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Bound Inequalities on Minimum Covering Energy of Graphs

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ABSTRACT: The sum of the absolute values of all Minimum Covering eigenvalues $E_{mc}\mathfrak{G}$, of graph $\mathfrak{G}$ represents the Minimum Covering energy of that graph. A few upper and lower constraints on the minimum Covering energy are obtained in this study.

Key Words: Minimum covering matrix, minimum covering eigenvalues, minimum covering energy of a graph.

Contents

1. Introduction

$E(\mathfrak{G}) = \sum_{k=1}^{n} |\Gamma_k|$, where Γ_k, $k = 1, 2, 3, ..., n$ are the eigenvalues of the adjacency matrix (AM), of the graph $\mathfrak{G}$, was defined in 1978 for graph energy. Ivan Gutman [5] conducted this research on $\mathfrak{G}$ for the first time. German researcher Erich Huckle, employed the energy of graphs technique in the early 1930s to develop approximations for solutions for a family of organic molecules known as conjugated hydro carbons [3],commonly known as Huckle molecular orbital (HMO) theory. Thousands of studies have been published since the beginning of graph energy. Numerous matrix types, including Incidence [8], Distance [7], Lapalcian [6], Maximum degree matrix [2], and others, are established and researched for graphs, with inspiration drawn from the AM of graph $\mathfrak{G}$.

Given n vertices $V = v_1, v_2, \ldots, v_n$ and edge set $E = e_1, e_2, \ldots, e_m$ of order m. Let $\mathfrak{G}(V, E)$ be a simple graph. If every edge in $\mathfrak{G}$ is incident to at least one vertex in C, then the subset C of V is referred to as a Covering set of $\mathfrak{G}$. Minimum Covering Set (MCS) is the Covering set with minimum cardinality. Let C be the graph $\mathfrak{G}'s$ MCS for any graph.
The following kind of matrix, known as the Minimum Covering Matrix (MCM) of a graph, was introduced by C. Adiga et al.in [1] and its eigenvalues and energy were examined.
The $n \times n$ matrix $M_c(\mathfrak{G}) = c_{kj}$, is the MCM of $\mathfrak{G}$,

$$c_{kj} = \begin{cases} 1 & \text{if } v_k \text{ and } v_j \text{ are adjacent,} \\ 1 & \text{if k=j } \mathit{and}\ v_i\ \mathit{in}\ C \\ 0 & \text{otherwise.} \end{cases}$$

$\chi(\mathfrak{G} : \eta) = det(\eta I - M_c(\mathfrak{G}))$, defines the characteristic polynomial of the MCM, $M_c(\mathfrak{G})$.

The minimum covering eigenvalues(McE) of the graph $\mathfrak{G}$ are represented by the eigenvalues $\eta_1, \eta_2, \ldots, \eta_n$ of $M_c(\mathfrak{G})$. The matrix $M_c(\mathfrak{G})$ is symmetric and real. The real numbers that make up the eigenvalues of $M_c(\mathfrak{G})$ are organized as follows: $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$.

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$$E_{mc}(\mathfrak{G}) = \sum_{k=1}^{n} |\eta_k|,$$

is the formula for a graph $\mathfrak{G}'s$ minimum covering energy (MCE).

Given that $M_c(\mathfrak{G})$ has $trace = |C|$, and $\sum_{k=1}^{n} \eta_k^2 = 2|E| + |C| = 2m + |C|$ and it is a real and symmetric matrix. We derive some upper and lower constraints for the MCE, $E_{mc}(\mathfrak{G})$, in this study.

2. Upper Bounds for Minimum Covering Energy of Graphs

In this section we study upper bounds for MCE of graphs.

