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Outer independent {2}-domination in trees

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ABSTRACT: An outer independent {2}-dominating function (OI{2}D-function) of a graph G is a function $f : V(G) \longrightarrow \{0,1,2\}$ such that no two vertices assigned 0 under f are adjacent, and $f(N[v]) \geq 2$ for all $v \in V(G)$, where $N[v]$ stands for the set of neighbors of v plus v. The weight of an OI{2}D-function is the value $\omega(f) = \Sigma_{u\in V(G)} f(u)$, and the minimum weight of an OI{2}D-function of G is the outer independent {2}-domination number $\gamma_{oi\{2\}}(G)$ of G. In this paper, we first determine the exact value of the outer independent-{2}-domination number for perfect binary trees, an then we provide a lower bound and an upper bound for the outer independent {2}-domination number for trees in terms of the covering number, the independence number, the number of leaves and the number of stems (support vertices).

Key Words: Outer-independent {2}-domination, {2}-domination, trees.

Contents

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V; E)$ be a graph of order $|V(G)| = n$. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$, or simply $\deg(v)$ if the graph G is clear from the context. A *leaf* is a vertex of degree 1 and a *stem* is a vertex adjacent to a leaf. Note that for a path on two vertices, both vertices can considered as stems and leaves. A *stem* is *strong* if it has at least two leaf neighbors and it is *weak* if it has exactly one leaf neighbor. The set of leaves and stems of G are denoted by $L(G)$ and $S(G)$, respectively, and let $\ell(G) = |L(G)|$ and $s(G) = |S(G)|$.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G while the *diameter,* $\text{diam}(G)$, of G is the maximum distance among all pairs of vertices in G.

We write P_n for the *path* of order n. A *tree* is an acyclic connected graph. A *star* is the graph $K_{1,t}$, with $t \geq 1$, where the vertex of degree t of the star is called the *center.* A *double star* $S_{r,s}$ is a tree obtained from two disjoint stars $K_{1,r}$ and $K_{1,s}$ by adding an edge joining their centers. A *rooted tree* T is a tree with a distinguished special vertex r, called the *root.* For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. We denote by $C(v)$ and $D(v)$ the set of children and descendants of v, respectively and we define $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\text{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* T_v at v is the subtree of T induced by $D[v]$.

A subset $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ has a neighbor in D. An *independent set* is any set $S \subseteq V(G)$ such that no edge of G has its two endvertices in S. The *independence number* $\alpha(G)$ is the maximum cardinality of an independent set in G. A *vertex cover* of a graph G is a set of vertices that covers all the edges, and the minimum cardinality of a vertex cover is the *vertex cover*

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number denoted by $\beta(G)$. Notice that the vertex cover number is sometimes called the outer-independent domination number and denoted by $\gamma_{oi}(G)$. For a comprehensive survey of domination in graphs and its variations, we refer the reader to [5] and [6].

A $\{2\}$-dominating function of G is a function $f : V(G) \longrightarrow \{0,1,2\}$ such that $f(N[v]) \geq 2$ for every vertex $v \in V(G)$. The weight of a $\{2\}$-dominating function is $\omega(f) = f(V) = \sum_{v\in V} f(v)$. For any $\{2\}$-dominating function f on G, let $V_i = \{v \in V \mid f(v_i) = i\}$ for $i \in \{0,1,2\}$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$. Note that $\omega(f) = |V_1| + 2|V_2|$. The concept of $\{2\}$-dominating functions was first defined by Domke, Hedetniemi, Laskar and Fricke [3] and investigated for example in [1], [2] and [7].

In this paper, we continue the study of a variant of $\{2\}$-dominating function recently investigated in [4], called outer-independent $\{2\}$-dominating function defined as follows. An *outer-independent* $\{2\}$-*dominating function* (OI$\{2\}$D-function) on a graph G is a $\{2\}$-dominating function $f = (V_0, V_1, V_2)$ such that V_0 is independent. The *outer-independent* $\{2\}$-*domination number* $\gamma_{oi\{2\}}(G)$ of G is the minimum weight of an OI$\{2\}$D-function. A $\gamma_{oi\{2\}}(G)$-function f is an OI$\{2\}$D-function of G with $f(V) = \gamma_{oi\{2\}}(G)$. For the sake of simplicity, we will write OI$\{2\}$-domination number instead of outer-independent $\{2\}$-domination number. It is worth noting that if G is a non-connected graph with components $G_1, \ldots, G_s$, then $\gamma_{oi\{2\}}(G) = \sum_{i=1}^{s} \gamma_{oi\{2\}}(G)$, and as a result we will only consider throughout this paper connected graphs.

In this paper, we first determine the exact value of the OI$\{2\}$-domination number for perfect binary trees. Then we provide a lower bound and an upper bound on the OI$\{2\}$-domination number for trees in terms of the covering number, the independence number, the number of leaves and the number of stems.

We close this section by the following results that are useful in what follows.

Proposition 1.1 *If v is a stem in a graph G, then there exists a $\gamma_{oi\{2\}}(G)$-function that assigns 2 to v and 0 to every leaf neighbor of w.*

Proposition 1.2 *If G is an isolate-free graph, then there exists a $\gamma_{oi\{2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that $S(G) \subset V_2$.*

Proposition 1.3 ([4]) *For every integer $n \geq 1$, $\gamma_{oi\{2\}}(P_n) = \left\lceil \frac{2(n+1)}{3} \right\rceil$.*

2. Perfect binary trees

In this section we provide the exact value of the OI$\{2\}$-domination number for perfect binary trees. A *binary tree* is an arborescence in which each vertex can have no more than 2 children. A *perfect binary tree* is a binary tree in which every internal vertex has two children and all leaves have the same depth or level, i.e. the distance from a leaf to the root is the same for any leaf of the tree. In the following, we consider a perfect binary tree T with root v and level $k \geq 0$. Since the number of vertices in the level t is 2^t, let $v_{t,1}, \ldots v_{t,2^t}$ denote the set of vertices in the t'th level from left to right. Note that when $t = 0$, $v_{0,1}$ is the root v of T. Thus for level $t \geq 1$, vertices $v_{t,2s-1}$ and $v_{t,2s}$ are the two children of $v_{t-1,s}$. Furthermore, for each $t \geq 1$ and $j \in \{1, 2, \ldots, 2^t\}$, let $T_{v_{t,j}}$ be the component of T containing $v_{t,j}$ after the removal of the parent of $v_{t,j}$. Note that if $t \in \{1, \ldots, k-1\}$, then $T_{v_{t,j}}$ is also a perfect binary tree.

Now we are ready to state the following.

