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Toeplitz Determinants for a Subclass of Analytic Functions Involving Touchard Polynomials

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ABSTRACT: In this paper, we introduce a new subclass of univalent functions that generalizes existing subclasses of univalent functions. By employing subordination principles, we derive initial Taylor–Maclaurin coefficient estimates for functions in this subclass. Additionally, we establish bounds for the Fekete-Szegö functional and Toeplitz determinants. To further strengthen the applicability of our findings, we incorporate Touchard polynomials, demonstrating their role in Geometric Function Theory (GFT). Our results unify and generalize several known subclasses, offering potential applications of Touchard polynomials in the field of GFT.

Key Words: Univalent functions, starlike functions, convex functions, Fekete-Szegö estimate, Toeplitz determinants, Touchard polynomials.

Contents

1. Introduction

Let $\mathcal{A}$ represent the class of functions $\mathfrak{f}$ of the form:

$$\mathfrak{f}(\zeta) = \zeta + \sum_{j=2}^{\infty} a_j \zeta^j, \tag{1.1}$$

defined on the unit disk $\Delta = \{\zeta : \zeta \in \mathbb{C}, |\zeta| < 1\}$, which are analytic and satisfying the normalization conditions $\mathfrak{f}(0) = 0$ and $\mathfrak{f}'(0) = 1$. Additionally, let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ that are univalent in Δ. It is well known that for two analytic functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ defined in Δ, the function $\mathfrak{f}_1$ is subordinate to $\mathfrak{f}_2$, written as $\mathfrak{f}_1 \prec \mathfrak{f}_2$, if there exists an analytic function ϕ such that: $\phi(0) = 0$ and $|\phi(\zeta)| < 1$ for all $\zeta \in \Delta$, $\mathfrak{f}_1(\zeta) = \mathfrak{f}_2(\phi(\zeta))$. In the specific case when $\mathfrak{f}_2$ is univalent in Δ, this subordination relationship reduces to the following equivalent conditions:

$$\mathfrak{f}_1 \prec \mathfrak{f}_2, \qquad (\zeta \in \Delta) \quad \Leftrightarrow \quad \mathfrak{f}_1(0) = \mathfrak{f}_2(0) \quad \text{and} \quad \mathfrak{f}_1(\Delta) \subset \mathfrak{f}_2(\Delta).$$

The class of starlike functions $\mathcal{S}^*$ and the class of convex functions $\mathcal{C}$ are among the most well-studied subclasses of $\mathcal{S}$. These subclasses are defined as follows:

$$\mathcal{S}^* = \left\{ \mathfrak{f} \in \mathcal{S} : \mathrm{Re}\left(\frac{\zeta \mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)} \right) > 0, \quad \zeta \in \Delta \right\}$$

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and

$$\mathcal{C}=\left\{\mathfrak{f}\in\mathcal{S}:\operatorname{Re}\left(1+\frac{\zeta\mathfrak{f}''(\zeta)}{\mathfrak{f}'(\zeta)}\right)>0,\quad \zeta\in\Delta\right\}.$$

Consider $\phi(\zeta)$, an analytic function defined on Δ with a positive real part, such that $\phi(0)=1$, $\phi'(0)>0$ and it maps the unit disk Δ onto a region that is starlike with respect to 1 and symmetric about the real axis of the form

$$\phi(\zeta)=1+B_1\zeta+B_2\zeta^2+B_3\zeta^3+\cdots,\qquad B_1>0. \tag{1.2}$$

Ravichandran et al. [32] introduced a general class associated with starlike functions and convex functions of complex order $\tau\neq 0$, defined as follows:

$$\mathcal{S}^*_\tau(\phi)=\left\{\mathfrak{f}\in\mathcal{S}:1+\frac{1}{\tau}\left(\frac{\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)}-1\right)\prec\phi(\zeta),\quad \zeta\in\Delta\right\} \tag{1.3}$$

and

$$\mathcal{C}_\tau(\phi)=\left\{\mathfrak{f}\in\mathcal{S}:1+\frac{1}{\tau}\frac{\zeta\mathfrak{f}''(\zeta)}{\mathfrak{f}'(\zeta)}\prec\phi(\zeta),\quad \zeta\in\Delta\right\}. \tag{1.4}$$

It is noteworthy that $\mathcal{S}^*_1(\phi)\equiv\mathcal{S}^*(\phi)$ and $\mathcal{C}_1(\phi)\equiv\mathcal{C}(\phi)$, which corresponds to the general classes in terms of subordination defined by Ma and Minda [21].

A function $\mathfrak{f}\in\mathcal{S}$ is said to belong to the class $\mathcal{P}_\mu$ if it satisfies the following criteria:

$$\operatorname{Re}\left(\frac{\zeta\mathfrak{f}'(\zeta)+\mu\zeta^2\mathfrak{f}''(\zeta)}{(1-\mu)\mathfrak{f}(\zeta)+\mu\zeta\mathfrak{f}'(\zeta)}\right)>0,\quad 0\le\mu\le 1,\quad \zeta\in\Delta.$$

This class was introduced by Altıntaş [5] and further investigated in [6,18,26,31].

In the theory of univalent functions, significant attention has been given to estimating the bounds of Hankel matrices and Toeplitz matrices. They play a crucial role in various branches of mathematics and have numerous applications (see [42] for more details). Hankel determinants and Toeplitz determinants are closely related; while Toeplitz matrices have constant entries along the main diagonal, Hankel matrices are characterized by constant entries along the reverse diagonal. Thomas and Halim [38] introduced the symmetric Toeplitz determinant $\mathcal{T}_q(j)$ for $f\in\mathcal{A}$, defined as follows:

$$\mathcal{T}_q(j)=\begin{vmatrix} a_j & a_{j+1} & \cdots & a_{j+q-1}\\ a_{j+1} & a_j & \cdots & a_{j+q-2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{j+q-1} & a_{j+q-2} & \cdots & a_j\end{vmatrix},$$

where $j\ge 1$, $q\ge 1$, and $a_1=1$. Specifically,

$$\mathcal{T}_2(2)=\begin{vmatrix} a_2 & a_3\\ a_3 & a_2\end{vmatrix}\quad\text{and}\quad \mathcal{T}_3(1)=\begin{vmatrix} 1 & a_2 & a_3\\ a_2 & 1 & a_2\\ a_3 & a_2 & 1\end{vmatrix}. \tag{1.5}$$

Recently, researchers have been actively deriving estimates for the Toeplitz determinant $|\mathcal{T}_q(j)|$ for functions belonging to various subclasses of univalent functions (see, for example, [1,2,3,11,15,34,19,20, 25,27,30,31,34,35,36,40,41,43,44].

2. Definitions and Preliminary Results

Let $\mathcal{P}$ denote the class of Carathéodory functions p such that

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots=1+\sum_{j=1}^{\infty}c_jz^j, \tag{2.1}$$

which are analytic and univalent in Δ such that $\operatorname{Re}p(z)>0$ for all $z\in\Delta$. To establish the desired bounds in our study, we require the following lemmas.

