Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.75950

Generalized contraction in b-metric spaces and application to nonlinear optimal control

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ABSTRACT: Fixed point theory plays a crucial role in addressing challenges in nonlinear systems, particularly in the context of optimal control. The complexity of these systems often complicates the establishment of solutions, especially when traditional methods are applied in varying metric spaces. This paper presents a novel approach by utilizing fixed point theorems specifically within b-metric spaces, offering effective solutions to these challenges. By demonstrating the existence and uniqueness of fixed points in this framework, we provide a pathway for ensuring convergence in optimal control strategies. Ultimately, our findings underscore the significance of integrating fixed point theory with b-metric spaces to enhance practical applications in nonlinear control.

Key Words: Fixed point, generalized contraction, b-metric spaces.

Contents

1. Introduction

The theory of fixed points has evolved significantly since its inception, playing a crucial role in the analysis of nonlinear systems, particularly in optimal control theory. A fixed point of a function $T : X \to X$ is an element $x^* \in X$ such that $T(x^*) = x^*$. Among the most significant results in fixed point theory are the Banach fixed point theorem [3] and the Brouwer fixed point theorem [5]. The Banach fixed point theorem, also known as the contraction mapping theorem, asserts that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, there exists a constant $k \in [0, 1)$ such that $d(T(x_1), T(x_2)) \leq kd(x_1, x_2)$ for all $x_1, x_2 \in X$ then T has a unique fixed point in X. This theorem underpins the existence of solutions to differential equations and optimization problems.

The development of fixed point theory can be traced back to the early 20th century, with Brouwer's fixed point theorem [5] establishing the existence of fixed points in compact convex sets. A contraction mapping is essential as it guarantees that iterative methods converge to a unique optimal solution when initial guesses are sufficiently close [15]. Completeness of the metric space (X, d) ensures that every cauchy sequence converges to a limit within X, which is vital for the application of the Banach fixed point theorem [16]. Metric fixed point theory extends classical results to more general settings, accommodating non-contractive mappings and various types of metrics, which is particularly useful in the study of nonlinear differential equations [6], [15].
In 1989, Bakhtin [2] introduced the concept of b-metric space, and in 1993, Czerwik [10] extensively utilized the idea of b-metric space, which generalizes traditional metric spaces by relaxing the triangle inequality. This framework enables a wider variety of mappings and applications in non-standard metric contexts.
In the realm of fixed point theory, several types of contractions have been defined within metric spaces.

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2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted March 01, 2025. Published September 25, 2025

Some of the significant contributions include the Meir-Keeler contraction [14], Suzuki contraction [21], Kannan contraction [13], weak contraction [18], Chatterjea contraction [7], and F-Suzuki contraction [21]. Chauhan et al. ([8], [9]) recently presented fascinating insights into F-contraction within partially ordered b-metric spaces. Similarly, Younis et al. ([24], [25]) explored diverse applications of fixed-point theory to establish the existence of solutions for various engineering models.
The past few years, advancements in fixed-point theory have been made by studying generalized contraction mappings by [1], [4], and also some interesting results have been introduced [22]. For further applications in this setup, one can see the noteworthy work done in [17].

In the context of optimal control for nonlinear second-order differential equations, fixed point theorems provide a robust framework for establishing the existence and uniqueness of optimal solutions, allowing for effective control strategies in complex systems. This interplay between fixed point theory and control optimization enriches the theoretical landscape and offers practical methodologies for addressing real-world problems [20].

2. Preliminaries

Before delving into the main result of the article, we review key terminology and concepts for further research.

Definition 2.1 *[11] A metric space is a set X together with a function $d : X \times X \to \mathbb{R}$ that assigns a non-negative real number $d(x, y)$ to each pair of elements $x, y \in X$. This function d, called a metric or distance function, satisfies the following properties for all $x, y, z \in X$*

(M1) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$,

(M2) $d(x, y) = d(y, x)$,

(M3) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a metric space.

Example: The set of real numbers $\mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ forms a metric space. Here, $X = \mathbb{R}$ and $d(x, y) = |x - y|$ satisfies all the conditions of a metric.

Definition 2.2 *[10] A b-metric space (X, d) is a set X equipped with a function $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions for all $x, y, z \in X$ and $b \geq 1$*

(b1) Non-negativity: $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$,

(b2) Symmetry: $d(x, y) = d(y, x)$,

(b3) Triangle Inequality: $d(x, z) \leq b \cdot \{d(x, y) + d(y, z)\}$.

