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Limits of recurrent operators

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ABSTRACT: An operator T acting on a complex, infinite-dimensional Hilbert space $\mathcal{H}$ is deemed recurrent (or super-recurrent) if, for each open subset $U \subset \mathcal{H}$, there exists an integer n (alternatively, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$) such that $T^nU \cap U \neq \emptyset$, (or $\lambda T^nU \cap U \neq \emptyset$ for the super-recurrent case). It is known that if T is recurrent, then the set of eigenvalues of T^*, the adjoint of T^*, is contained in the unit circle $\mathbb{T}$ and that the union of the spectrum of T and $\mathbb{T}$ is a connected set. By these results, we gave a complete spectral characterization of the norm closure of the class $REC(\mathcal{H})$, which consists of all recurrent operators acting on $\mathcal{H}$. Furthermore, analogous results are obtained for the closely related class $SREC(\mathcal{H})$, the set of all super-recurrent operators on $\mathcal{H}$.

Key Words: Hypercyclic operators, recurrent operators, super-recurrent operators.

Contents

1. Introduction

This paper will use the symbol $\mathcal{H}$ to represent a Hilbert space over the complex field $\mathbb{C}$. An operator refers to a linear and continuous mapping that operates on $\mathcal{H}$ and the space of operators on $\mathcal{H}$ shall be denoted by $\mathcal{B}(\mathcal{H})$.

Given an operator T belonging to the space of bounded linear operators on $\mathcal{H}$, the pair $(\mathcal{H}, T)$ forms a linear topological dynamical system. In the field of linear dynamical systems, the notions of hypercyclicity and supercyclicity hold significant importance and have been extensively studied.

If an operator T acting on $\mathcal{H}$ satisfies the condition that there exists a vector x whose orbit under T, denoted by $\mathrm{Orb}(T, x)$, is a dense subset of $\mathcal{H}$, then T is referred to as hypercyclic. The orbit $\mathrm{Orb}(T, x)$ is defined as the set $\{T^nx : n \in \mathbb{N}\}$. The vector x itself is called a hypercyclic vector for T. The set of all hypercyclic vectors for the operator T is denoted by $HC(T)$ while the notation $HC(\mathcal{H})$ represents the set of all hypercyclic operators that act on $\mathcal{H}$.
Birkhoff established a necessary and sufficient condition for hypercyclicity in [10], demonstrating that an operator T is hypercyclic if and only if it is topologically transitive. Topological transitivity means that for any pair (U, V) of nonempty and open subsets of $\mathcal{H}$, there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

Likewise, an operator T is defined as supercyclic if a vector $x \in \mathcal{H}$ exists such that its projective orbit is denoted by $\mathbb{C}.\mathrm{Orb}(T, x)$, is a dense subset of $\mathcal{H}$. The projective orbit consists of all scalar multiples of the elements in the orbit of x under T, i.e., $\{\lambda T^nx : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$. A vector x satisfying the given condition is referred to as a supercyclic vector for the operator T. The set of all supercyclic vectors for the operator T is represented by $\mathrm{SC}(T)$, and $SC(\mathcal{H})$ denotes the set of all supercyclic operators that act on $\mathcal{H}$. Similarly, the operator T is considered supercyclic if and only if, for any pair (U, V) of nonempty and open subsets of $\mathcal{H}$, there exist positive integer $n \in \mathbb{N}$ and a complex number $\lambda \in \mathbb{C}$ such that,

$$\lambda T^n(U) \cap V \neq \emptyset.$$

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For more information see [3,4,6,26,27].

Recurrence is a fundamental concept in dynamical systems, initially introduced by Poincaré in [29]. The systematic study of this class of operators traces its roots back to the influential works of Gottschalk and Hedlund [21], as well as Furstenberg [18]. More recently, there has been ongoing research on recurrent operators, as evidenced by studies conducted in [2,5,7,8,9,12,13,25,30,31].
A vector $x \in X$ is considered recurrent for the operator T if there exists a strictly increasing sequence (n_k) of positive integers such that the sequence $(T^{n_k}x)$ converges to x. The set of all recurrent vectors for T is denoted by $\mathrm{Rec}(T)$. An operator T is called a recurrent operator if it has a dense set of recurrent vectors, meaning that $\overline{\mathrm{Rec}(T)} = \mathcal{H}$. H. Furstenberg, in [18], and G. Costakis et al., in [12, Proposition 2.1], have proven that an operator acting on X is recurrent if and only if, for every nonempty open subset U of X, there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \emptyset.$$