Theorem 2.1 *For any perfect binary tree T with level $k \geq 1$,*

$$\gamma_{oi\{2\}}(T) = \begin{cases} 2^k + 3(2^2 + 2^5 + \cdots + 2^{k-4}) + 2 & \text{if} \quad k \equiv 0 \pmod 3 \\ 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}) & \text{if} \quad k \equiv 1 \pmod 3 \\ 2^k + 3(2 + 2^4 + \cdots + 2^{k-4}) & \text{if} \quad k \equiv 2 \pmod 3. \end{cases}$$

Moreover, T has a unique $\gamma_{oi\{2\}}(T)$-function when $k \equiv 1, 2 \pmod 3$ and T has exactly four $\gamma_{oi\{2\}}(T)$-functions when $k \equiv 0 \pmod 3$.

Proof: Let T be a perfect binary tree with level $k \geq 1$, and let v be the root of T. Define on T the following functions according to the values of k.

(i) If $k \equiv 0 \pmod 3$, then consider the four functions defined as follows: let $f_1(v) = 2$, $f_1(u) = 2$ for all u such that $d(u, v) = k - 1$, $f_1(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 0, 2 \pmod 3$ and $f_1(x) = 0$ otherwise. Let $f_2(v_{1,1}) = f_2(v_{1,2}) = 1$, $f_2(u) = 2$ for all u such that $d(u, v) = k - 1$, $f_2(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 0, 2 \pmod 3$ and $f_2(x) = 0$ otherwise. Let $f_3(v_{1,1}) = f_3(v) = 1$, $f_3(u) = 2$ for all u such that $d(u, v) = k - 1$, $f_3(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 0, 2 \pmod 3$ and $f_3(x) = 0$ otherwise. Let $f_4(v_{1,2}) = f_4(v) = 1$, $f_4(u) = 2$ for all u such that $d(u, v) = k - 1$, $f_4(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 0, 2 \pmod 3$ and $f_4(x) = 0$ otherwise.
(ii) If $k \equiv 1 \pmod 3$, then let $f(u) = 2$ for all u such that $d(u, v) = k - 1$, $f(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 0, 1 \pmod 3$ and $f(x) = 0$ otherwise.
(iii) If $k \equiv 2 \pmod 3$, then let $f(u) = 2$ for all u such that $d(u, v) = k - 1$, $f(u) = 1$ for all u such that $d(u, v) \leq k - 3$ and $d(u, v) \equiv 1, 2 \pmod 3$ and $f(x) = 0$ otherwise.

It is easy to verify that each of the functions defined above is an OI{2}D-function of T of weight as indicated in the statement according to the possible values of k, thereby yielding the upper bound.

Our aim in the following is to simultaneously prove the inverse inequality and that the functions defined previously in items (ii) and (iii) are the unique $\gamma_{oi\{2\}}(T)$-functions for their respective values of k and the four functions defined previously in item (i) are the only $\gamma_{oi\{2\}}(T)$-functions when $k \equiv 0 \pmod 3$. We proceed by induction on k. Since the result is true when $k \in \{1, 2, 3\}$, establishing the base case, let us assume that $k \geq 4$. Let f be a $\gamma_{oi\{2\}}(T)$-function, and consider the following three cases.

Case 1. $k \equiv 1 \pmod 3$.
First assume that $f(v) = 0$. Note that $k \geq 4$ and $k - 1 \equiv 0 \pmod 3$. Then the function f restricted to $V(T_{v_{1,i}})$ for $i \in \{1, 2\}$ is an OI{2}D-function of $T_{v_{1,i}}$, and thus the induction hypothesis on $T_{v_{1,1}}$ and $T_{v_{1,2}}$ leads to

$$\omega(f) \geq 2\left(2^{k-1} + 3(2^2 + 2^5 + \cdots + 2^{k-5}) + 2\right) = 2^k + 3(1 + 2^3 + 2^6 + \cdots + 2^{k-4}) + 1,$$

and the desired equality follows.

Assume now that $f(v) = 1$. Since one of the two children must be assigned at least 1, assume, without loss of generality, that $f(v_{1,1}) = \max\{f(v_{1,1}), f(v_{1,2})\} \geq 1$. We distinguish three situations.

1. $f(v_{1,1}) = 2$.
   Since $k \geq 4$ and f is an OI{2}D-function of T we must have $f(N[v_{2,1}]) \geq 3$, and likewise $f(N[v_{2,2}]) \geq 3$. But then reassigning $v_{1,1}$ the value 1 instead of 2 provides an OI{2}D-function of T of weight less than $\omega(f)$ leading to a contradiction.

2. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 0$.
   Since f is an OI{2}D-function we must have $f(v_{2,3}) \geq 1$ and $f(v_{2,4}) \geq 1$ and thus the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$. If $f(v_{2,i}) \geq 1$ for some $i \in \{1, 2\}$, then the function $f_{1,1}$ is an OI{2}D-function of $T_{v_{1,1}}$ and applying the induction hypothesis on each $T_{1,i}$ for $i \in \{1, 2\}$, we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{1,1}) + \omega(f_{1,2}) + 1\\ &\geq 2\left(2^{k-1} + 3(2^2 + 2^5 + \cdots + 2^{k-5}) + 2\right) + 1\\ &= 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}) + 2.\end{aligned}$$

   Hence we assume that $f(v_{2,i}) = 0$ for each $i \in \{1, 2\}$, leading to $f(v_{3,i}) \geq 1$ for each $i \in \{1, 2, 3, 4\}$. Then the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for $i = 1, 2$. Using the induction hypothesis we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{2,1}) + \omega(f_{2,2}) + \omega(f_{1,2}) + 2\\ &\geq 2\left(2^{k-2} + 3(2 + 2^4 + \cdots + 2^{k-6})\right) + \left(2^{k-1} + 3(2^2 + 2^5 + \cdots + 2^{k-5}) + 2\right) + 2\\ &= 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}) + 1.\end{aligned}$$

3. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 1$.
   If $f(v_{2,i}) \geq 1$ for some $i \in \{1,2\}$ and $f(v_{2,i}) \geq 1$ for some $i \in \{3,4\}$, then reassigning v the value 0, provides an OI{2}D-function of T of weight less than $\omega(f)$, a contradiction. Hence, without loss of generality, we may assume that $f(v_{2,1}) = f(v_{2,2}) = 0$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1,2,3,4\}$ and thus the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for $i \in \{1,2\}$. Now, assume that $f(v_{2,i}) \geq 1$ for some $i \in \{3,4\}$. Then the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$ and as before we get $\omega(f) \geq 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}) + 1$. Hence we can assume that $f(v_{2,i}) = 0$ for each $i \in \{3,4\}$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{5,6,7,8\}$, and thus $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{3,4\}$. Applying the induction hypothesis on $T_{v_{2,i}}$ for $i \in \{1,2,3,4\}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{2,1}) + \omega(f_{2,2}) + \omega(f_{2,3}) + \omega(f_{2,4}) + 3\\ &\geq 4\big(2^{k-2} + 3(2 + 2^4 + \cdots + 2^{k-6})\big) + 3\\ &= 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}).\end{aligned}$$

Consequently, $\gamma_{oi\{2\}}(T) = 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4})$, and therefore the function f restricted to each $T_{v_{2,i}}$ is a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function which is additionally unique. Hence f is a unique $\gamma_{oi\{2\}}(T)$-function.