Definition 2.3 *[12] A b-metric space (X, d) is said to be complete if every cauchy sequence in X converges to a limit in X.*

F-contraction is a more generalized form of contraction mapping in which the contraction condition is a continuous and arbitrary function T, which is typically non-decreasing. In order to establish a fixed point result as a generalization of the Banach contraction principle, Wardowski [23] introduced and examined a novel contraction known as the F-contraction.

Definition 2.4 *[23] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:*

(F1) F *is strictly increasing,*

(F2) For all sequence $\alpha_n \subseteq \mathbb{R}^+$, $\lim_{n\to+\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to+\infty} F(\alpha_n) = -\infty$,

(F3) There exists a constant k with $0 < k < 1$ such that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.

Definition 2.5 *[23] Let (X, d) be a b-metric space. A mapping $T : X \to X$ is said to be a generalized-contraction if there exists a function $F : X \times X \to [0, 1]$ such that for all $x, y \in X$:*

$$d(T(x), T(y)) \leq F(x, y) \cdot d(x, y) + \gamma \cdot d(x, y).$$

where $F(x, y)$ is a non-negative function that satisfies the following properties:

(I) Non-negativity: $F(x, y) \geq 0$ for all $x, y \in X$,

(II) Symmetry: $F(x, y) = F(y, x)$ for all $x, y \in X$,

(III) Boundedness: There exist constants $\alpha, \beta \in (0, 1)$ such that:

$$F(x, y) \leq \alpha d(x, y) + \beta.$$

γ is a constant satisfying $\gamma \in [0, 1)$.

Lemma 2.1 *[19] In a complete metric space, every cauchy sequence converges to a limit within that space.*

Nex, we utilize the concept of generalized-contraction in b-metric space and proved some fixed point results in n optimal control system.

3. Main Result

The concept of generalized contraction in b-metric space was introduced in this section, and several fixed point theorems were demonstrated. The primary outcome is as follows.

Theorem 3.1 *Let (X, d) be a complete b-metric space, and let $F : X \times X \to [0, 1]$ be a function satisfying the following conditions:*

(I) $F(x, y) = F(y, x)$ for all $x, y \in X$,

(II) $\alpha, \beta \in (0, 1)$ such that for all $x, y \in X$; $F(x, y) \leq \alpha d(x, y) + \beta$.

Let $T : X \to X$ be a mapping such that for any $x, y \in X$:

$$d(T(x), T(y)) \leq F(x, y) \cdot d(x, y) + \gamma \cdot d(x, y),$$

where γ is a constant satisfying $\gamma \in [0, 1)$.
Then, T has a unique fixed point in X.

Proof: Rewrite the inequality for $d(T(x), T(y))$

$$d(T(x), T(y)) \leq (F(x, y) + \gamma) \cdot d(x, y).$$

If we let $\delta = \alpha + \gamma$, we can express it as

$$d(T(x), T(y)) \leq \delta d(x, y) + \beta.$$

For T to be a contraction, we need $\delta < 1$. Let an arbitrary point $x_0 \in X$ and define a sequence (x_n) by

$$x_n = T(x_{n-1}).$$

We will show that this sequence is cauchy. For $n \geq m$:

$$d(x_n, x_m) = d(T(x_{n-1}), T(x_{m-1})).$$

Using the contraction condition

$$d(x_n, x_m) \leq (F(x_{n-1}, x_{m-1}) + \gamma) \cdot d(x_{n-1}, x_{m-1}).$$

By applying the mapping T repeatedly, we have

$$d(x_n, x_{n-1}) \leq (F(x_{n-1}, x_{n-2}) + \gamma) \cdot d(x_{n-1}, x_{n-2}).$$

Continuing this process, we have

$$d(x_n, x_{n-1}) \leq \delta^{n-1} d(x_1, x_0) + (\text{terms involving } \beta).$$

Since X is complete, the cauchy sequence (x_n) converges to some limit $x^* \in X$. Next to show that $T(x^*) = x^*$. Since $x_n \to x^*$ as $n \to +\infty$, we can express

$$d(T(x_n), T(x^*)) \to 0 \text{ as } n \to +\infty.$$

Using continuity of T (which follows from the contraction condition), we have

$$T(x^*) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_n = x^*.$$

Assume there are two fixed points x^* and y^* such that $T(x^*) = x^*$ and $T(y^*) = y^*$.
Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq (F(x^*, y^*) + \gamma) \cdot d(x^*, y^*).$$