We use the notation $REC(\mathcal{H})$ to represent the set of all recurrent operators that act on $\mathcal{H}$.
In recent years, recurrent operators have garnered significant attention from various authors. Notable contributions in this area include the works of Galán, Martínez-Giménez, Oprocha, and Peris in [19], Grivaux, Matheron, and Menet in [22], Yin and Wei in [33], Bonilla, Grosse-Erdmann, López-Martínez, and Peris in [11], Furstenberg and Weiss in [17], Glasner and Maon in [20], and Eisner and Grivaux in [16,15].

In a similar vein, a vector $x \in \mathcal{H}$ is referred to as a "super-recurrent vector" for the operator T if there exists a strictly increasing sequence (n_k) of positive integers and a sequence (λ_k) of complex numbers such that the sequence $(\lambda_k T^{n_k}x)$ converges to x. The set of all super-recurrent vectors for T is denoted by $\mathrm{SRec}(T)$. A super-recurrent operator T is characterized by the property that for every nonempty and open subset U of X, there exist λ and n such that

$$\lambda T^n(U) \cap V \neq \emptyset.$$

The set of all super-recurrent operators that act on $\mathcal{H}$ is denoted by $SREC(\mathcal{H})$. For a more comprehensive understanding, please refer to [5].

In 2012, Costakis and Manoussos proposed a new concept within the domain of linear dynamics. This concept can be considered as a form of "localization" of the notion of hypercyclicity [14]. An operator T on $\mathcal{H}$ is classified as a J-class operator when there exists a vector x such that for every neighborhood U of x and any nonempty open subset V of X, there exists a positive integer n satisfying the condition:

$$T^n(U) \cap V \neq \emptyset.$$

We use the notation $J(\mathcal{H})$ to denote the set of all J-class operators acting on $\mathcal{H}$.

One of the fundamental ideas in the field of dynamics is to achieve a spectral characterization of the closure of the class of operators mentioned earlier. This particular study has already been undertaken for $HC(\mathcal{H})$, $SC(\mathcal{H})$, and $J(\mathcal{H})$. In this paper, we aim to present a characterization for the sets $REC(\mathcal{H})$ and $SREC(\mathcal{H})$. Let us first collect established findings regarding the closures of sets containing hypercyclic, supercyclic, and J-class operators that act on a separable complex Hilbert space $\mathcal{H}$. This reminder primarily aims to compare the above results and our main findings. By doing so, we aim to facilitate accessibility to this information for the reader, making it readily available and easy to comprehend.

Let us introduce some notations and terminology related to our study:

- For $T \in \mathcal{B}(\mathcal{H})$, denote the kernel of T and the range of T by $KerT$ and $RanT$ respectively.
- Let $\mathcal{K}(\mathcal{H})$ be the set of compact operators on H, and note by $\mathcal{A}(\mathcal{H})$ the quotient space $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.
- Denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_w(T)$ and $\sigma_{lre}(T)$ ($\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : \ \lambda - \pi(T)$ is neither left invertible nor right invertible in the Calkin algebra $\mathcal{A}(\mathcal{H})$ $\}$) the spectrum, the point spectrum, the Weyl spectrum and the Wolf spectrum of T respectively. For $\lambda \in \rho_{s-F}(T) := \mathbb{C} \setminus \sigma_{lre}(T)$, $ind(\lambda - T) = dimKer(\lambda - T) - dimKer(\lambda - T)^*, \min\{dimKer(\lambda - T), dimKer(\lambda - T)^*\}$

- Denote $\rho_{s-F}^{(n)}(T) = \{\lambda \in \rho_{s-F}(T) : \ ind(\lambda - T) = n\}$, where $-\infty \leq n \leq +\infty$, $\rho_{s-F}^{(+)}(T) = \{\lambda \in \rho_{s-F}(T) : \ ind(\lambda - T) > 0\}$ and $\rho_{s-F}^{(+)}(T) = \{\lambda \in \rho_{s-F}(T) : \ ind(\lambda - T) < 0\}$.
- Denote by $\sigma_0(T)$ the set of isolated points of $\sigma(T) \setminus \sigma_e(T)$.
- Denote by E^- the norm closure of a set $E \subset \mathcal{B}(\mathcal{H})$.