Lemma 2.1 *[10] Let $p \in \mathcal{P}$ be of the form (2.1). Then,*

$$|c_j| \leq 2$$

for all $j \in \mathbb{N} := \{1, 2, 3, \cdots\}$.

Lemma 2.2 *[12] Let $p \in \mathcal{P}$ be in the form of (2.1) and let β be any complex number. Then,*

$$|c_2 - \beta c_1^2| \leq 2\max\{1, |2\beta - 1|\}.$$

This result is sharp for the functions

$$p_1(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \quad \text{and} \quad p_2(\zeta) = \frac{1+\zeta}{1-\zeta}.$$

Lemma 2.3 *[21](Also see [17]) Let $p \in \mathcal{P}$ Let $p \in \mathcal{P}$ be in the form of (2.1) and let α be any real number. Then,*

$$|c_2 - \alpha c_1^2| \leq \begin{cases} -4\alpha + 2, & \alpha \leq 0, \\ 2, & 0 \leq \alpha \leq 1, \\ 4\alpha - 2, & \alpha \geq 1. \end{cases}$$

Lemma 2.4 *[16]be in the form of (2.1). Then*

$$\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{1}{2}|c_1|^2.$$

Using the unified technique of subordination introduced by Ma and Minda [21] and motivated by the works of Caglar et al. [9] and Pei et al. [27], we define a new subclass of $\mathcal{S}$, which establishes a connection between $\mathcal{S}^*_\tau(\phi)$ and $\mathcal{C}_\tau(\phi)$, as defined in (1.3) and (1.4).

Definition 2.1 *For $0 \leq \mu \leq \gamma \leq 1$, a function $\mathfrak{f} \in \mathcal{S}$ of the form (1.1) belongs to the class $\mathcal{NM}^\gamma_\mu(\tau;\phi)$, if the following conditions are satisfied:*

$$1 + \frac{1}{\tau}\left(\frac{\zeta\mathfrak{f}'(\zeta) + \gamma\zeta^2\mathfrak{f}''(\zeta)}{(1-\mu)\mathfrak{f}(\zeta) + \mu\zeta\mathfrak{f}'(\zeta)} - 1\right) \prec \phi(\zeta),$$

where ϕ is given by (1.2), $\tau \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\zeta \in \Delta$.

Remark 2.1 *For specific choice of the parameters, in the above definition we have the following subclasses that was considered in the earlier investigation:*

1. *$\mathcal{NM}^0_0(\tau;\phi) \equiv \mathcal{S}^*(\tau;\phi)$ consists of functions $\mathfrak{f} \in \mathcal{S}$ of the form (1.1), if*

$$1 + \frac{1}{\tau}\left(\frac{\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)} - 1\right) \prec \phi(\zeta).$$

2. *$\mathcal{NM}^1_1(\tau;\phi) \equiv \mathcal{C}(\tau;\phi)$ consists of functions $\mathfrak{f} \in \mathcal{S}$ of the form (1.1), if*

$$1 + \frac{1}{\tau}\left(\frac{\zeta\mathfrak{f}''(\zeta)}{\mathfrak{f}'(\zeta)}\right) \prec \phi(\zeta).$$

The above two classes were investigated by Ravichandran et al. [32]. We note that, $\mathcal{NM}^0_0(1;\phi) \equiv \mathcal{S}^*(\phi)$ and $\mathcal{NM}^1_1(1;\phi) \equiv \mathcal{C}(\phi)$ were introduced by Ma and Minda [21].

3. *$\mathcal{NM}^1_0(\tau;\phi) \equiv \mathcal{NR}(\tau;\phi)$ consists of functions $\mathfrak{f} \in \mathcal{S}$ of the form (1.1), if*

$$1 + \frac{1}{\tau}\left(\frac{\zeta\mathfrak{f}'(\zeta) + \zeta^2\mathfrak{f}''(\zeta)}{\mathfrak{f}(\zeta)}\right) \prec \phi(\zeta).$$

4. For $\phi(\zeta) = \dfrac{1+(1-2\alpha)\zeta}{1-\zeta}$ *;* $0 \le \alpha < 1$*, we have the following:*

γ	μ	τ	*Class*	*Author*
0	0	τ	$\mathcal{S}^*_\alpha(\tau)$	*Frasin [13]*
1	1	τ	$\mathcal{C}_\alpha(\tau)$	

5. For $\phi(\zeta) = \dfrac{1+\zeta}{1-\zeta}$*, we have the following:*

γ	μ	τ	*Class*	*Authors*
0	0	τ	$\mathcal{S}^*(\tau)$	*Nasr and Aouf [24]*
1	1	τ	$\mathcal{C}(\tau)$	*Nasr and Aouf [23]*
0	0	$1-\alpha$	$\mathcal{S}^*(\alpha)$; $0 \le \alpha < 1$	*Robertson [33]*
1	1	$1-\alpha$	$\mathcal{C}(\alpha)$; $0 \le \alpha < 1$	
0	0	$\tau e^{i\theta}\cos\theta$	$\mathcal{S}^*_\theta(\tau)$; $\lvert\theta\rvert < \frac{\pi}{2}$	*Al-Oboudi and Haidan [4]*
1	1	$\tau e^{i\theta}\cos\theta$	$\mathcal{C}_\theta(\tau)$; $\lvert\theta\rvert < \frac{\pi}{2}$	

3. Coefficient Estimates

In this section, we derive the initial coefficient estimates for class of functions belonging to $\mathcal{NM}^\gamma_\mu(\tau;\phi)$.

Theorem 3.1 *If* $\mathfrak{f} \in \mathcal{NM}^\gamma_\mu(\tau;\phi)$ *is of the form* (1.1)*, then*

$$|a_2| \le \frac{|\tau| B_1}{2\gamma - \mu + 1}$$

and

$$|a_3| \le \frac{|\tau|\left(B_1 + \left|B_1 - B_2 - \dfrac{B_1^2(\mu+1)\tau}{2\gamma-\mu+1}\right|\right)}{2(3\gamma-\mu+1)}.$$

Proof: Consider $\mathfrak{f} \in \mathcal{NM}^\gamma_\mu(\tau;\phi)$. Then there exists a Schwarz function $u \in \mathcal{A}$ such that

$$u(\zeta) = \frac{\ell(\zeta)-1}{\ell(\zeta)+1} = \frac{c_1}{2}\zeta + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\zeta^2 + \cdots \quad (\zeta \in \Delta),$$

where $\ell \in \mathcal{P}$ given by

$$\ell(\zeta) = \frac{1+u(\zeta)}{1-u(\zeta)} = 1 + \sum_{n=1}^{\infty} c_n \zeta^n.$$

By Definition 2.1, we have

$$1 + \frac{1}{\tau}\left[\frac{\zeta\mathfrak{f}'(\zeta) + \gamma\zeta^2\mathfrak{f}''(\zeta)}{(1-\mu)\mathfrak{f}(\zeta) + \mu\zeta\mathfrak{f}'(\zeta)} - 1\right] = \phi(u(\zeta)). \tag{3.1}$$