If we denote $d(x^*, y^*) = r$, we have

$$r \leq (F(x^*, y^*) + \gamma) r.$$

If $r > 0$, we can divide both sides by r

$$1 \leq F(x^*, y^*) + \gamma.$$

Since $\gamma < 1$. This contradicts the boundedness of F (which is less than 1). Therefore, $r = 0$, implying $x^* = y^*$. Thus T has a unique fixed point x^* in X. □

Next, we provide an example that illustrates the validity of the hypotheses and the degree of generality of Theorem 3.1.

Example 3.1 *Consider,* $X = [0, 1]$ *with the following b-metric defined for all* $x, y \in X$*:*

$$d(x, y) = |x - y| + \frac{1}{2}|x^2 - y^2|.$$

This b-metric combines the standard metric with a quadratic term, ensuring it satisfies the properties of a b-metric space.
Define the function $F : X \times X \to [0, 1]$ *as follows:*

$$F(x, y) = \frac{|x - y|}{1 + |x - y|} \quad \text{for all } x, y \in X.$$

Clearly, $F(x, y) \geq 0$ *and* $|x - y| \geq 0$*.*
Also $F(x, y) = F(y, x)$ *due to the absolute value.*
For all $x, y \in X$*:* $F(x, y) \leq 1$*, Define the mapping* $T : X \to X$ *by*

$$T(x) = \frac{x}{2} + \frac{1}{4}.$$

For any $x, y \in X$

$$d(T(x), T(y)) = d\left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{4}\right) = \left|\frac{x - y}{2}\right| + \frac{1}{2}\left|\left(\frac{x}{2} + \frac{1}{4}\right)^2 - \left(\frac{y}{2} + \frac{1}{4}\right)^2\right|.$$

Calculating the quadratic term

$$\left(\frac{x}{2}+\frac{1}{4}\right)^2-\left(\frac{y}{2}+\frac{1}{4}\right)^2=\left(\frac{x-y}{2}\right)\left(\frac{x+y}{2}+\frac{1}{2}\right).$$

Thus,

$$d(T(x),T(y))=\frac{|x-y|}{2}+\frac{1}{2}\cdot\frac{|x-y|}{2}\cdot\left(\frac{x+y}{2}+\frac{1}{2}\right).$$

Now, we can express this as

$$d(T(x),T(y))\leq\left(\frac{1}{2}+\frac{1}{4}\right)|x-y|=\frac{3}{4}|x-y|\leq F(x,y)\cdot d(x,y)+\gamma\cdot d(x,y),$$

where $\gamma=0$.
Therefore, the mapping T is a generalized-contraction in the defined b-metric space. Thus from theorem 3.1, T has a unique fixed point. To find the fixed point, set $T(x)=x$,

$$x=\tfrac{x}{2}+\tfrac{1}{4}\ \textit{implies}\ \tfrac{x}{2}=\tfrac{1}{4}\ \textit{implies}\ x=\tfrac{1}{2}.$$

Thus, the unique fixed point is $x^*=\frac{1}{2}$.

Further, we can extend the fixed point theorem in b-metric spaces with dual generalized-contractions in b-metric spaces.

Theorem 3.2 *Let (X,d) be a complete b-metric space. Define two functions $F_1,F_2:X\times X\to[0,1]$ satisfying the following properties*

(I) $F_1(x,y)\geq 0$ *and* $F_2(x,y)\geq 0$ *for all* $x,y\in X$,

(II) $F_1(x,y)=F_1(y,x)$ *and* $F_2(x,y)=F_2(y,x)$ *for all* $x,y\in X$,

(III) There exist constants $\alpha_1,\alpha_2,\beta_1,\beta_2\in(0,1)$ *such that for all* $x,y\in X$*:*

$$F_1(x,y)\leq\alpha_1 d(x,y)+\beta_1,$$

$$F_2(x,y)\leq\alpha_2 d(x,y)+\beta_2.$$

Let $T:X\to X$ be a mapping such that for any $x,y\in X$

$$d(T(x),T(y))\leq(F_1(x,y)+F_2(x,y))\cdot d(x,y).$$

If there exists a constant $\gamma\in[0,1)$ such that

$$F_1(x,y)+F_2(x,y)\leq\gamma,$$

then T has a unique fixed point in X.