Let us begin by considering the set of hypercyclic operators that act on $\mathcal{H}$. In this context, Herrero proved the following theorem.

Theorem 1.1 *[24, Theorem 2.1] $HC(\mathcal{H})^-$ is the class of all those operators T in $\mathcal{B}(\mathcal{H})$ satisfying the conditions*

1. *$\sigma_w(T) \cup \mathbb{T}$ is connected,*
2. *$\sigma_0(T) = \emptyset$, and*
3. *$ind(\lambda - T) \geq 0$ for all $\lambda \in \rho_{s-F}(T)$.*

Furthermore, $HC(\mathcal{H})^- + \mathcal{K}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T$ satisfies (1) and (3) $\}$ is a closed subset of $\mathcal{B}(\mathcal{H})$, and,for each $T \in \mathcal{B}(\mathcal{H})$, the distance from T to $HC(\mathcal{H})^- + \mathcal{K}(\mathcal{H})$ is equal to

$$\kappa_{HC}(T) = \min\{\gamma \geq 0 : (1')\sigma_w(T) \cup \mathbb{T} \cup \Delta_\gamma(T) \textit{ is connected, } \\ \textit{and } (3')ind(\lambda - T) \geq 0 \textit{ for all } \lambda \in \rho_{s-F}(T) \setminus \Delta_\gamma(T)\}$$

and

$$distance(T, HC(\mathcal{H})) = \min\{\kappa_{HC}(T), \delta_0(T)\}$$

where

$$\delta_0(T) = \inf\{\|K\| : K \in \mathcal{K}(\mathcal{H}),\ \sigma_0(T - K) = \emptyset\}.$$

In the same paper, Herrero provides a similar characterization of the set of supercyclic operators that act on $\mathcal{H}$. The following theorem demonstrates this characterization.

Theorem 1.2 *[24, Theorem 3.3] $SC(\mathcal{H})^-$ is the class of all those operators T in $\mathcal{B}(\mathcal{H})$ satisfying the conditions*

1. *$\sigma(T) \cup \partial(r\mathbb{D})$ is connected (for some $r \geq 0$),*
2. *$\sigma_w(T) \cup \partial(r\mathbb{D})$ is connected (for some $r \geq 0$),*
3. *either $\sigma_0(T) = \emptyset$, or $\sigma_0(T) = \{\alpha\}$ for some $\alpha \neq 0$. In the second case, T is similar to $S \oplus \alpha I_{\mathbb{C}}$, where $\alpha \notin \sigma(S)$, and*
4. *$ind(\lambda - T) \geq 0$ for all $\lambda \in \rho_{s-F}(T)$.*

Furthermore, $SC(\mathcal{H})^- + \mathcal{K}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T$ satisfies (2) and (4) $\}$ is a closed subset of $\mathcal{B}(\mathcal{H})$, and, for each $T \in \mathcal{B}(\mathcal{H})$, the distance from T to $SC(\mathcal{H})^- + \mathcal{K}(\mathcal{H})$ is equal to

$$\kappa_{SC}(T) = \min\{\gamma \geq 0 : (2')\sigma_w(T) \cup r\mathbb{T} \cup \Delta_\gamma(T) \textit{ is connected for some } ,r \geq 0 \\ \textit{and } (4')ind(\lambda - T) \geq 0 \textit{ for all } \lambda \in \rho_{s-F}(T) \setminus \Delta_\gamma(T)\}$$

and

$$distance(T, SC(\mathcal{H}) = \min\{\max[\gamma, \delta_{0,\gamma}(T)]\}$$

where

$$\delta_{0,\gamma}(T) = \inf\{\|K\| : K \in \mathcal{K}(\mathcal{H}),\ \textit{either } \sigma_0(T - K) \subset \Delta_\gamma(T),\ \textit{or } \sigma_0(T - K) \setminus \Delta_\gamma(T) = \{\alpha\}$$

$$\textit{for some non-zero } \alpha \textit{ such that } |\alpha| \in \{|\lambda| : \ \lambda \in \Delta_\gamma(T)\}, \textit{ and the corresponding Riesz}$$

$$\textit{spectral invariant subspace of } T - K \textit{ is one-dimensional } \}.$$

Finally, let us recall the characterization of the set of J-class operators that act on $\mathcal{H}$, as given by Geng Tian and Bingzhe Hou.

