function. Also, the same type of results can be worked out for interesting orthogonal polynomial. The study also suggests that by employing q-calculus for values of $0 < q < 1$, along with functions with bounded boundary and bounded radius rotation, one can define a functional class.

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