Finally assume that $f(v) = 2$. First, suppose that $f(v_{1,1}) = f(v_{1,2}) = 0$. Since f is an OI{2}D-function we have $f(v_{2,i}) \geq 1$ for each $i \in \{1,2,3,4\}$ and thus the function $f_{1,i}$ is an OI{2}D-function of $T_{v_{1,i}}$ for each $i \in \{1,2\}$. Using the induction hypothesis on $T_{v_{1,i}}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{1,1}) + \omega(f_{1,2}) + 2\\ &\geq 2\big(2^{k-1} + 3(2^2 + 2^5 + \cdots + 2^{k-5}) + 2\big) + 2\\ &= 2^k + 3(2^0 + 2^3 + \cdots + 2^{k-4}) + 3.\end{aligned}$$

We now suppose, without loss of generality, that $f(v_{1,1}) \geq 1$. Since f is an OI{2}D-function of T we must have $f(v_{1,2}) \geq 1$ or $f(v_{2,i}) \geq 1$ for each $i \in \{3,4\}$. Thus reassigning v the value 1, provides an OI{2}D-function of T of weight less that $\omega(f)$ leading to a contradiction.

Case 2. $k \equiv 2 \pmod 3$.
We first note that since $k \equiv 2 \pmod 3$ and k is assumed to at least 4, we have $k \geq 5$. As before, three situations will be considered depending on $f(v) \in \{0,1,2\}$. So we start by assuming that $f(v) = 0$. Then the function f restricted to each $T_{v_{1,i}}$ for $i \in \{1,2\}$ is an OI{2}D-function of $T_{v_{1,i}}$. Since $k-1 \equiv 1 \pmod 3$, we deduce by using the induction hypothesis on each $T_{v_{1,i}}$ that

$$\omega(f) \geq 2\big(2^{k-1} + 3(2^0 + 2^3 + \cdots + 2^{k-5})\big) = 2^k + 3(2 + 2^4 + \cdots + 2^{k-4}).$$

Consequently, the desired equality is obtained, that is $\gamma_{oi\{2\}}(T) = 2^k + 3(2 + 2^4 + \cdots + 2^{k-4})$, and therefore the function f restricted to $T_{v_{1,i}}$ is a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function which is additionally unique. Hence f is a unique $\gamma_{oi\{2\}}(T)$-function.

Assume now that $f(v) = 1$. As seen in Case 1, we have $\max\{f(v_{1,1}), f(v_{1,2})\} = 1$. Hence, without loss of generality, let $f(v_{1,1}) = 1$. We distinguish two situations.

1. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 0$.
   As for Item (2) of Case 1, the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$. Moreover, if $f(v_{2,i}) \geq 1$ for some $i \in \{1,2\}$, then the function $f_{1,1}$ is an OI{2}D-function of $T_{v_{1,1}}$ and by applying the induction hypothesis on $T_{1,i}$ for each $i \in \{1,2\}$, we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{1,1}) + \omega(f_{1,2}) + 1\\ &\geq 2\big(2^{k-1} + 3(2^0 + 2^3 + \cdots + 2^{k-5})\big) + 1\\ &= 2^k + 3(2^1 + 2^4 + \cdots + 2^{k-4}) + 1.\end{aligned}$$

   So now we can assume that $f(v_{2,i}) = 0$ for each $i \in \{1,2\}$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1,2,3,4\}$, and thus the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{1,2\}$. Now, using the induction hypothesis, we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{2,1}) + \omega(f_{2,2}) + \omega(f_{1,2}) + 2\\ &\geq 2\big(2^{k-2} + 3(2^2 + 2^5 + \cdots + 2^{k-6}) + 2\big) + \big(2^{k-1} + 3(2^0 + 2^3 + \cdots + 2^{k-5})\big) + 2\\ &= 2^k + 3(2^1 + 2^4 + \cdots + 2^{k-4}) + 3.\end{aligned}$$

2. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 1$.
   If $f(v_{2,i}) \geq 1$ for some $i \in \{1,2\}$ and $f(v_{2,i}) \geq 1$ for some $i \in \{3,4\}$, then reassigning v the value 0 instead of 1 provides an OI{2}D-function of T of weight less than $\omega(f)$, a contradiction. Thus, without loss of generality, we may assume that $f(v_{2,1}) = f(v_{2,2}) = 0$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1,2,3,4\}$ and so the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{1,2\}$. On the other hand, if $f(v_{2,i}) \geq 1$ for some $i \in \{3,4\}$, then the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$ and as before we get $\omega(f) \geq 2^k + 3(2^1 + 2^4 + \cdots + 2^{k-4}) + 3$. Hence we will assume that $f(v_{2,i}) = 0$ for each $i \in \{3,4\}$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{5,6,7,8\}$, and thus $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{3,4\}$. Now, applying the induction hypothesis on $T_{v_{2,i}}$ for each $i \in \{1,2,3,4\}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{2,1}) + \omega(f_{2,2}) + \omega(f_{2,3}) + \omega(f_{2,4}) + 3\\ &\geq 4\big(2^{k-2} + 3(2^2 + 2^5 + \cdots + 2^{k-6}) + 2\big) + 3\\ &= 2^k + 3(2^1 + 2^4 + \cdots + 2^{k-4}) + 5.\end{aligned}$$

Finally, assume that $f(v) = 2$. As in Case 1, we can see that $f(v_{1,1}) = f(v_{1,2}) = 0$. Since vertices assigned 0 under f is an independent set, we have $f(v_{2,i}) \geq 1$ for each $i \in \{1,2,3,4\}$ and so the function $f_{1,i}$ is an OI{2}D-function of $T_{v_{1,i}}$ for $i \in \{1,2\}$. Using the induction hypothesis on $T_{v_{1,i}}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{1,1}) + \omega(f_{1,2}) + 2\\ &\geq 2\big(2^{k-1} + 3(2^0 + 2^3 + \cdots + 2^{k-5})\big) + 2\\ &= 2^k + 3(2^1 + 2^4 + \cdots + 2^{k-4}) + 2.\end{aligned}$$

Case 3. $k \equiv 0 \pmod 3$.
As with the two previous cases, we will consider three situations depending on whether $f(v) \in \{0,1,2\}$. Suppose first that $f(v) = 2$. As in Case 1, we can see that $f(v_{1,1}) = f(v_{1,2}) = 0$. Also, since vertices assigned 0 under f is an independent set, $f(v_{2,i}) \geq 1$ for each $i \in \{1,2,3,4\}$. Thus the function $f_{1,i}$ is an OI{2}D-function of $T_{v_{1,i}}$ for $i \in \{1,2\}$, and by using the induction hypothesis on $T_{v_{1,i}}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{1,1}) + \omega(f_{1,2}) + 2\\ &\geq 2\big(2^{k-1} + 3(2 + 2^4 + \cdots + 2^{k-5})\big) + 2\\ &= 2^k + 3(2^2 + 2^5 + \cdots + 2^{k-4}) + 2.\end{aligned}$$

Consequently, $\gamma_{oi\{2\}}(T) = 2^k + 3(2^2 + 2^5 + \cdots + 2^{k-4}) + 2$, and therefore the function f restricted to each $T_{v_{1,i}}$ is a $\gamma_{oi\{2\}}(T_{v_{1,i}})$-function which is additionally unique. Hence f is considered as one of the $\gamma_{oi\{2\}}(T)$-functions. For the remaining three $\gamma_{oi\{2\}}(T)$-functions, simply consider the assignments of the vertices $v, v_{1,1}$ and $v_{1,2}$ as follows: 1,1,0 or 1,0,1 or 0,1,1, respectively.