Theorem 1.3 *[32, Theorem 2.7] Let E be the set $\{T \in \mathcal{B}(\mathcal{H}) : [\sigma_{lre}(T) \cup \rho^{(+)}_{s-F}(T)] \cap \partial\mathbb{D} = \emptyset, \mathbb{D}^{-} \cap \rho^{(-)}_{s-F}(T) = \emptyset$, there does not exist a component of $\sigma(T) \setminus \rho^{(0)}_{s-F}(T)$ included in $\mathbb{D}$ and $\sigma_0(T) \cap \mathbb{D} = \emptyset\}$. Then $\overline{J(\mathcal{H})} = E$.*

This paper aims to present a similar characterization of the closure of the set of recurrent and super-recurrent operators that act on $\mathcal{H}$. In the second section, we focus on the set of recurrent operators, while in the third section, we address the set of super-recurrent operators.

2. Norm Closure of $REC(H)$

In the following section, we will present our central result. To lay the groundwork for its proof, we shall introduce a set of essential auxiliary lemmas, sourced from the paper [12]. These lemmas will serve as fundamental building blocks, facilitating the demonstration of our main finding and enhancing the clarity of our arguments.

Lemma 2.1 (Proposition 2.15 [12]) *Let T be an operator on $\mathcal{H}$. If $\sigma_p(T^*) \cap \mathbb{T}^c \neq \emptyset$, then T is not recurrent.*

Lemma 2.2 (Propositions 2.9 and 2.11 [12]) *Let T be an operator on a Banach space X. If $r(T) < 1$ or $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| > 1\}$, then T cannot be recurrent.*

Theorem 2.1 *The class $REC(\mathcal{H})^{-}$ consists of all operators T in $\mathcal{B}(\mathcal{H})$ that satisfy the following conditions:*

1. $[\sigma_{lre}(T) \cup \rho^{(+)}_{s-F}(T)] \cap \mathbb{T} \neq \emptyset$;
2. $\rho^{(-)}_{s-F}(T) \cap \mathbb{T}^c = \emptyset$;
3. *there is no component of $\sigma(T)\setminus\rho^{(0)}_{s-F}(T)$ included in $\mathbb{D}$ or in $\mathbb{C} \setminus \bar{\mathbb{D}}$;*
4. $\sigma_0(T) \subset \mathbb{T}$.

Proof: Let us denote the set of operators satisfying conditions (1), (2), (3), and (4) as E.

Our primary aim is to prove that $REC(\mathcal{H})^{-}$ is a subset of E. Due to the stability properties of semi-Fredholm operators, we find that E is closed in the norm topology. As a result, our task is simplified to proving the inclusion $REC(\mathcal{H}) \subset E$ to conclude.

For any chosen $T \in E^c$, we encounter one of the following cases:

1. there exists a component of $\sigma(T)\setminus\rho^{(0)}_{s-F}(T)$ that is either included in $\mathbb{D}$ or in $\mathbb{C} \setminus \bar{\mathbb{D}}$;
2. there are no components of $\sigma(T)\setminus\rho^{(0)}_{s-F}(T)$ located within the open unit disk $\mathbb{D}$ or in the complement of the closed unit disk $\mathbb{C} \setminus \bar{\mathbb{D}}$. However, there exists a point $\lambda_0 \notin \mathbb{T}$ such that $\mathrm{ind}(\lambda_0 - T) < 0$;
3. there are no component of $\sigma(A)\setminus\rho^{(0)}_{s-F}(A)$ in $\mathbb{D}$ or in $\mathbb{C} \setminus \bar{\mathbb{D}}$ and $\rho^{(-)}_{s-F}(T) \cap \mathbb{T}^c = \emptyset$. However, $[\sigma_{lre}(T) \cup \rho^{(+)}_{s-F}(T)] \cap \mathbb{T} = \emptyset$;
4. there are no component of $\sigma(T)\setminus\rho^{(0)}_{s-F}(T)$ contained in $\mathbb{D}$ or in $\mathbb{C} \setminus \bar{\mathbb{D}}$, $\rho^{(-)}_{s-F}(T) \cap \mathbb{T}^c = \emptyset$ and $[\sigma_{lre}(T) \cup \rho^{(+)}_{s-F}(T)] \cap \mathbb{T} \neq \emptyset$. However, $\sigma_0(T) \not\subset \mathbb{T}$.