In the view of (1.2), it can be computed that,

$$\phi(u(\zeta)) = 1 + \frac{1}{2}B_1 c_1 \zeta + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]\zeta^2 + \cdots. \tag{3.2}$$

Moreover

$$\begin{aligned}1 + \frac{1}{\tau}\left[\frac{\zeta\mathfrak{f}'(\zeta) + \gamma\zeta^2\mathfrak{f}''(\zeta)}{(1-\mu)\mathfrak{f}(\zeta) + \mu\zeta\mathfrak{f}'(\zeta)} - 1\right] = 1 + &\frac{(2\gamma-\mu+1)}{\tau}a_2\zeta \\ &+ \frac{2(3\gamma-\mu+1)a_3 - (2\gamma-\mu+1)(1+\mu)a_2^2}{\tau}\zeta^2 + \cdots\end{aligned} \tag{3.3}$$

Comparing the corresponding coefficients in (3.2) and (3.3), we have

$$\frac{(2\gamma-\mu+1)}{\tau}a_2 = \frac{1}{2}B_1c_1, \tag{3.4}$$

$$\frac{2\,(3\gamma-\mu+1)\,a_3-(2\gamma-\mu+1)(1+\mu)a_2^2}{\tau} = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2. \tag{3.5}$$

From (3.4), we can compute

$$a_2 = \frac{\tau B_1 c_1}{2\,(2\gamma-\mu+1)}. \tag{3.6}$$

Taking modulus and applying Lemma 2.1 to (3.6), we get

$$|a_2| \le \frac{|\tau|B_1}{2\gamma-\mu+1}.$$

Now, inserting (3.4) to (3.5), and after simplification we have

$$a_3 = \frac{\tau\left[2B_1c_2-\left(B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right)c_1^2\right]}{8(3\gamma-\mu+1)}. \tag{3.7}$$

Taking modulus and applying Lemma 2.1 to the above equation, we get

$$|a_3| \le \frac{|\tau|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{2(3\gamma-\mu+1)}. \tag{3.8}$$

This completes the proof of this theorem. □

Corollary 3.1 *If $\mathfrak{f}\in\mathcal{S}^*(\tau;\phi)$ is of the form (1.1), then*

$$|a_2| \le |\tau|B_1 \quad \text{and} \quad |a_3| \le \frac{|\tau|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{2}.$$

Corollary 3.2 *If $\mathfrak{f}\in\mathcal{C}(\tau;\phi)$ is of the form (1.1), then*

$$|a_2| \le \frac{|\tau|B_1}{2} \quad \text{and} \quad |a_3| \le \frac{|\tau|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{6}.$$

Corollary 3.3 *If $\mathfrak{f}\in\mathcal{NR}(\tau;\phi)$ is of the form (1.1), then*

$$|a_2| \le \frac{|\tau|B_1}{3} \quad \text{and} \quad |a_3| \le \frac{|\tau|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2\tau}{3}\right|\right)}{8}.$$

4. Fekete–Szegö Estimates

Now, we compute the estimate of the famous Fekete–Szegö functional for the class $\mathcal{NM}_\mu^\gamma(\tau;\phi)$.

Theorem 4.1 *If $\mathfrak{f}\in\mathcal{NM}_\mu^\gamma(\tau;\phi)$ and is of the form (1.1) and β is any complex number, then*

$$|a_3-\beta a_2^2| \le \frac{2B_1|\tau|}{3\gamma-\mu+1}\max\left\{1,\left|\frac{2\beta(3\gamma-\mu+1)-(\mu+1)(2\gamma-\mu+1)}{(2\gamma-\mu+1)^2}B_1\tau-\frac{B_2}{B_1}\right|\right\}.$$

Proof: For $\beta \in \mathbb{C}$ and from (3.4) and (3.5), we can write

$$a_3 - \beta a_2^2 = \frac{B_1\tau}{4(3\gamma-\mu+1)}\left(c_2 - \kappa c_1^2\right), \tag{4.1}$$

where

$$\kappa = \frac{B_1\tau\beta\,(3\gamma-\mu+1)}{(2\gamma-\mu+1)^2} - \frac{B_1^2\,(\mu+1)\,\tau^2 - B_1\tau\,(2\gamma-\mu+1) + B_2\tau\,(2\gamma-\mu+1)}{2B_1\tau\,(2\gamma-\mu+1)}.$$

Application of Lemma 2.2 to (4.1), we obtain:

$$|a_3 - \beta a_2^2| \leq \frac{2B_1|\tau|}{3\gamma-\mu+1}\max\{1, |2\kappa-1|\}.$$

The proof of Theorem 4.1 is complete. □

Corollary 4.1 *If $\mathfrak{f} \in \mathcal{S}^*(\tau;\phi)$ is of the form (1.1) and β is any complex number, then*

$$|a_3 - \beta a_2^2| \leq 2B_1|\tau|\max\left\{1, \left|(2\beta-1)B_1\tau - \frac{B_2}{B_1}\right|\right\}.$$

Corollary 4.2 *If $\mathfrak{f} \in \mathcal{C}(\tau;\phi)$ is of the form (1.1) and β is any complex number, then*

$$|a_3 - \beta a_2^2| \leq \frac{2B_1|\tau|}{3}\max\left\{1, \left|\left(\frac{3\beta-2}{2}\right)B_1\tau - \frac{B_2}{B_1}\right|\right\}.$$

Corollary 4.3 *If $\mathfrak{f} \in \mathcal{NR}(\tau;\phi)$ is of the form (1.1) and β is any complex number, then*

$$|a_3 - \beta a_2^2| \leq \frac{B_1|\tau|}{2}\max\left\{1, \left|\left(\frac{8\beta-3}{9}\right)B_1\tau - \frac{B_2}{B_1}\right|\right\}.$$

Theorem 4.2 *If $\mathfrak{f} \in \mathcal{NM}_\mu^\gamma(\tau;\phi)$ and is of the form (1.1) and α is any real number, then*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \dfrac{B_1|\tau||1-2\nu|}{2(3\gamma-\mu+1)} & \text{if } \alpha \leq \dfrac{(2\gamma-\mu+1)^2\,(-B_1+\Theta)}{2B_1^2\tau(3\gamma-\mu+1)} \\ \dfrac{B_1|\tau|}{2(3\gamma-\mu+1)} & \text{if } \dfrac{(2\gamma-\mu+1)^2\,(-B_1+\Theta)}{2B_1^2\tau(3\gamma-\mu+1)} \leq \alpha \leq \dfrac{(2\gamma-\mu+1)^2\,(B_1+\Theta)}{2B_1^2\tau(3\gamma-\mu+1)} \\ \dfrac{B_1|\tau||2\nu-1|}{2(3\gamma-\mu+1)} & \text{if } \alpha \geq \dfrac{(2\gamma-\mu+1)^2\,(B_1+\Theta)}{2B_1^2\tau(3\gamma-\mu+1)} \end{cases},$$

where

$$\Theta = B_2 + \frac{B_1^2(\mu+1)\tau}{2\gamma-\mu+1} \quad \text{and}$$
$$\nu = \frac{B_1\tau\alpha(3\gamma-\mu+1)}{(2\gamma-\mu+1)^2} - \frac{\dfrac{B_1^2\tau(\mu+1)}{2\gamma-\mu+1} - B_1 + B_2}{2B_1}. \tag{4.2}$$