Suppose now that $f(v) = 1$. As in Case 1, we have $\max\{f(v_{1,1}), f(v_{1,2})\} = 1$. Without loss of generality, let $f(v_{1,1}) = 1$. We distinguish two situations.

1. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 0$.
   As for Item (2) of Case 1, the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$. For the sequel, we start by assuming that $f(v_{2,i}) = 0$ for each $i \in \{1,2\}$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1,2,3,4\}$, and thus the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{1,2\}$ which are not $\gamma_{oi\{2\}}(T_{2,i})$-functions (since by looking at Case 1, when the root is assigned 0, the weight of $T_{v_{2,i}}$ equals

$\gamma_{oi\{2\}}(T_{2,i})+1)$. Now, using the induction hypothesis we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{2,1})+\omega(f_{2,2})+\omega(f_{1,2})+2\\ &\geq 2\big(2^{k-2}+3(2^0+2^3+\cdots+2^{k-6})+1\big)+\big(2^{k-1}+3(2+2^4+\cdots+2^{k-5})\big)+2\\ &= 2^k+3(2^2+2^5+\cdots+2^{k-4})+4.\end{aligned}$$

Henceforth, we will assume that $f(v_{2,i})\geq 1$ for some $i\in\{1,2\}$, say $i=1$, and consider the following.

- $f(v_{2,2})=0$.
  Clearly, $f(v_{3,3})\geq 1$ and $f(v_{3,4})\geq 1$, since no two vertices assigned 0 are adjacent. Hence $f_{2,2}$ is an OI{2}D-function of $T_{v_{2,2}}$ which is not a $\gamma_{oi\{2\}}(T_{v_{2,2}})$-function (based on the same argument as above). In addition, define the function g on $T_{v_{2,1}}$ by $g(v_{2,1})=2$ and $g(x)=f(x)$ for any other vertex of $T_{v_{2,1}}$. Clearly, g is an OI{2}D-function of $T_{v_{2,1}}$ which is not a $\gamma_{oi\{2\}}(T_{v_{2,1}})$-function (see the corresponding situation in Case 1). Now, applying the induction hypothesis we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{2,1})+\omega(f_{2,2})+\omega(f_{1,2})+1\\ &\geq 2\big(2^{k-2}+3(2^0+2^3+\cdots+2^{k-6})+1\big)+\big(2^{k-1}+3(2+2^4+\cdots+2^{k-5})\big)+1\\ &= 2^k+3(2^2+2^5+\cdots+2^{k-4})+3.\end{aligned}$$

- $f(v_{2,2})\geq 1$.
  If $f(v_{2,2})=2$, then the function $f_{2,2}$ is an OI{2}D-function of $T_{v_{2,2}}$ which is not a $\gamma_{oi\{2\}}(T_{v_{2,2}})$-function (see Case 1, when $f(v)=2$). Also, considering the OI{2}-function g on $T_{v_{2,1}}$ as defined for the previous item, we obtain $\omega(f)\geq 2^k+3(2^2+2^5+\cdots+2^{k-4})+3$. Hence, we assume that $f(v_{2,2})=1$. Since $f(v_{1,2})$ was assumed to be at least 1, so by analogy to the previous situation when $f(v_{2,2})=2$, we have to assume that $f(v_{1,2})=1$. Now, if $\max\{f(v_{3,2i-1}),f(v_{3,2i})\}\geq 1$ for each $i\in\{1,2\}$, then the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i\in\{1,2\}$, and by using the induction hypothesis we have

$$\begin{aligned}\omega(f) &= \omega(f_{2,1})+\omega(f_{2,2})+\omega(f_{1,2})+2\\ &\geq 2\big(2^{k-2}+3(2^0+2^3+\cdots+2^{k-6})\big)+\big(2^{k-1}+3(2+2^4+\cdots+2^{k-5})\big)+2\\ &= 2^k+3(2^2+2^5+\cdots+2^{k-4})+2.\end{aligned}$$

  Consequently, $\gamma_{oi\{2\}}(T)=2^k+3(2^2+2^5+\cdots+2^{k-4})+2$, and therefore the function f restricted to each $T_{v_{1,2}}$, $T_{v_{2,1}}$ and $T_{v_{2,2}}$ is a $\gamma_{oi\{2\}}(T_{v_{1,2}})$-function, $\gamma_{oi\{2\}}(T_{v_{2,1}})$-function and $\gamma_{oi\{2\}}(T_{v_{2,2}})$-function respectively, which are additionally unique. Hence f is considered as one of the four $\gamma_{oi\{2\}}(T)$-functions already provided above.
  Henceforth we can assume in the following that $\max\{f(v_{3,2i-1}),f(v_{3,2i})\}=0$ for some $i\in\{1,2\}$, say $i=1$. Since no two vertices assigned 0 under f are adjacent, we have $f(v_{4,1})\geq 1$ and $f(v_{4,2})\geq 1$, and so the function $f_{3,i}$ is an OI{2}D-function of $T_{v_{3,i}}$ for each $i\in\{1,2\}$. Moreover, if $\max\{f(v_{3,3}),f(v_{3,4})\}\geq 1$, then the function $f_{2,2}$ is an OI{2}D-function of $T_{v_{2,2}}$ and by using the induction hypothesis we obtain

$$\begin{aligned}\omega(f) &= \omega(f_{3,1})+\omega(f_{3,2})+\omega(f_{2,2})+\omega(f_{1,2})+3\\ &\geq 2\big(2^{k-3}+3(2^2+2^5+\cdots+2^{k-7})+2\big)+\big(2^{k-2}+3(2^0+2^3+\cdots+2^{k-6})\big)\\ &\quad+\big(2^{k-1}+3(2^1+2^4+\cdots+2^{k-5})\big)+3\\ &= 2^k+3(2^2+2^5+\cdots+2^{k-4})+4.\end{aligned}$$