For (1), we have two possibilities:

- There exists a component σ of $\sigma(T)$ that lies within the open unit disk $\mathbb{D}$ or in the complement of the closed unit disk $\mathbb{C} \setminus \bar{\mathbb{D}}$.

- There is a point $\lambda_0 \in \mathbb{D}$ such that the kernel dimension of the adjoint operator $(T-\lambda_0)^*$ is non-zero.

For the first case, applying Riesz's decomposition theorem, we have the following outcome:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix} = \begin{pmatrix} T_1 & * \\ 0 & \widetilde{T_2} \end{pmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix} \sim \begin{pmatrix} T_1 & 0 \\ 0 & \widetilde{T_2} \end{pmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix},$$

where the symbol $\sim$ is used to indicate the similarity between operators, $\sigma(T_1) = \sigma$ and $\sigma(T_2) = \sigma(\widetilde{T_2}) = \sigma(T)\backslash\sigma(T_1)$.

If $\sigma \subset \mathbb{D}$, then since $r(T_1) < 1$, T is not recurrent. If $\sigma \subset \mathbb{C} \setminus \bar{\mathbb{D}}$, then $\sigma(T_1) \subset \{\lambda \in C : \ |\lambda| > 1\}$, in this case also $\widetilde{T_1}$ cannot be recurrent.

In the case where $\lambda_0 \in \mathbb{D}$ and $\mathrm{dimKer}(A-\lambda_0)^* > 0$, Lemma 2.1 demonstrates that the operator T is not recurrent.

Lemma 2.1 establishes that for both (2) and (4), the operator T is non-recurrent.

For (3), either there exists a point $\lambda_0 \in \mathbb{T}^c$ such that $\mathrm{dimKer}(T-\lambda_0)^* > 0$ or $\sigma(T) \cap \mathbb{T} = \emptyset$. Based on Lemma 2.1 and Lemma 2.2, it can be concluded that the operator T is non-recurrent.

To sum up, we have shown that T does not belong to the set of recurrent operators. Consequently, it follows that the set of recurrent operators, $REC(\mathcal{H})$, is a subset of E.

Secondly, we aim to demonstrate that E is a subset of $REC(\mathcal{H})$. To be more precise, for any $T \in E$ and given an arbitrary $\varepsilon > 0$, we can find an operator K such that $\|K\| < \varepsilon$ and $T + K$ belongs to $REC(\mathcal{H})$.

Pick any element T from the set E. Using a common approximation method, we can discover an operator denoted as $S \in E$ such that $\|T - S\| < \delta_1$, and the spectrum of S ($\sigma(S)$) only consists of finite components, $\rho_{s-F}(S) \subset \rho_{s-F}(T)$ and $\mathrm{ind}(\lambda - S) = \mathrm{ind}(\lambda - T)$ for all $\lambda \in \rho_{s-F}(S)$, $\sigma_{lre}(S)$ is the closure of an analytic Cauchy domain Ω, $\sigma_0(S)$ has only finite points and $\sigma_0(S) \subset \mathbb{T}$. Consider σ_1 as the union of components σ belonging to $\sigma(S)$ that fulfill the following conditions simultaneously: $\sigma \cap \mathbb{T} \neq \emptyset$, $\sigma \cap \rho_{s-F}^{(-)}(S) \neq \emptyset$ and $\sigma \cap \rho_{s-F}^{(+)}(S) \neq \emptyset$. As per Riesz's decomposition theorem, we have the following conclusion:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{matrix} H(S,\sigma_1) \\ H(S,\sigma(S)\setminus\sigma_1) \end{matrix}.$$