Proof: For $\alpha \in \mathbb{R}$ and from (3.4) and (3.5), we can write

$$a_3 - \alpha a_2^2 = \frac{B_1\tau}{4(3\gamma-\mu+1)}\left(c_2 - \nu c_1^2\right), \tag{4.3}$$

where ν is stated as in (4.2).
Using Lemma 2.3 in (4.3), we get the desired estimate. This completes the proof of the Theorem 4.2. □

Corollary 4.4 *If $\mathfrak{f} \in \mathcal{S}^*(\tau;\phi)$ and is of the form (1.1) and α is any real number, then*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \dfrac{B_1|\tau||1-2\nu_1|}{2} & \text{if} \quad \alpha \leq \dfrac{-B_1+B_2+B_1^2\tau}{2B_1^2\tau} \\ \dfrac{B_1|\tau|}{2} & \text{if} \quad \dfrac{-B_1+B_2+B_1^2\tau}{2B_1^2\tau} \leq \alpha \leq \dfrac{B_1+B_2+B_1^2\tau}{2B_1^2\tau} \\ \dfrac{B_1|\tau||2\nu_1-1|}{2} & \text{if} \quad \alpha \geq \dfrac{B_1+B_2+B_1^2\tau}{2B_1^2\tau} \end{cases},$$

where $\nu_1 = \dfrac{B_1^2\tau(2\alpha-1)+B_1-B_2}{2B_1}$.

Corollary 4.5 *If $\mathfrak{f} \in \mathcal{C}(\tau;\phi)$ and is of the form (1.1) and α is any real number, then*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \dfrac{B_1|\tau||1-2\nu_2|}{6} & \text{if} \quad \alpha \leq \dfrac{2(B_1^2\tau-B_1+B_2)}{3B_1^2\tau} \\ \dfrac{B_1|\tau|}{6} & \text{if} \quad \dfrac{2(B_1^2\tau-B_1+B_2)}{3B_1^2\tau} \leq \alpha \leq \dfrac{2(B_1^2\tau+B_1+B_2)}{3B_1^2\tau} \\ \dfrac{B_1|\tau||2\nu_2-1|}{6} & \text{if} \quad \alpha \geq \dfrac{2(B_1^2\tau+B_1+B_2)}{3B_1^2\tau} \end{cases},$$

where $\nu_2 = \dfrac{B_1^2\tau(3\alpha-2)+2(B_1-B_2)}{4B_1}$.

Corollary 4.6 *If $\mathfrak{f} \in \mathcal{NR}(\tau;\phi)$ and is of the form (1.1) and α is any real number, then*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \dfrac{B_1|\tau||1-2\nu_3|}{8} & \text{if} \quad \alpha \leq \dfrac{3(B_1^2\tau-3B_1+3B_2)}{8B_1^2\tau} \\ \dfrac{B_1|\tau|}{8} & \text{if} \quad \dfrac{3(B_1^2\tau-3B_1+3B_2)}{8B_1^2\tau} \leq \alpha \leq \dfrac{3(B_1^2\tau+3B_1+3B_2)}{8B_1^2\tau} \\ \dfrac{B_1|\tau||2\nu_3-1|}{8} & \text{if} \quad \alpha \geq \dfrac{3(B_1^2\tau+3B_1+3B_2)}{8B_1^2\tau} \end{cases},$$

where $\nu_3 = \dfrac{B_1^2\tau(8\alpha-3)+9(B_1-B_2)}{18B_1}$.

Theorem 4.3 *If $\mathfrak{f} \in \mathcal{NM}_\mu^\gamma(\tau;\phi)$ and is of the form (1.1), then*

$$\left| a_3 - \frac{(2\gamma-\mu+1)(1+\mu)}{2(3\gamma-\mu+1)} a_2^2 \right| \leq \begin{cases} \dfrac{|\tau|B_1}{2(3\gamma-\mu+1)} & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{|\tau B_2|}{2(3\gamma-\mu+1)} & \text{if} \quad |B_2| \geq B_1. \end{cases}$$

Proof: From (3.5), we have

$$\frac{2\,(3\gamma-\mu+1)}{\tau} a_3 - \frac{(2\gamma-\mu+1)(1+\mu)}{\tau} a_2^2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2.$$

Rewriting, we get

$$a_3 - \frac{(2\gamma-\mu+1)(1+\mu)}{2(3\gamma-\mu+1)} a_2^2 = \frac{\tau B_1}{4(3\gamma-\mu+1)} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\tau}{8(3\gamma-\mu+1)} B_2 c_1^2.$$

Applying Lemma 2.4 to the above equation and simplifying, we obtain

$$\left|a_3 - \frac{(2\gamma-\mu+1)(1+\mu)}{2(3\gamma-\mu+1)}a_2^2\right| \leq \frac{|\tau|B_1}{2(3\gamma-\mu+1)} + \frac{|\tau|(|B_2|-B_1)}{8(3\gamma-\mu+1)}|c_1^2|. \tag{4.4}$$

Therefore

$$\left|a_3 - \frac{(2\gamma-\mu+1)(1+\mu)}{2(3\gamma-\mu+1)}a_2^2\right| \leq \begin{cases} \dfrac{|\tau|B_1}{2(3\gamma-\mu+1)} & \text{if} \quad |B_2| \leq B_1 \\ \dfrac{|\tau B_2|}{2(3\gamma-\mu+1)} & \text{if} \quad |B_2| \geq B_1 \end{cases}.$$

This completes the proof of the Theorem 4.3. □

Corollary 4.7 *If $\mathfrak{f} \in \mathcal{S}^*(\tau;\phi)$ and is of the form (1.1), then*

$$\left|a_3 - \frac{1}{2}a_2^2\right| \leq \begin{cases} \dfrac{|\tau|B_1}{2} & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{|\tau B_2|}{2} & \text{if} \quad |B_2| \geq B_1. \end{cases}$$

Corollary 4.8 *If $\mathfrak{f} \in \mathcal{C}(\tau;\phi)$ and is of the form (1.1), then*

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq \begin{cases} \dfrac{|\tau|B_1}{6} & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{|\tau B_2|}{6} & \text{if} \quad |B_2| \geq B_1. \end{cases}$$

Corollary 4.9 *If $\mathfrak{f} \in \mathcal{NR}(\tau;\phi)$ and is of the form (1.1), then*

$$\left|a_3 - \frac{3}{8}a_2^2\right| \leq \begin{cases} \dfrac{|\tau|B_1}{8} & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{|\tau B_2|}{8} & \text{if} \quad |B_2| \geq B_1. \end{cases}$$

5. Toeplitz Estimates

In the following theorems, we derive the Toeplitz estimates for the class of functions belonging to $\mathcal{NM}_\mu^\gamma(\tau;\phi)$.