  Finally, we assume that $\max\{f(v_{3,3}),f(v_{3,4})\}=0$. As seen above for $f_{3,1}$ and $f_{3,2}$, we will similarly have the functions $f_{3,3}$ and $f_{3,4}$ as OI{2}D-functions, and by using the induction hypothesis we will have

$$\begin{aligned}\omega(f) &= \omega(f_{3,1}) + \omega(f_{3,2}) + \omega(f_{3,3}) + \omega(f_{3,4}) + \omega(f_{1,2}) + 4\\ &\geq 4\big(2^{k-3} + 3(2^2 + 2^5 + \cdots + 2^{k-7}) + 2\big) + \big(2^{k-1} + 3(2^1 + 2^4 + \cdots + 2^{k-5})\big) + 4\\ &= 2^k + 3(2^2 + 2^5 + \cdots + 2^{k-4}) + 6.\end{aligned}$$

2. $f(v_{1,1}) = 1$ and $f(v_{1,2}) = 1$.
If $f(v_{2,i}) \geq 1$ for some $i \in \{1, 2\}$ and $f(v_{2,i}) \geq 1$ for some $i \in \{3, 4\}$, then reassigning v the value 0 instead of 1 provides an OI{2}D-function of T of weight less than $\omega(f)$, a contradiction. Hence, without loss of generality, assume that $f(v_{2,1}) = f(v_{2,2}) = 0$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1, 2, 3, 4\}$ and so the function $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ but it is not a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function for each $i \in \{1, 2\}$ (see Case 1, when $f(v) = 0$). Now, if $f(v_{2,3}) \geq 1$ or $f(v_{2,4}) \geq 1$, then the function $f_{1,2}$ is an OI{2}D-function of $T_{v_{1,2}}$ which is not a $\gamma_{oi\{2\}}(T_{1,2})$-function (see Case 2 when $f(v) = 1$). But then replacing $f_{1,2}$ by a minimum OI{2}D-function provides an OI{2}D-function of T with weight less than $\omega(f)$ leading to a contradiction. Therefore, we assume that $f(v_{2,i}) = 0$ for each $i \in \{3, 4\}$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{5, 6, 7, 8\}$, and thus $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ for each $i \in \{3, 4\}$ which is not, as above, of minimum weight. Now, applying the induction hypothesis on $T_{v_{2,i}}$ for each $i \in \{1, 2, 3, 4\}$, we get

$$\begin{aligned}\omega(f) &= \omega(f_{2,1}) + \omega(f_{2,2}) + \omega(f_{2,3}) + \omega(f_{2,4}) + 3\\ &\geq 4\big(2^{k-2} + 3(2^0 + 2^3 + \cdots + 2^{k-6}) + 1\big) + 3\\ &\geq 2^k + 3(2^2 + 2^5 + \cdots + 2^{k-4}) + 7.\end{aligned}$$

Finally, assume that $f(v) = 0$. Since vertices assigned 0 under f are not adjacent, we must have $\min\{f(v_{1,1}), f(v_{1,2})\} \geq 1$. Without loss of generality, assume that $f(v_{1,1}) = \max\{f(v_{1,1}), f(v_{1,2})\}$. First let $f(v_{1,1}) = 2$. If $f(v_{2,i}) \geq 1$ for some $i \in \{1, 2\}$, then by reassigning $v_{1,1}$ the value 1 instead of 2, we obtain an OI{2}D-function on T with weight less than $\omega(f)$, a contradiction. Hence $f(v_{2,1}) = f(v_{2,2}) = 0$. It follows that $f(v_{3,i}) \geq 1$ for each $i \in \{1, 2, 3, 4\}$, and thus $f_{2,i}$ is an OI{2}D-function of $T_{v_{2,i}}$ which is not a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function for each $i \in \{1, 2\}$ (see Case 1 when $f(v) = 0$). But then by considering a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function instead of $f_{2,i}$ we obtain an OI{2}D-function of T with weight less than $\omega(f)$, a contradiction. Thus $f(v_{1,1}) = 1$ and by our earlier assumption we also have $f(v_{1,2}) = 1$. Since $v_{1,1}$ must satisfy $f(N[v_{1,1}]) \geq 2$, it follows that $\min\{f(v_{2,1}), f(v_{2,2})\} \geq 1$ and likewise $\min\{f(v_{2,3}), f(v_{2,4})\} \geq 1$ because of $f(v_{1,2}) = 1$. Assume, without loss of generality, that $f(v_{2,1}) \geq 1$ and $f(v_{2,3}) \geq 1$. If $f(v_{2,2}) = 0$, then $\min\{f(v_{3,3}), f(v_{3,4}\} \geq 1$ and thus the function $f_{2,2}$ is an OI{2}D-function of $T_{v_{2,2}}$ which is not a $\gamma_{oi\{2\}}(T_{v_{2,2}})$-function (see Case 1 when $f(v) = 0$). But then considering a $\gamma_{oi\{2\}}(T_{v_{2,2}})$-function instead of $f_{2,2}$ we obtain an OI{2}D-function of T of weight less than $\omega(f)$, a contradiction. Hence $f(v_{2,2}) \geq 1$, and likewise $f(v_{2,4}) \geq 1$. Now, if $f(v_{2,i}) = 2$ for some $i \in \{1, 2, 3, 4\}$, then it follows from $k \geq 4$ that reassigning such a $v_{2,i}$ the value 1 instead of 2 provides an OI{2}D-function of T with weight less than $\omega(f)$, a contradiction. Therefore, $f(v_{2,i}) = 1$ for each $i \in \{1, 2, 3, 4\}$. On the other hand, if $f(v_{3,1}) = f(v_{3,2}) = 0$, then, since vertices assigned 0 are not adjacent, we have $f(v_{4,i}) \geq 1$ for $i \in \{1, 2, 3, 4\}$. In this case, the function $f_{3,i}$ is an OI{2}D-function of $T_{v_{3,i}}$ for each $i \in \{1, 2\}$ and by the induction hypothesis we have

$$\begin{aligned}\omega(f_{2,1}) &= \omega(f_{3,1}) + \omega(f_{3,2}) + 1\\ &\geq 2\big(2^{k-3} + 3(2^2 + 2^5 + \cdots + 2^{k-7}) + 2\big) + 1\\ &= 2^{k-2} + 3(2^0 + 2^3 + 2^6 + \cdots + 2^{k-6}) + 2.\end{aligned}$$

Clearly, in this case, considering a $\gamma_{oi\{2\}}(T_{v_{2,1}})$-function instead of $f_{2,1}$ would provide an OI{2}D-function of T with weight less than $\omega(f)$, a contradiction. Therefore, $\max\{f(v_{3,1}), f(v_{3,1})\} \geq 1$. It follows that $f_{2,1}$ is an OI{2}D-function of $T_{v_{2,1}}$, and we deduce from the minimality of f that $f_{2,1}$ is a $\gamma_{oi\{2\}}(T_{v_{2,1}})$-function which is additionally unique. Likewise, $f_{2,i}$ is a $\gamma_{oi\{2\}}(T_{v_{2,i}})$-function which is additionally unique for each $i \in \{2, 3, 4\}$. Hence f is a one of the four $\gamma_{oi\{2\}}(T)$-functions already provided above. This completes the proof. □

3. Trees

In this section, we present two sharp bounds on the OI{2}-domination number for trees in terms of the covering number, the independence number, the number of stems and the number of leaves. We start by giving the following lower bound.