Theorem 2.1 *Let $\mathfrak{G}$ be non-empty graph with n vertices and m edges then*

$$E_{mc}(\mathfrak{G}) \leq \sqrt{\frac{1}{2}(n^2 + |C|^2) + 2m(m + |C|)}.$$

Proof. Let e_k, f_k, g_k and h_k are sequences of real number and p_k and q_k are non negative for $k = 1, 2, \ldots, n$ then the following inequality is valid (see [4])

$$\sum_{k=1}^{n} p_k e_k^2 \sum_{k=1}^{n} q_k f_k^2 + \sum_{k=1}^{n} p_k g_k^2 \sum_{k=1}^{n} q_k h_k^2 \geq 2 \sum_{k=1}^{n} p_k e_k g_k \sum_{k=1}^{n} q_k f_k h_k \tag{2.1}$$

for $e_k = f_k = p_k = q_k = 1$ and $g_k = h_k = |\eta_k|$, $k = 1, 2, ..., n$, inequality (2.1) becomes

$$\sum_{k=1}^{n} 1 \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} |\eta_k|^2 \sum_{k=1}^{n} |\eta_k|^2 \geq 2 \sum_{k=1}^{n} |\eta_k| \sum_{k=1}^{n} |\eta_k|$$

using, $\sum_{k=1}^{n} |\eta_k|^2 = \sum_{k=1}^{n} \eta_k^2 = 2m + |C|$ in above inequality we deduce that,

$$n \cdot n + (2m + |C|)(2m + |C|) \geq 2E_{mc}(\mathfrak{G}) \cdot E_{mc}(\mathfrak{G})$$
$$2E_{mc}(\mathfrak{G})^2 \leq n^2 + (2m + |C|)^2$$

Hence,

$$E_{mc}(\mathfrak{G}) \leq \sqrt{\frac{1}{2}(n^2 + |C|^2) + 2m(m + |C|)}.$$

Theorem 2.2 *Let $\mathfrak{G}$ be non-empty graph with n vertices and m edges then*

$$E_{mc}(\mathfrak{G}) \leq \frac{1}{2}(n + |C|) + m.$$

Proof. Let e_k, f_k, g_k and h_k are sequences of real number and p_k and q_k are non negative for $k = 1, 2, \ldots, n$ then the following inequality is valid (see [4])

$$\sum_{k=1}^{n} p_k e_k^2 \sum_{k=1}^{n} q_k f_k^2 + \sum_{k=1}^{n} p_k g_k^2 \sum_{k=1}^{n} q_k h_k^2 \geq 2 \sum_{k=1}^{n} p_k e_k g_k \sum_{k=1}^{n} q_k f_k h_k \tag{2.2}$$

for $e_k = f_k = h_k = p_k = q_k = 1$ and $g_k = |\eta_k|$, $k = 1, 2, ..., n$, inequality (2.2) becomes

$$\sum_{k=1}^{n} 1 \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} |\eta_k|^2 \sum_{k=1}^{n} 1 \geq 2 \sum_{k=1}^{n} |\eta_k| \sum_{k=1}^{n} 1$$

$$n^2 + \left(\sum_{k=1}^{n} |\eta_k|^2 \right) n \geq 2 \left(\sum_{k=1}^{n} |\eta_k| \right) n$$

$$n + 2m + |C| \geq 2E_{mc}(\mathfrak{G})$$

Hence,

$$E_{mc}(\mathfrak{G}) \leq \frac{1}{2}(n + |C|) + m$$

3. Lower Bounds for Minimum Covering Energy of Graphs

In this section we study lower bounds for MCE of graphs.

Theorem 3.1 *Let $\mathfrak{G}$ be non-empty bipartite graph of order at least 2 with m edges and having spectral radius η_1 , then*

$$E_{mc}(\mathfrak{G}) \geq \frac{2m+|C|}{\eta_1}.$$

Proof. Let e_k, f_k are decreasing non-negative sequences with $e_k, f_k \neq 0$ and j_k a non-negative sequence for $k = 1, 2, \ldots, n$. Then the following inequality is valid (see [4])

$$\sum_{k=1}^{n} j_k e_k^2 \sum_{k=1}^{n} j_k f_k^2 \leq max\left\{ f_1 \sum_{k=1}^{n} j_k e_k^2, e_1 \sum_{k=1}^{n} j_k f_k^2 \right\} \sum_{k=1}^{n} j_k e_k f_k \tag{3.1}$$

for $e_k = f_k = |\eta_k|$ and $j_k = 1$, $k = 1, 2, \ldots, n$, the inequality (3.1) becomes,