In reference to [1] (also see [32, Theorem 2.5]), there exists an operator denoted as $T_1 \in B(H(S,\sigma_1))$ with the property that $\|T_1\| < \delta_2$, where δ_2 is a given value, such that that $S_1 + T_1$ is a simple model, $T_1 + A_1 \in E$, $\sigma_{lre}(S_1 + T_1)$ is the closure of an analytic Cauchy domain, $\sigma(S_1 + A_1) \cap \sigma(S_2) = \emptyset$ and for any component σ of $\sigma(S_1 + T_1)$ satisfying $\sigma \cap \mathbb{T} \neq \emptyset$ and $\rho_{s-F}^{(+)}(S_1+T_1) \cap \sigma \neq \emptyset$, $\rho_{s-F}^{(-)}(S_1+T_1) \cap \sigma = \emptyset$. Let

$$A = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} H(S,\sigma_1) \\ H(S,\sigma(S)\setminus\sigma_1) \end{matrix}.$$

The straightforward observation is that for any connected component σ of $\sigma(S+T)$, which satisfies $\sigma \cap \mathbb{T} \neq \emptyset$, the following conditions hold: $\sigma \cap \mathbb{T} \neq \emptyset$, we have $\rho_{s-F}^{(-)}(S+T) \cap \sigma = \emptyset$ and $\rho_{s-F}^{(+)}(S+T) \cap \sigma \neq \emptyset$.

Consider σ_2 as the union of components of $\sigma(S+T)$ that intersect $\mathbb{T}$. It follows that $\sigma(S+T) \setminus \sigma_2$ is contained within $\mathbb{T}$. By applying Riesz's decomposition theorem, we obtain the following result:

$$S + T = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{matrix} H(S+T,\sigma_2) \\ H(S+T,\sigma(S+T)\setminus\sigma_2) \end{matrix}.$$

Take note that for the operator B_1, there are three properties:

- $\sigma_w(B_1) \cup \mathbb{T}$ is a connected set,
- $\sigma_0(B_1) = \emptyset$,
- $\mathrm{ind}(\lambda - B_1) \geq 0$ for $\lambda \in \rho_{s-F}(B_1)$.

Hence, as per Theorem 2.1 in [23], there exists an operator C_1 such that $|C_1| < \delta_3$, and $B_1 + C_1$ becomes a mixing operator.

For the operator B_2, the following properties hold:

- $\sigma_W(B_2) \cup \mathbb{T}$ is a connected set,
- $\text{ind}(\lambda - B_2) \leq 0$ for $\lambda \in \rho_{s-F}(B_2)$.

As a result, there exists an operator C_2 such that $|C_2| < \delta_4$, and $B_2 + C_2$ becomes a lower triangular operator. The diagonal of this lower triangular operator is represented by some $\alpha \in \mathbb{T}$. Consequently, we have the relationship:

$$H = \overline{span\{\ker(B_2 + C_2 - \alpha)^k : \ k \in \mathbb{N}\}}.$$

The stability of the semi-Fredholm index implies that:

$$H = \overline{span\{\ker(B_2 + C_2 - \alpha_k) : \ k \in \mathbb{N}\}},$$

where $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of distinct points in $\mathbb{T}$ that are in a neighborhood of α. This observation further indicates that $B_2 + C_2$ is recurrent.

Let

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{matrix} H(S+T, \sigma_2) \\ H(S+T, \sigma(S+T) \setminus \sigma_2) \end{matrix}.$$

Then $S + T + C$ is a recurrent operator. Thus, one can obtain the result if $\delta_1, \delta_2, \delta_3$ and δ_4 are chosen small enough. □

Now let $m_e(R)$ be the minimal essential modulus of an operator R in $\mathcal{B}(\mathcal{H})$ defined by $m_e(R) = \min\{r \in \sigma_e([R^*R]^{1/2})\}$. For $\gamma \geq 0$, we denote

$$\Delta_\gamma(R) = \{\lambda \in \mathbb{C} : \ m_e(\lambda - R) \leq \gamma\}.$$