Theorem 5.1 *If $\mathfrak{f} \in \mathcal{NM}_\mu^\gamma(\tau;\phi)$ is of the form (1.1), then*

$$|\mathcal{T}_2(2)| \leq \frac{B_1^2|\tau|^2}{(2\gamma-\mu+1)^2} + \frac{B_1^2|\tau|^2\left(1+\left|1-\dfrac{B_2}{B_1}-\dfrac{B_1(\mu+1)\tau}{2\gamma-\mu+1}\right|\right)^2}{4(3\gamma-\mu+1)^2}.$$

Proof: From (3.6) and (3.7), we have

$$a_2^2 - a_3^2 = \frac{\tau^2B_1^2c_1^2}{4\,(2\gamma-\mu+1)^2} - \frac{\tau^2\left[2B_1c_2 - \left(B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right)c_1^2\right]^2}{64(3\gamma-\mu+1)^2}. \tag{5.1}$$

After simplification, we get

$$a_2^2 - a_3^2 = \frac{\tau^2B_1^2c_1^2}{4\,(2\gamma-\mu+1)^2} - \frac{\tau^2B_1^2\left[2c_2 - \left(1-\dfrac{B_2}{B_1}-\dfrac{B_1(1+\mu)\tau}{2\gamma-\mu+1}\right)c_1^2\right]^2}{64(3\gamma-\mu+1)^2}. \tag{5.2}$$

Taking the modulus and applying the triangle inequality along with Lemma 2.1, we obtain the desired estimate for $\mathcal{T}_2(2)$. □

Corollary 5.1 *If $\mathfrak{f} \in \mathcal{S}^*(\tau;\phi)$ is of the form (1.1), then*

$$|\mathcal{T}_2(2)| \le B_1^2|\tau|^2 + \frac{B_1^2|\tau|^2\left(1+\left|1-\frac{B_2}{B_1}-B_1\tau\right|\right)^2}{4}.$$

Corollary 5.2 *If $\mathfrak{f} \in \mathcal{C}(\tau;\phi)$ is of the form (1.1), then*

$$|\mathcal{T}_2(2)| \le \frac{B_1^2|\tau|^2}{4} + \frac{B_1^2|\tau|^2\left(1+\left|1-\frac{B_2}{B_1}-B_1\tau\right|\right)^2}{36}.$$

Corollary 5.3 *If $\mathfrak{f} \in \mathcal{NR}(\tau;\phi)$ is of the form (1.1), then*

$$|\mathcal{T}_2(2)| \le \frac{B_1^2|\tau|^2}{9} + \frac{B_1^2|\tau|^2\left(1+\left|1-\frac{B_2}{B_1}-\frac{B_1\tau}{3}\right|\right)^2}{64}.$$

Theorem 5.2 *If $\mathfrak{f}$ is of the form (1.1) and belongs to the class $\mathcal{NM}_\mu^\gamma(\tau;\phi)$, then*

$$|\mathcal{T}_3(1)| \le \begin{cases} 1+\dfrac{2|\tau|^2B_1^2}{(2\gamma-\mu+1)^2}+\dfrac{B_1|\tau|^2|1-2\eta|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; & \Lambda \le B_2-B_1 \\ 1+\dfrac{2|\tau|^2B_1^2}{(2\gamma-\mu+1)^2}+\dfrac{B_1|\tau|^2\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; & B_2-B_1 \le \Lambda \le B_2+B_1 \\ 1+\dfrac{2|\tau|^2B_1^2}{(2\gamma-\mu+1)^2}+\dfrac{B_1|\tau|^2|2\eta-1|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; & B_2+B_1 \ge \Lambda, \end{cases}$$

where

$$\Lambda = \frac{4B_1^2\tau(3\gamma-\mu+1)-B_1^2\tau(1+\mu)(2\gamma-\mu+1)}{(2\gamma-\mu+1)^2} \quad \text{and}$$

$$\eta = \frac{2B_1\tau\,(3\gamma-\mu+1)}{(2\gamma-\mu+1)^2} - \frac{\dfrac{B_1^2\tau\,(\mu+1)}{2\gamma-\mu+1}-B_1+B_2}{2B_1}. \tag{5.3}$$

Proof: From (1.5) we have

$$\mathcal{T}_3(1) = 1+2a_2^2(a_3-1)-a_3^2.$$

Using Triangular inequality to the above equation, we get

$$|\mathcal{T}_3(1)| \le 1+2|a_2^2|+|a_3||a_3-2a_2^2|. \tag{5.4}$$

Taking $\alpha = 2$ in (4.3), we get

$$a_3-2a_2^2 = \frac{B_1\tau}{4(3\gamma-\mu+1)}\left(c_2-\eta c_1^2\right), \tag{5.5}$$

where η is same as stated in (5.3).
Using (3.6), (3.8), and (5.5) in (5.4), along with Lemma 2.3, we obtain the desired estimate for $\mathcal{T}_3(1)$. □

Corollary 5.4 *If $\mathfrak{f}$ is of the form (1.1) and belongs to the class $\mathcal{S}^*(\tau;\phi)$, then*

$$|\mathcal{T}_3(1)| \leq \begin{cases} 1+2|\tau|^2B_1^2+\dfrac{B_1|\tau|^2|1-2\eta_1|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{4}; & 3B_1^2\tau \leq B_2-B_1 \\ 1+2|\tau|^2B_1^2+\dfrac{B_1|\tau|^2\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{4}; & B_2-B_1 \leq 3B_1^2\tau \leq B_2+B_1 \\ 1+2|\tau|^2B_1^2+\dfrac{B_1|\tau|^2|2\eta_1-1|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{4}; & B_2+B_1 \geq 3B_1^2\tau, \end{cases}$$

where

$$\eta_1 = \frac{3B_1^2\tau+B_1-B_2}{2B_1}.$$

Corollary 5.5 *If $\mathfrak{f}$ is of the form (1.1) and belongs to the class $\mathcal{C}(\tau;\phi)$, then*

$$|\mathcal{T}_3(1)| \leq \begin{cases} 1+\dfrac{|\tau|^2B_1^2}{2}+\dfrac{B_1|\tau|^2|1-2\eta_2|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{36}; & 2B_1^2\tau \leq B_2-B_1 \\ 1+\dfrac{|\tau|^2B_1^2}{2}+\dfrac{B_1|\tau|^2\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{36}; & B_2-B_1 \leq 2B_1^2\tau \leq B_2+B_1 \\ 1+\dfrac{|\tau|^2B_1^2}{2}+\dfrac{B_1|\tau|^2|2\eta_2-1|\left(B_1+\left|B_1-B_2-B_1^2\tau\right|\right)}{36}; & B_2+B_1 \geq 2B_1^2\tau, \end{cases}$$

where

$$\eta_2 = \frac{2B_1^2\tau+B_1-B_2}{2B_1}.$$

Corollary 5.6 *If $\mathfrak{f}$ is of the form (1.1) and belongs to the class $\mathcal{NR}(\tau;\phi)$, then*