$$\sum_{k=1}^{n} 1 \cdot |\eta_k|^2 \sum_{k=1}^{n} 1 \cdot |\eta_k|^2 \leq max\left\{ \eta_1 \sum_{k=1}^{n} |\eta_k|, \eta_1 \sum_{k=1}^{n} |\eta_k| \right\} \sum_{k=1}^{n} |\eta_k|^2$$

$$\sum_{k=1}^{n} |\eta_k|^2 \leq \eta_1 \sum_{k=1}^{n} |\eta_k|$$

$$\eta_1 E_{mc}(\mathfrak{G}) \geq \sum_{k=1}^{n} |\eta_k|^2$$

Hence,

$$E_{mc}(\mathfrak{G}) \geq \frac{2m+|C|}{\eta_1}$$

Lemma 1 [9] *Let $n \geq 1$ be an integer and $l_1, l_2 \ldots, l_n$ be some non-negative real numbers such that $l_1 \geq l_2 \geq \ldots \geq l_n$. then*

$$(l_1 + l_2 + \cdots + l_n)(l_1 + l_n) \geq l_1^2 + \cdots + l_n^2 + n l_1 l_n$$

Moreover,the inequality holds if and only if for some $r \in \{1, \ldots, n\}$, $l_1 = \cdots = l_r$ and $l_{r+1} = \cdots = l_n$

Theorem 3.2 *Let $\mathfrak{G}$ be a graph with $n \geq 2$ vertices and $m \geq 1$ edges. Assume that $\eta_1, \ldots, \eta_n$ are all minimum covering eigenvalues of $\mathfrak{G}$ such that $|\eta_n| \geq \ldots \geq |\eta_1| \geq 0$, then*

$$E_{mc}(\mathfrak{G}) \geq \frac{2\sqrt{|\eta_1 \eta_n|}}{|\eta_1| + |\eta_n|} \sqrt{2|\mathfrak{C}_2| n}.$$

Proof. we know that, since $\mathfrak{G}$ has at least one edge, $\mathfrak{G}$ has at least one non zero eigenvalue. Using Lemma1 we get,

$$(|\eta_1| + \cdots + |\eta_n|)(|\eta_1| + |\eta_n|) \geq |\eta_1|^2 + \cdots + |\eta_n|^2 + n|\eta_1||\eta_n| \tag{3.2}$$

and the equality holds if and only if $|\eta_1| = \cdots = |\eta_r|$ and $|\eta_{r+1}| = \cdots = |\eta_n|$ for some $r \in \{1, \ldots, n\}$, since, $|\eta_1|^2 + \cdots + |\eta_n|^2 = 2m + |C|$. By equation (3.2) we get,

$$E_{mc}(\mathfrak{G})(|\eta_1| + |\eta_n|) \geq 2m + |C| + n|\eta_1||\eta_n|$$

$$E_{mc}(\mathfrak{G}) \geq \frac{2m + |C| + n|\eta_1||\eta_n|}{|\eta_1| + |\eta_n|} \tag{3.3}$$

and the equality holds if and only if $|\eta_1| = \cdots = |\eta_r|$ and $|\eta_{r+1}| = \cdots = |\eta_n|$ for some $r \in \{1, \ldots, n\}$. We all know that for every real number $a \geq 0$ and $b \geq 0$ we have $a + b \geq 2\sqrt{ab}$ and equality holds if and only if $a = b$, using this equation (3.3) becomes

$$E_{mc}(\mathfrak{G}) \geq \frac{(2m + |C|) + n|\eta_1||\eta_n|}{|\eta_1| + |\eta_n|} \geq \frac{2\sqrt{(2m + |C|)n|\eta_1||\eta_n|}}{|\eta_1| + |\eta_n|} = \frac{2\sqrt{(2m + |C|)n}\sqrt{|\eta_1 \eta_n|}}{|\eta_1| + |\eta_n|}$$

Hence,

$$E_{mc}(\mathfrak{G}) \geq \frac{2\sqrt{(2m + |C|)n}\sqrt{|\eta_1 \eta_n|}}{|\eta_1| + |\eta_n|}.$$

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