Proof: Let $F_1(x,y)+F_2(x,y)\leq\gamma$, we can rewrite the mapping condition

$$d(T(x),T(y))\leq(F_1(x,y)+F_2(x,y))\cdot d(x,y)\leq\gamma d(x,y).$$

Since $\gamma<1$, it follows that T is a contraction mapping. Let an arbitrary point $x_0\in X$ and define a sequence (x_n) by

$$x_n=T(x_{n-1}).$$

For any $n>m$

$$d(x_n,x_m)=d(T(x_{n-1}),T(x_{m-1})).$$

Using the contraction property

$$d(x_n,x_m)\leq\gamma d(x_{n-1},x_{m-1}).$$

We can apply this repeatedly

$$d(x_n, x_m) \leq \gamma d(x_{n-1}, x_{m-1}) \leq \gamma^2 d(x_{n-2}, x_{m-2}) \leq \ldots \leq \gamma^{n-m} d(x_{m-1}, x_{m-1}).$$

Since $d(x_{m-1}, x_{m-1}) = 0$, we have

$$d(x_n, x_m) \leq \gamma^{n-m} d(x_{m-1}, x_{m-1}).$$

As n and m increase, $\gamma^{n-m} \to 0$. Thus, (x_n) is Cauchy. Since (X, d) is complete, the Cauchy sequence (x_n) converges to some limit $x^* \in X$

$$x_n \to x^* \text{ as } n \to +\infty.$$

To show $T(x^*) = x^*$. Using the continuity of T

$$T(x^*) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} T(x_n) = x^*.$$

Thus, x^* is a fixed point of T.
Assume there are two fixed points x^* and y^* such that $T(x^*) = x^*$ and $T(y^*) = y^*$. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq (F_1(x^*, y^*) + F_2(x^*, y^*)) \cdot d(x^*, y^*).$$

By the assumption $F_1(x^*, y^*) + F_2(x^*, y^*) \leq \gamma$

$$d(x^*, y^*) \leq \gamma d(x^*, y^*).$$

Since $\gamma < 1$, we have

$$d(x^*, y^*) = 0 \text{ implies } x^* = y^*.$$

Thus, the fixed point is unique. □

Here, we set an example in favour of Theorem 3.2.

Example 3.2 *Let $X = \mathbb{R}$ (the set of real numbers) equipped with the b-metric defined as:*

$$d(x, y) = |x - y| + |x - y|^2.$$

Define two functions F_1 and F_2 as follows: $F_1(x, y) = \frac{1}{2}|x - y|$, $F_2(x, y) = \frac{1}{3}|x - y|$.
Since $F_1(x, y) \geq 0$ and $F_2(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
Also $F_1(x, y) = F_1(y, x)$ and $F_2(x, y) = F_2(y, x)$ for all $x, y \in \mathbb{R}$.
For F_1:

$$F_1(x, y) = \frac{1}{2}|x - y| \leq \frac{1}{2}d(x, y) + 0,$$

thus $\alpha_1 = \frac{1}{2}$ and $\beta_1 = 0$.
For F_2:

$$F_2(x, y) = \frac{1}{3}|x - y| \leq \frac{1}{3}d(x, y) + 0,$$

thus $\alpha_2 = \frac{1}{3}$ and $\beta_2 = 0$. Now define the mapping $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \frac{2}{3}x + 1.$$

For any $x, y \in \mathbb{R}$

$$d(T(x), T(y)) = \left|\frac{2}{3}x + 1 - \left(\frac{2}{3}y + 1\right)\right| + \left|\frac{2}{3}x + 1 - \left(\frac{2}{3}y + 1\right)\right|^2.$$

This simplifies to

$$d(T(x),T(y)) = \left|\frac{2}{3}(x-y)\right| + \left|\frac{2}{3}(x-y)\right|^2 = \frac{2}{3}|x-y| + \frac{4}{9}|x-y|^2.$$

Now,

$$F_1(x,y)+F_2(x,y) = \frac{1}{2}|x-y| + \frac{1}{3}|x-y| = \left(\frac{3}{6}+\frac{2}{6}\right)|x-y| = \frac{5}{6}|x-y|.$$