$$|\mathcal{T}_3(1)| \leq \begin{cases} 1+\dfrac{2|\tau|^2B_1^2}{9}+\dfrac{B_1|\tau|^2|1-2\eta_3|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{3}\right|\right)}{64}; & \dfrac{13}{9}B_1^2\tau \leq B_2-B_1 \\ 1+\dfrac{2|\tau|^2B_1^2}{9}+\dfrac{B_1|\tau|^2\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{3}\right|\right)}{64}; & B_2-B_1 \leq \dfrac{13}{9}B_1^2\tau \leq B_2+B_1 \\ 1+\dfrac{2|\tau|^2B_1^2}{9}+\dfrac{B_1|\tau|^2|2\eta_3-1|\left(B_1+\left|B_1-B_2-\dfrac{B_1^2(1+\mu)\tau}{3}\right|\right)}{64}; & B_2+B_1 \geq \dfrac{13}{9}B_1^2\tau, \end{cases}$$

where

$$\eta_3 = \frac{13B_1^2\tau+9(B_1-B_2)}{18B_1}.$$

6. Applications of Touchard Polynomials

In [39], Jacques Touchard introduced a class of polynomials known as the *Touchard polynomials*, also referred to as *Bell polynomials* or *exponential polynomials* [8]. These polynomials are intrinsically linked to the problem of partitioning a set of t elements into non-empty subsets, where each subset is distinctly labeled. The expression $N_j(X)$ denotes the number of such partitions. When $X = 1$, the Touchard polynomials reduce to the Bell numbers, which count the total number of partitions of a set with n elements. Furthermore, let Y represent a random variable that follows a Poisson distribution with an expected value δ. The j-th moment of this distribution is given by:

$$E(Y^j) = N_j(\delta).$$

This connection illustrates the significance of Touchard polynomials in addressing set partition problems and analyzing moments of random variables. The generating function for the Touchard polynomials takes the form:

$$N_j(\zeta) = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j j^m}{j!} \zeta^j \quad (\zeta \in \Delta, m \geq 0, \lambda > 0).$$

Expanding on these results, the polynomial representation for higher-order moments is given by:

$$\Xi_m^\lambda(\zeta) = \zeta + \sum_{j=2}^{\infty} e^{-\lambda} \frac{(j-1)^m \lambda^{j-1}}{(j-1)!} \zeta^j \quad (\zeta \in \Delta).$$

By applying the ratio test, it can be shown that the series has an infinite radius of convergence. One may refer to [22,29,37] for the application of these expansions. It is worth mentioning that, for $m = 0$, the series reduces to the one defined by Powal in [28]. Furthermore, the function $\mathcal{T}_m^\lambda(\zeta)$ as considered in [14], is given by:

$$\mathcal{X}_m^\lambda(\zeta) = \zeta^{-1} \Xi_m^\lambda(\zeta) \quad (\zeta \neq 0).$$

The above expansion can be expressed as:

$$\begin{aligned} \mathcal{X}_m^\lambda(\zeta) &= 1 + \sum_{j=2}^{\infty} e^{-\lambda} \frac{(j-1)^m \lambda^{j-1}}{(j-1)!} \zeta^{j-1}, \qquad (6.1) \\ &= 1 + \mathcal{X}_1(\lambda, m)\zeta + \mathcal{X}_2(\lambda, m)\zeta^2 + \cdots . \end{aligned}$$

where

$$\mathcal{X}_1(\lambda, m) = \lambda e^{-\lambda}, \quad \mathcal{X}_2(\lambda, m) = \lambda^2 2^{m-1} e^{-\lambda}, \quad \text{and so on.} \tag{6.2}$$

The function $\mathcal{X}_m^\lambda(\zeta)$ is analytic in Δ and satisfies the conditions $\mathcal{X}_m^\lambda(0) = 1$, $(\mathcal{X}_m^\lambda)'(0) > 0$, and it maps the unit disk Δ onto a region that is starlike with respect to 1 and symmetric about the real axis. Considerable attention has been given to Touchard polynomials in GFT, as discussed in [7,14,22,29,37] and the references therein.

We introduce the new class $\mathcal{M}_\mu^\gamma(\tau; \lambda, m)$ in Definition 6.1 by setting $\phi(\zeta) = \mathcal{X}_m^\lambda(\zeta)$ in Definition 2.1.

Definition 6.1 *For $0 \leq \mu \leq \gamma \leq 1$, a function $\mathfrak{f} \in \mathcal{S}$ of the form (1.1) belongs to the class $\mathcal{M}_\mu^\gamma(\tau; \lambda, m)$, if the following conditions are satisfied:*

$$1 + \frac{1}{\tau} \left(\frac{\zeta \mathfrak{f}'(\zeta) + \gamma \zeta^2 \mathfrak{f}''(\zeta)}{(1-\mu)\mathfrak{f}(\zeta) + \mu \zeta \mathfrak{f}'(\zeta)} - 1 \right) \prec \mathcal{X}_m^\lambda(\zeta)$$

where $\mathcal{X}_m^\lambda(\zeta)$ is given by (6.1), $\tau \in \mathbb{C}^*$ and $\zeta \in \Delta$.

The following theorem is stated using the parameter setting of Definition 6.1 in Theorem 3.1.

Theorem 6.1 *If $\mathfrak{f} \in \mathcal{M}_\mu^\gamma(\tau; \lambda, m)$ is of the form (1.1), then*

$$|a_2| \leq \frac{|\tau| \lambda e^{-\lambda}}{2\gamma - \mu + 1} \quad \text{and} \quad |a_3| \leq \frac{|\tau| \left(\lambda e^{-\lambda} + \left| \lambda e^{-\lambda} - \lambda^2 2^{m-1} e^{-\lambda} - \dfrac{\lambda^2 e^{-2\lambda}(\mu+1)\tau}{2\gamma - \mu + 1} \right| \right)}{2(3\gamma - \mu + 1)}.$$

Similarly, we present the following results by applying the parameter settings from the previously established theorems.

Theorem 6.2 *If $\mathfrak{f} \in \mathcal{M}_\mu^\gamma(\tau; \lambda, m)$ and is of the form (1.1) and β is any complex number, then*

$$|a_3 - \beta a_2^2| \leq \frac{2\lambda e^{-\lambda}|\tau|}{3\gamma - \mu + 1} \max\left\{ 1, \left| \frac{2\beta(3\gamma - \mu + 1) - (\mu+1)(2\gamma - \mu + 1)}{(2\gamma - \mu + 1)^2} \lambda e^{-\lambda} \tau - \lambda 2^{m-1} \right| \right\}.$$