Theorem 3.1 *For any tree T,*
$$\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil.$$

Proof: We proceed by induction on the order n of T. If $n=1$, then clearly $\beta(T)=0$ and $\gamma_{oi\{2\}}(T) = 2 > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. If $n=2$, then $T \cong P_2$ and $\gamma_{oi\{2\}}(T) = 2 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Moreover, if $n=3$, then $T \cong P_3$ and we have $\gamma_{oi\{2\}}(T) = 2 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. These establish the base cases. Let $n \geq 4$ and assume that if T' is a tree of order n', with $n' < n$, then $\gamma_{io\{2\}}(T') \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil$. Let T be a tree of order n. If $\text{diam}(T)=2$, then T is a star $K_{1,n-1}$, where $\gamma_{oi\{2\}}(T) = 2 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. If $\text{diam}(T)=3$, then T is a double star $S_{r,s}$ $(r \geq s \geq 1)$, where $\gamma_{oi\{2\}}(T) = 4 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Hence, in the following we may assume that $\text{diam}(T) \geq 4$, and let f be a $\gamma_{oi\{2\}}(T)$-function.

If T has a strong stem u with at least two leaves, say u_1 and u_2, then let $T' = T - u_1$. By Observation 1.1, we may assume that $f(u)=2$ and $f(u_1)=0$. Therefore, the function f restricted to T' is an OI{2}D-function of T' and thus $\gamma_{oi\{2\}}(T') \leq \gamma_{oi\{2\}}(T)$. Moreover, it is easy to see that $\beta(T)=\beta(T')$. Now, by the induction hypothesis on T' we obtain $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T') \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Henceforth, we can assume in the sequel that T has no strong stem.

Let $v_1v_1 \ldots v_k$ be a diametral path in T and root T in v_k. Since T has no strong stem, $\deg_T(v_2)=2$ and any child of v_3 is either a leaf or a stem of degree 2.

First assume that $\deg_T(v_3) \geq 3$. Observe that if v_3 is a stem, then v_3 has exactly one leaf neighbor, say w. By Observation 1.1 we can assume that $f(w)=f(v_1)=0$ and $f(v_2)=f(v_3)=2$. While, if v_3 is not a stem, then let u_2 be a child of v_3 other than v_2 and u_1 the leaf adjacent to u_2. Again by Observation 1.1, we assume that $f(v_2)=f(u_2)=2$ and $f(v_1)=f(u_1)=0$. Now, consider the tree $T' = T - \{v_1, v_2\}$. In either of the above two situations, the restriction of f on T' is an OI{2}D-function on T' of weight $\omega(f)-2$. Furthermore, again in either situation, it is easily observed that $\beta(T)=\beta(T')+1$. Therefore, it follows from the induction hypothesis that $\gamma_{oi\{2\}}(T)-2 = \omega(f)-2 \geq \gamma_{oi\{2\}}(T') \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil$, leading to $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4(\beta(T)-1)+2}{3} \right\rceil + 2 \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$, as desired.

Hence, from now on, we assume that $\deg_T(v_3)=2$. As a result, we may assume that the maximal subtree rooted at any child of v_4 with depth 2 is a path P_3 attached by one of its leaves at v_4. Now, assume that v_4 is a stem and let w be a leaf neighbor of v_4. By Observation 1.1, we assume that $f(w)=f(v_3)=f(v_1)=0$ and $f(v_4)=f(v_2)=2$. Let $T' = T - T_{v_3}$. Clearly, the restriction of f on T' is an OI{2}D-function on T' of weight $\gamma_{oi\{2\}}(T)-2$, and applying the induction hypothesis it follows that $\gamma_{oi\{2\}}(T)-2 \geq \gamma_{oi\{2\}}(T') \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil$. Moreover, since v_4 is a stem, v_4 belongs to some $\beta(T')$-set and thus one can easily see that $\beta(T)=\beta(T')+1$. Therefore, $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T')+2 \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil + 2 > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$, as desired. In the following, v_4 is assumed to be different from a stem. If v_4 has a child w_3 with depth 2 other than v_3, then let $v_4w_3w_2w_1$ be a pendant path in T. Since w_3 plays the same role as v_3, we have $\deg(w_3)=\deg(w_2)=2$. By Observation 1.1, we may assume that $f(v_2)=f(w_2)=2$, and thus $f(v_i)=f(w_i)=0$ for $i \in \{1,3\}$. Also, since no two vertices assigned 0 are adjacent, we deduce that $f(v_4) \neq 0$. Now, let $T' = T - T_{v_3}$. Clearly, the restriction of f to T' is an OI{2}D-function of T' with weight $\gamma_{oi\{2\}}(T)-2$. Also as before, one can see that $\beta(T)=\beta(T')+1$. Applying the induction hypothesis, it follows that $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T')+2 \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Hence, we can assume that v_4 has no child with depth 2 other than v_3. Moreover, we recall that v_4 is not a stem. Now, if $\deg_T(v_4) \geq 4$, then v_4 has at least two children w_2 and z_2 that are stems. Let w_1 and z_1

be leaf neighbors of w_2 and z_2, respectively. By Observation 1.1, we assume that $f(w_2) = f(z_2) = 2$ and $f(w_1) = f(z_1) = 0$. It follows that the restriction of f to the tree $T' = T - \{z_1, z_2\}$ is an OI{2}D-function of T' with weight $\gamma_{oi\{2\}}(T) - 2$. Since $\beta(T) = \beta(T') + 1$, it follows from the induction hypothesis that $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T')+2 \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. If $\deg_T(v_4) = 3$, then v_4 has exactly one child w_2 as a stem. Let w_1 be a leaf adjacent to w_2. By the definition of an OI{2}D-function, we have $f(T_{v_4}) \geq 5$. In this case, consider the tree $T' = T - T_{v_4}$, and define the function g on $V(T')$ as follows: if $f(T_{v_4}) = 5$, then let $g(v_5) = \min\{2, f(v_5)+1\}$ and $g(x) = f(x)$ for $x \in V(T') - \{v_5\}$, while if $f(T_{v_4}) \geq 6$, then let $g(v_5) = 2$ and $g(x) = f(x)$ for $x \in V(T') - \{v_5\}$. Clearly, g is an OI{2}D-function on T' of weight at most $\omega(f) - 4$, and thus $\gamma_{oi\{2\}}(T') \leq \gamma_{oi\{2\}}(T) - 4$. On the other hand, one can see that $\beta(T) = \beta(T')+3$. Now, applying the inductive hypothesis, we obtain $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T') + 4 \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil + 4 = \left\lceil \frac{4(\beta(T)-3)+2}{3} \right\rceil + 4 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$, as desired.