Thus,

$$d(T(x),T(y)) \le \left(\frac{5}{6}\right)|x-y| \le \left(\frac{5}{6}\right)d(x,y),$$

which satisfies the contraction condition since $\frac{5}{6} < 1$.
Since $F_1(x,y)+F_2(x,y) \le \gamma$ *where* $\gamma = \frac{5}{6}$,
To find the fixed point x^*

$$x^* = T(x^*) = \frac{2}{3}x^* + 1.$$

Rearranging,

$$x^* - \tfrac{2}{3}x^* = 1 \textit{ implies } \tfrac{1}{3}x^* = 1 \textit{ implies } x^* = 3.$$

Thus, the unique fixed point of the mapping T *in this example is* $x^* = 3$. *The extended enhanced fixed point theorem is demonstrated in this example using dual generalized* F*-contractions in a* b*-metric space.*

Theorem 3.1 and Theorem 3.2 can be applied in optimal control for nonlinear second-order differential equations

4. Application

Consider a second-order nonlinear differential equation of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + f(y) = 0,$$

where, $y(t)$ represents the state variable.
$p(t)$ represents the damping function (can be linear or nonlinear).
$f(y)$ represents the a nonlinear function representing forces acting on the system.
We want to minimize a cost functional defined as

$$J[y] = \int_0^T \left(y^2 + \lambda\left(\frac{dy}{dt}\right)^2\right) dt,$$

where λ is a weighting factor.
Express the solution using Green's function as

$$y(t) = \int_0^T G(t,s)f(s,y(s))ds,$$

where $f(s,y(s))$ is the nonlinear term evaluated at the state variable.
Now define a mapping T as

$$T[y](t) = \int_0^T G(t,s)f(s,y(s))ds.$$

Since $\|T[y_1]-T[y_2]\| \le k\|y_1-y_2\|$, for some $0<k<1$.
Therefore,

$$T[y_1](t) - T[y_2](t) = \int_0^T G(t,s)\left(f(s,y_1(s)) - f(s,y_2(s))\right)ds.$$

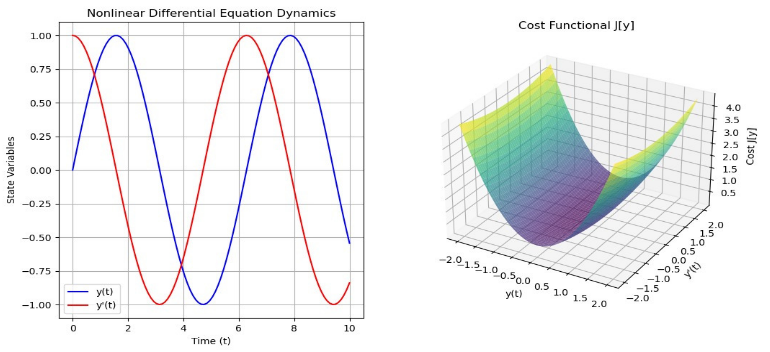

Figure 1: Optimal Control System Dynamic

If f is Lipschitz continuous with respect to y, there exists a constant L such that

$$|f(s, y_1) - f(s, y_2)| \leq L|y_1 - y_2|.$$

Then we have

$$|T[y_1](t) - T[y_2](t)| \leq \int_0^T |G(t,s)|L|y_1(s) - y_2(s)|ds.$$

Now assuming $G(t, s)$ is bounded by M

$$|T[y_1](t) - T[y_2](t)| \leq ML \int_0^T |y_1(s) - y_2(s)|ds.$$

If $\|y_1 - y_2\|$ is sufficiently small, we can choose $k < 1$ such that

$$\|T[y_1] - T[y_2]\| \leq k\|y_1 - y_2\|.$$

Apply the enhanced fixed point theorem 3.1, there exists a unique fixed point y^* such that $T[y^*] = y^*$. Inevitably, the cost functional $J[y]$ is minimized by the control input that corresponds to the fixed point, thus validating the optimal control solution and the graphical depiction shown in Figure 1.

5. Conclusion

Our results demonstrates the use of fixed point theorems in solving an optimal control problem represented by a second-order nonlinear differential equation. By establishing that the mapping is a contraction, we confirm the existence and uniqueness of the optimal solution, validating the results of the optimization process. The methods discussed can be adapted to various nonlinear systems, making them versatile in the field of optimization.

Acknowledgments

The authors are acknowledged by the Department of Applied Sciences, National Institute of Technical Teachers' Training and Research, Bhopal, 462002, Madhya Pradesh, India, under the Internal Research Funding Project ID: $CFP2022O4$.

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