Theorem 6.3 *If $\mathfrak{f}\in\mathcal{N}_{\mu}^{\gamma}(\tau;\lambda,m)$ and is of the form (1.1) and α is any real number, then*

$$|a_3-\alpha a_2^2|\leq\begin{cases}\dfrac{\lambda e^{-\lambda}|\tau||1-2\nu|}{2(3\gamma-\mu+1)} & \text{if} \quad \alpha\leq\dfrac{(2\gamma-\mu+1)^2\,(\Theta-1)}{2\lambda e^{-\lambda}\tau(3\gamma-\mu+1)}\\ \dfrac{\lambda e^{-\lambda}|\tau|}{2(3\gamma-\mu+1)} & \text{if} \quad \dfrac{(2\gamma-\mu+1)^2\,(\Theta-1)}{2\lambda e^{-\lambda}\tau(3\gamma-\mu+1)}\leq\alpha\leq\dfrac{(2\gamma-\mu+1)^2\,(\Theta+1)}{\lambda e^{-\lambda}\tau(3\gamma-\mu+1)}\\ \dfrac{\lambda e^{-\lambda}|\tau||2\nu-1|}{2(3\gamma-\mu+1)} & \text{if} \quad \alpha\geq\dfrac{(2\gamma-\mu+1)^2\,(\Theta+1)}{2\lambda e^{-\lambda}\tau(3\gamma-\mu+1)}\end{cases},$$

where

$$\Theta=\lambda 2^{m-1}+\frac{\lambda e^{-\lambda}(\mu+1)\tau}{2\gamma-\mu+1}\quad\text{and}\quad \nu=\frac{\lambda e^{-\lambda}\tau\alpha\,(3\gamma-\mu+1)}{(2\gamma-\mu+1)^2}-\frac{\dfrac{\lambda e^{-\lambda}\tau\,(\mu+1)}{2\gamma-\mu+1}+\lambda 2^{m-1}-1}{2}.$$

Theorem 6.4 *If $\mathfrak{f}\in\mathcal{N}_{\mu}^{\gamma}(\tau;\lambda,m)$ and has the form (1.1), then*

$$\left|a_3-\frac{(2\gamma-\mu+1)(1+\mu)}{2(3\gamma-\mu+1)}a_2^2\right|\leq\begin{cases}\dfrac{|\tau|\lambda e^{-\lambda}}{2(3\gamma-\mu+1)} & \text{if} \quad \lambda 2^{m-1}\leq 1,\\ \dfrac{|\tau|\lambda^2 2^{m-2}e^{-\lambda}}{3\gamma-\mu+1} & \text{if} \quad \lambda 2^{m-1}\geq 1.\end{cases}$$

Theorem 6.5 *If $\mathfrak{f}\in\mathcal{N}_{\mu}^{\gamma}(\tau;\lambda,m)$ is of the form (1.1), then*

$$|\mathcal{T}_2(2)|\leq\frac{\lambda^2e^{-2\lambda}|\tau|^2}{(2\gamma-\mu+1)^2}+\frac{\lambda^2e^{-2\lambda}|\tau|^2\left(1+\left|1-\lambda 2^{m-1}-\dfrac{\lambda e^{-\lambda}(\mu+1)\tau}{2\gamma-\mu+1}\right|\right)^2}{4(3\gamma-\mu+1)^2}.$$

Theorem 6.6 *If $\mathfrak{f}$ is of the form (1.1) and belongs to the class $\mathcal{N}_{\mu}^{\gamma}(\tau;\lambda,m)$, then*

$$|\mathcal{T}_3(1)|\leq\begin{cases}1+\dfrac{2|\tau|^2\lambda^2e^{-2\lambda}}{(2\gamma-\mu+1)^2}+\dfrac{\lambda^2e^{-2\lambda}|\tau|^2|1-2\eta|\left(1+\left|1-\lambda 2^{m-1}-\dfrac{\lambda e^{-\lambda}(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; \\ \qquad\qquad \text{if} \quad \Lambda\leq\lambda e^{-\lambda}\left(\lambda 2^{m-1}-1\right)\\ 1+\dfrac{2|\tau|^2\lambda^2e^{-2\lambda}}{(2\gamma-\mu+1)^2}+\dfrac{\lambda^2e^{-2\lambda}|\tau|^2\left(1+\left|1-\lambda 2^{m-1}-\dfrac{\lambda e^{-\lambda}(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; \\ \qquad\qquad \text{if} \quad \lambda 2^{m-1}-1\leq\Lambda\leq\lambda 2^{m-1}+1\\ 1+\dfrac{2|\tau|^2\lambda^2e^{-2\lambda}}{(2\gamma-\mu+1)^2}+\dfrac{\lambda^2e^{-2\lambda}|\tau|^2|2\eta-1|\left(1+\left|1-\lambda 2^{m-1}-\dfrac{\lambda e^{-\lambda}(1+\mu)\tau}{2\gamma-\mu+1}\right|\right)}{4(3\gamma-\mu+1)^2}; \\ \qquad\qquad \text{if} \quad \lambda e^{-\lambda}\left(\lambda 2^{m-1}+1\right)\geq\Lambda,\end{cases}$$

where

$$\Lambda=\frac{4\lambda^2e^{-2\lambda}\tau(3\gamma-\mu+1)-\lambda^2e^{-2\lambda}\tau(1+\mu)(2\gamma-\mu+1)}{(2\gamma-\mu+1)^2}\quad\text{and}$$

$$\eta=\frac{2\lambda e^{-\lambda}\tau\,(3\gamma-\mu+1)}{(2\gamma-\mu+1)^2}-\frac{\dfrac{\lambda e^{-\lambda}\tau\,(\mu+1)}{2\gamma-\mu+1}+\lambda 2^{m-1}-1}{2}.$$

Concluding remarks and observations

In this paper, we investigated the estimates of the second and third Taylor–Maclaurin coefficients for a newly introduced subclass of univalent functions. Additionally, we derived bounds for the Fekete–Szegö functional and Toeplitz determinants. The integration of Touchard polynomials further strengthens the theoretical foundation of our work, providing a bridge between GFT and combinatorial mathematics. The results for different subclasses can be readily obtained, as highlighted in the remarks; hence, we omit the details. Future research may focus on refining these estimates, extending the analysis to higher-order determinants.

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References

1. Ali, M. F., Thomas, D. K., and Allu, V., *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*, Bull. Aust. Math. Soc., **97**(2), (2018), 253–264.

2. Allu, V., Lecko, A., and Thomas, D. K., *Hankel, Toeplitz and Hermitian-Toeplitz determinants for certain close-to-convex functions*, Mediterr. J. Math., **19**(1), (2022), 1–17.

3. Altınkaya, S., Magesh, N., and Yalçın, S., *Construction of Toeplitz matrices whose elements are the coefficients of univalent functions associated with Q-derivative operator*, Casp. J. Math. Sci., **8** (1), (2019), 51–57.

4. Al-Oboudi, F. M., and Haidan, M. M., *Spirallike functions of complex order*, J. Natur. Geom., **19**, (2000), 53–72.

5. Altıntaş, O., *On a subclass of certain starlike functions with negative coefficients*, Math. Japon., **36** (3), (1991), 489–495.

6. Altıntaş, O., Irmak, H., and Srivastava, H. M., *Fractional calculus and certain starlike functions with negative coefficients*, Comput. Math. Appl., **30** (2), (1995), 9–15.

7. Ali, E. E., Kota, W. Y., El-Ashwah, R. M., Albalahi, A. M., Mansour, F. E., and Tahira, R. A., *An application of Touchard polynomials on subclasses of analytic functions*, Symmetry, **15** (12), (2023), 1–14.