From now on, we will assume that $\deg(v_4) = 2$. We distinguish the following cases.

Case 1. $f(v_5) = 2$ or $f(v_4) = 0$.
Let $T' = T - T_{v_4}$. It is not hard to see that $\beta(T) = \beta(T')+2$, and that $\sum_{i=1}^{3} f(v_i) = 3$. So, the restriction of f to T' is an OI{2}D-function on T' of weight $\omega(f) - 3$. It follows from the induction hypothesis that $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T') + 3 \geq \left\lceil \frac{4\beta(T')+1}{3} \right\rceil + 3 \geq \left\lceil \frac{4(\beta(T)-2)+2}{3} \right\rceil + 3 > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$.

Case 2. $\deg(v_5) \geq 3$ and $f(v_5) = 0$.
Let $T' = T - T_{v_4}$. As in Case 1, $\beta(T) = \beta(T') + 2$ and it is easy to see that $\sum_{i=1}^{3} f(v_i) = 4$. Also, since no two vertices assigned 0 are adjacent, all neighbors of v_5 are assigned a non-zero value, and thus $\sum_{x \in N(v_5)-\{v_4\}} f(x) \geq 2$ because of $\deg(v_5) \geq 3$. Consequently, the function f restricted to T' is an OI{2}D-function of T' of weight $\omega(f) - 4$ and the induction hypothesis leads to $\gamma_{oi\{2\}}(T) > \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$.

Case 3. $\deg(v_5) \geq 3$, $f(v_5) = 1$ and $f(v_4) \geq 1$.
If $\sum_{x \in N(v_5)-\{v_4\}} f(x) \geq 1$, then the function f restricted to T' is an OI{2}D-function of $T' = T - T_{v_4}$ of weight $\omega(f)-3$ and as before we have $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Hence we assume that $\sum_{x \in N(v_5)-\{v_4\}} f(x) = 0$. Therefore, v_5 is not a stem, and has no child (besides v_4) with depth 1 and 3, and so any child of v_5 besides v_4 has depth 2. Let z_3 be a child of v_5 with depth 2 and let $v_5 z_3 z_2 z_1$ be a path in T. Since T has no strong stem, we have $\deg(z_2) = 2$. Also, note that z_3 cannot be a stem, for otherwise it will be assigned 2 under f. Now, if $\deg(z_3) \geq 3$, then all children of z_3 are stems of degree two, and in this case, we can consider the tree T' obtained from T by removing vertices z_1 and z_2. Clearly similar situations have been already considering yielding the desired result. Thus we can assume $\deg(z_3) = \deg(z_2) = 2$. By Observation 1.1, $f(z_2) = 2$ and $f(z_3) = f(z_1) = 0$. Let $T'' = T - T_{z_3}$. Then $\beta(T'') + 1 = \beta(T)$, and the restriction of f to T'' is an OI{2}RDF of T'' of weight $\omega(f) - 2$. As before, using the induction hypothesis we get $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+1}{3} \right\rceil$.

According to Cases 1,2, and 3, we may assume that $\deg(v_5) = 2$, $f(v_5) \neq 2$ and $f(v_4) \geq 1$. We proceed by further cases.

Case 4. $f(v_6) = 2$ or $f(v_5) = 0$.
Let $T' = T - T_{v_5}$. It is not hard to see that $\beta(T) \leq \beta(T') + 3$, $\sum_{i=1}^{4} f(v_i) = 4$, and thus the restriction of f to T' is an OI{2}D-function on T' of weight $\omega(f) - 4$. It follows from the induction hypothesis that $\gamma_{oi\{2\}}(T) \geq \gamma_{oi\{2\}}(T') + 4 \geq \left\lceil \frac{4\beta(T')+2}{3} \right\rceil + 4 \geq \left\lceil \frac{4(\beta(T)-3)+2}{3} \right\rceil + 4 = \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$.

Case 5. $\deg(v_6) \geq 3$ and $f(v_6) = 0$.
Since $f(v_6) = 0$, $f(N(v_6) - \{v_5\}) \geq 2$. Consider the tree $T' = T - T_{v_5}$, and observe that the function f restricted to T' is an OI{2}D-function on T' with weight $\omega(f) - 4$, leading to $\gamma_{oi\{2\}}(T') + 4 \leq \gamma_{oi\{2\}}(T)$. Using the induction hypothesis, we can see as before that $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$.

Case 6. $\deg(v_6) \geq 3$ and $f(v_6) = 1$.
Then clearly $\sum_{i=1}^{4} f(v_i) = 4$. Now, if $\sum_{x \in N(v_6)-\{v_5\}} f(x) \geq 1$, then the function f restricted to T' is an OI{2}D-function of T' of weight $\omega(f) - 4$ and as before we have $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$. Hence assume

that $\sum_{x\in N(v_6)-\{v_5\}} f(x) = 0$. It follows that v_6 has no child with depth 0, 1,3 and 4. If v_6 has a child z_3 with depth 2, then as in Case 3, we can see that $\gamma_{oi\{2\}}(T) \geq \left\lceil \frac{4\beta(T)+2}{3} \right\rceil$.

By the discussion before Case 3 and using Cases 4,5 and 6, we may assume that $\deg(v_6) = 2$, $f(v_6) \neq 2$, $f(v_5) \geq 1$ and $f(v_4) \geq 1$. First let $f(v_6) = 0$. Then the function f restricted to $T' = T - T_{v_6}$ is an OI{2}D-function of T' of weight at most $\omega(f) - 4$ and since $\beta(T) = \beta(T') + 3$, the desired result follows by using the induction hypothesis. Hence assume that $f(v_6) = 1$. Since $f(v_5) \geq 1$ and $\sum_{i=1}^{4} f(v_i) = 3$, then the restriction of f to the tree $T' = T - T_{v_4}$ is an OI{2}D-function of T' of weight $\omega(f) - 3$. Since $\beta(T) = \beta(T') + 2$, the desired result follows by applying the induction hypothesis. This completes the proof. □

Stars and double stars show that Theorem 3.1 is sharp.

In the next, we present an upper bound on the OI{2}-domination number for trees in terms of the independence number, the number of stems and the number of leaves. It should be noted that in the proof of the result that follows, relating the independence number of a tree T and the independence number of a subtree of T will be given without straightforward proof, but which are in fact based on the following remark. In any tree, there is a maximum independent set that contains all leaves. Moreover, we recall that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$.

Theorem 3.2 *If T is a nontrivial tree of order n, then*

$$\gamma_{oi\{2\}}(T) \leq \left\lfloor \frac{3\alpha(G) + 4s(T) - 3\ell(T)}{2} \right\rfloor.$$

Furthermore, this bound is sharp for stars and double stars.