Corollary 2.1 (Distance to the set of recurrent operators) *We have, $REC(\mathcal{H})^- + \mathcal{K}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \ T$ satisfies $(1), (2)$ and $(3)\}$ is a closed subset of $\mathcal{B}(\mathcal{H})$, and, for each R in $\mathcal{B}(\mathcal{H})$, the distance from R to $REC(\mathcal{H})^- + \mathcal{K}(\mathcal{H})$ is equal to*

$$\kappa_{REC}(R) = min\{\gamma \geq 0 : \ (1') \ (\sigma_{lre}(R) \cup \rho^{(+)}_{s-F}(R) \cup \Delta_\gamma(R)) \cap \mathbb{T} \neq \emptyset,$$

$$(2') \ (\rho^{(-)}_{s-F}(R) \backslash \Delta_\gamma(R)) \cap \mathbb{T}^c = \emptyset,$$

$$\text{and } (3') \text{ there is no component of } (\sigma(R) \backslash \rho^{(0)}_{s-F}(R)) \cup \Delta_\gamma(R) \text{ included in } \mathbb{D} \text{ or in } \mathbb{C} \setminus \bar{\mathbb{D}}\}$$

and

$$distance[R, REC(\mathcal{H})] = max\{\kappa_{REC}(R), \delta_1(R)\},$$

where $\delta_1(R) = \inf\{\|K\| : \ K \in \mathcal{K}(\mathcal{H}), \ \sigma_0(R - K) \subset \mathbb{T}\}$.

3. Norm closure of $SREC(\mathcal{H})$

Theorem 3.1 *The set $SREC(\mathcal{H})^-$ consists of all operators $T \in \mathcal{B}(\mathcal{H})$ that satisfy the following conditions:*

1. $[\sigma_{lre}(T) \cup \rho^{(+)}_{s-F}(T)] \cap (r\mathbb{T}) \neq \emptyset$;
2. $\rho^{(-)}_{s-F}(T) \cap (r\mathbb{T})^c = \emptyset$;
3. *there is no component of* $\sigma(T) \backslash \rho^{(0)}_{s-F}(T)$ *included in* $(r\mathbb{D})$ *or in* $\mathbb{C} \setminus (r\bar{\mathbb{D}})$;
4. $\sigma_0(T) \subset (r\mathbb{T})$.

Proof: The necessary implication can be established by employing a similar reasoning as demonstrated in Theorem 2.1.

Let us now establish the sufficient implication. Assuming that conditions (1), (2), (3), and (4) are satisfied for an operator T. Then $(1/r)T \in \mathrm{REC}(H)^-$, which implies that $T \in \mathrm{SREC}(H)^-$. □

Corollary 3.1 (Distance to the set of super-recurrent operators) *We have, $SREC(\mathcal{H})^- + \mathcal{K}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) :\ T$ satisfies $(1), (2)$ and $(3)\}$ is a closed subset of $L(H)$, and, for each R in $\mathcal{B}(\mathcal{H})$, the distance from R to $SREC(\mathcal{H})^- + \mathcal{K}(\mathcal{H})$ is equal to*

$$\kappa_{SREC}(R) = min\{\gamma \geq 0 :\ (1')\ (\sigma_{lre}(R) \cup \rho_{s-F}^{(+)} \cup \Delta_\gamma(R)) \cap r\mathbb{T} \neq \emptyset,\ \textit{for some } r \geq 0,$$

$$(2')\ (\rho_{s-F}^{(-)}(R) \backslash \Delta_\gamma(R)) \cap (r\mathbb{T})^c = \emptyset,\ \textit{for some } r \geq 0,$$

and $(3')$ there is no component of $(\sigma(T)\backslash\rho_{s-F}^{(0)}(T))\ \cup\ \Delta_\gamma(R)$ included in $r\mathbb{D}$ or in $\mathbb{C} \ \backslash \ r\bar{\mathbb{D}}\}$, for some $r \geq 0$, and

$$distance[R, SREC(\mathcal{H})] = max\{\kappa_{SREC}(R), \delta_2(R)\},$$

where $\delta_2(R) = \inf\{\|K\| :\ K \in \mathcal{K}(\mathcal{H}),\ \sigma_0(R-K) \subset r\mathbb{T},\ \textit{for some } r \geq 0\}$.

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