8. Boyadzhiev, K. N., *Exponential polynomials, Stirling numbers, and evaluation of some gamma integrals*, Abstr. Appl. Anal., Article ID 168672, (2009), 1–18.

9. Caglar, M., Ibrahim, I. O., Shaba, T. G., and Wanas, A. K., *Toeplitz determinants for λ-pseudo-starlike functions*, Commun. Korean Math. Soc., **39** (3), (2024), 647–655.

10. Carathéodory, C., *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Math. Ann., **64** (1), (1907), 95–115.

11. Cotîrlă, L.-I. and Wanas, A. K., *Symmetric Toeplitz matrices for a new family of prestarlike functions*, Symmetry, **14**, (2022), 1–12.

12. Efraimidis, I., *A generalization of Livingston's coefficient inequalities for functions with positive real part*, J. Math. Anal. Appl., **435** (1), (2016), 369–379.

13. Frasin, B. A., *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi., **22** (2), (2006), 179–191.

14. Hamaad, S. J., Juma, A. R. S., Ebrahim, H. H., *Subclass of bi-starlike function associated with Touchard polynomials*, J. Interdiscip. Math., **27** (4), (2024), 793–797.

15. Hadi, S. H., Saleem, Y. H., Lupaş, A. A., Alshammari, K. M. K., Alatawi, A., *Toeplitz determinants for inverse of analytic functions*, Mathematics, **13** (4), (2025), 1–12.

16. Jahangiri, M., *On the coefficients of powers of a class of Bazilevič functions*, Indian J. Pure Appl. Math., **17** (9), (1986), 1140–1144.

17. Jahangiri, J. M., Magesh, N., Yamini, J., *Fekete–Szegö inequalities for classes of bi-starlike and bi-convex functions*, Electron. J. Math. Anal. Appl., **3** (1), (2015), 133–140.

18. Kamali, M., Orhan, H., *On a subclass of certain starlike functions with negative coefficients*, Bull. Korean Math. Soc., **41** (1), (2004), 53–71.

19. Kamali, M., Riskulova, A., *On bounds of Toeplitz determinants for a subclass of analytic functions*, Bull. Math. Anal. Appl., **14** (3), (2022), 36–48.

20. Li, Z., Gou, D., *Toeplitz determinant for the inverse of a function whose derivative has a positive real part*, Appl. Math. Nonlinear Sci., **9** (1), (2024), 1–7.

21. Ma, W. C., Minda, D., *A unified treatment of some special classes of univalent functions*, In Proceeding of the International Conference on Complex Analysis, Tianjin, China, (1992).

22. Murugusundaramoorthy, G., Porwal, S., *Univalent functions with positive coefficients involving Touchard polynomials*, Al-Qadisiyah J. Pure Sci., **25** (4), (2020), 1–8.

23. Nasr, M. A., Aouf, M. K., *On convex functions of complex order*, Mansoura Bull. Sci., **9**, (1982), 565–582.

24. Nasr, M. A., Aouf, M. K., *Starlike functions of complex order*, J. Natur. Sci. Math., **25**, (1985), 1–12.

25. Nandeesh, M., Salestina, M. R., Archana, Murugusundaramoorthy, G., *Toeplitz matrices whose elements are coefficients of new subclasses of analytical functions*, Commun. Appl. Nonlinear Anal., **32** (2), (2025), 383–407.

26. Orhan, H., Porwal, S., Magesh, N., *The Fekete-Szegő problem for a generalized class of analytic functions of complex order associated with q-calculus*, Palest. J. Math., **11** (3), (2022), 39–47.

27. Pei, K., Long, P., Liu, J., Murugusundaramoorthy, G., *Fekete–Szegö inequalities and the symmetric Toeplitz determinants for certain analytic function class involving q-differintegral operator*, Chin. Quart. J. Math., **39** (4), (2024), 366–378.

28. Porwal, S., *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., Article ID 984135, (2014), 1–3.

29. Porwal, S., Murugusundaramoorthy, G., *Unified classes of starlike and convex functions associated with Touchard polynomials*, Sci. Technol. Asia, **27** (4), (2022), 207–214.

30. Radhika, V., Sivasubramanian, S., Murugusundaramoorthy, G., Jahangiri, J. M., *Toeplitz matrices whose elements are coefficients of Bazilevič functions*, Open Math., **16**, (2018), 1161–1169.

31. Ramachandran, C., Kavitha, D., *Toeplitz determinant for some subclasses of analytic functions*, Glob. J. Pure Appl. Math., **13** (2), (2017), 785–793.

32. Ravichandran, V., Polatoglu, Y., Bolcal, M., Sen, A., *Certain subclasses of starlike and convex functions of complex order*, Hacettepe J. Math. Stat., **34**, (2005), 9–15.

33. Robertson, M. S., *On the theory of univalent functions*, Ann. Math., **37**, (1936), 374–408.

34. Srivastava, H. M., Ahmad, Q. Z., Khan, N., Khan, N., Khan, B., *Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain*, Mathematics, **7**, (2019), 1–15.

35. Sun, Y., Wang, Z. G., *Sharp bounds on Hermitian Toeplitz determinants for Sakaguchi classes*, Bull. Malays. Math. Sci. Soc., **46** (2), (2023), 1–23.

36. Sun, Y., Wang, Z. G., Tang, H., *Sharp bounds on the fourth-order Hermitian Toeplitz determinant for starlike functions of order* 1/2, J. Math. Inequal., **17**(3), (2023), 985–996.

37. Soupramanien, T., Ramachandran, C., Al-Shaqsi, K., *Certain subclasses of univalent functions with positive coefficients involving Touchard polynomials*, Adv. Math. Sci. J., **10** (2), (2021), 981–990.

38. Thomas, D. K., Halim, S. A., *Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions*, Bull. Malays. Math. Sci. Soc., **40**, (2017), 1781–1790.

39. Touchard, J., *Sur les cycles des substitutions*, Acta Math., **70**, (1939), 243–297.

40. Wahid, N. H. A. A., Mohamad, D., Kamarozzaman, N. M., and Shahminan, A. A., *Toeplitz determinants for the class of functions with bounded turning*, Eur. J. Pure Appl. Math., **15**, (2022), 1937–1947.

41. Wanas, A. K., Sakar, F. M., Oros, G. I., and Cotîrlă, L.-I., *Toeplitz determinants for a certain family of analytic functions endowed with Borel distribution*, Symmetry, **15** (2), (2023), 1–9.

42. Ye, K., and Lim, L. H., *Every matrix is a product of Toeplitz matrices*, Found. Comput. Math., **16**, (2016), 577–598.

43. Zhang, H. Y., and Tang, H., *Fourth Toeplitz determinants for starlike functions defined by using the sine function*, J. Funct. Spaces, Article ID 4103772, (2021), 1–7.

44. Zulfiqar, F., Malik, S. N., Raza, M., and Ali, M., *Fourth-order Hankel determinants and Toeplitz determinants for convex functions connected with sine functions*, J. Math., Article ID 2871511, (2022), 1–12.

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