Proof: Since $\gamma_{oi\{2\}}(T)$ is an integer, it is enough to show that for any nontrivial tree T, $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. We will proceed by induction on order $n \geq 2$. If $n = 2$, then $T \cong P_2$ and $\gamma_{oi2}(T) = 2 < \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$, establishing the base case. Let $n \geq 3$ and assume that every tree T' of order n', with $2 \leq n' < n$ satisfies $\gamma_{oi\{2\}}(T') \leq \frac{3\alpha(T')+4s(T')-3\ell(T')}{2}$. Let T be a tree of order n. If T is a star, then the function that assigns 2 to the center vertex and 0 to leaves of the star is an OI{2}D-function of T of weight 2. Since $\alpha(T) = \ell(T) = n - 1$ and $s(T) = 1$, we have $\gamma_{oi\{2\}}(T) = 2 = \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. If $\operatorname{diam}(T) = 3$, then T is a double star $T \cong S_{r,s}$ with $r \geq s \geq 1$. In this case, let u and v be the stems of the double star having respectively r and s leaf neighbors. Then $s(T) = 2, \alpha(T) = \ell(T) = r + s$ and the function that assigns 2 to u and v and 0 to the leaves of T is a $\gamma_{oi\{2\}}(T)$-function of weight 4, and so $\gamma_{oi\{2\}}(T) = 4 = \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence, we may assume that $\operatorname{diam}(T) \geq 4$ and thus $n \geq 5$.

If T has a strong stem v, then consider the tree $T' = T - u$, where u is a leaf neighbor of v. Clearly, $s(T') = s(T), \ell(T) = \ell(T') + 1$ and it is easy to see that $\alpha(T) = \alpha(T') + 1$. Now if f' is an $\gamma_{oi\{2\}}(T')$-function such that $f(v) = 2$ (Observation 1.2), then f' can be extended to an OI{2}D-function of T by assigning 0 to u, leading to $\gamma_{oi\{2\}}(T) \leq \gamma_{oi\{2\}}(T')$. Applying the induction hypothesis, it follows that $\gamma_{oi\{2\}}(T) \leq \gamma_{oi\{2\}}(T') \leq \frac{3\alpha(T')+4s(T')-3\ell(T')}{2} = \frac{3(\alpha(T)-1)+4s(T)-3(\ell(T)-1)}{2} = \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence, in the sequel, we assume that T has no strong stem. If $\Delta(T) = 2$, then T is a path and by Proposition 1.3, we have $\gamma_{oi\{2\}}(T) = \lceil \frac{2(n+1)}{3} \rceil$. Since $\alpha(P_n) = \lceil \frac{n}{2} \rceil$ and $s(P_n) = \ell(P_n) = 2$, $\frac{3\lceil \frac{n}{2} \rceil+4s(T)-3\ell(T)}{2} = \frac{3\lceil \frac{n}{2} \rceil+2}{2} \geq \left\lceil \frac{2(n+1)}{3} \right\rceil$, as desired. Hence suppose that T has maximum degree at least 3. Let z be a vertex with maximum degree, and root T at z.

First, assume that z is the unique vertex in T having degree at least three. Thus any other vertex of T has degree 1 or 2. Let $A = \{x \in C(z) \mid \operatorname{depth}(x) \equiv 0 \pmod 3\}$, $B = \{x \in C(z) \mid \operatorname{depth}(x) \equiv 1 \pmod 3\}$, $C = \{x \in C(z) \mid \operatorname{depth}(x) \equiv 2 \pmod 3\}$ and set $|A| = a, |B| = b, |C| = c$. Clearly, $a+b+c \geq 3$ because of the degree of z. Also, since each component of $T-z$ is a path, let $P_1, \ldots, P_a$ denote all paths of $T-z$ whose endvertices are in A. Note that P_i may have order 1. Similarly, let $Q_1, \ldots, Q_b$ and $W_1, \ldots, W_c$

denote all paths of $T-z$ whose endvertices are in B and C, respectively. Note that $\ell(T)=a+b+c$ and $s(T)\le a+b+c$. In addition, let $m_a=|V(P_1)|+\cdots+|V(P_a)|$, and likewise m_b and m_c are defined. Also, let p_a be the number of paths in $\{P_1,\ldots,P_a\}$ whose orders are odd, and likewise we define p_b and p_c. Clearly, every maximum independent set S of T contains $\lceil\frac{|V(H)|}{2}\rceil$ vertices of each path H whether P_i's, Q_i's or W_i's. Also z belongs to S if $p_a+p_b+p_c=0$. Therefore, we have

$$\begin{aligned}\alpha(T)&\le\sum_{i=1}^{a}\lceil\frac{|V(P_i)|}{2}\rceil+\sum_{i=1}^{b}\lceil\frac{|V(Q_i)|}{2}\rceil+\sum_{i=1}^{c}\lceil\frac{|V(W_i)|}{2}\rceil\\&\le\frac{m_a+m_b+m_c+p_a+p_b+p_c}{2}+t,\end{aligned}$$

where $t=1$ if $p_a+p_b+p_c=0$ and $t=0$ otherwise. On the other hand, let us define the function f on $V(T)$ as follows:

$$f(z)=\begin{cases}2 & \textit{if } a=b=0,\\ 0 & \textit{if } a=c=0,\\ 1 & \textit{othewise},\end{cases}$$

and for any vertex x in T, $f(x)=0$ if depth$\,(x)\equiv 2 \pmod 3$ and $f(x)=1$, otherwise. It can be seen that f is an OI{2}D-function on T of weight

$$\omega(f)=\begin{cases}\frac{2m_c}{3}+2 & \textit{if } a=b=0,\\ \frac{2m_b+2b}{3} & \textit{if } a=c=0,\\ \frac{2m_a+2m_b+2m_c+a+2b}{3}+1 & \textit{othewise},\end{cases}$$

and by calculation, we can see that $\gamma_{oi\{2\}}(T)=\omega(f)\le\frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$.

In the following we can assume that z is not the unique vertex in T with degree at least three. Let v be a vertex of degree at least three in T at maximum distance from z in the rooted tree. Clearly, every vertex in T_v besides v is of degree at most two. Now, let $A_i=\{x\in C(v)\mid \text{depth}\,(x)\equiv i \pmod 6\}$ for $i\in\{0,1,\ldots,5\}$ and set $|A_i|=a_i$. If $v_1\in A_1$ and $vv_1v_2\ldots v_{6t+2}$ is a pendant path in T, for some integer $t\ge 0$, then consider the tree $T'=T-T_{v_1}$. Obviously, $\alpha(T')\le\alpha(T)-3t-1$, $s(T')=s(T)-1$ and $\ell(T')=\ell(T)-1$. On the other hand, any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $4t+2$ by assigning 0 to every vertex x with $d(v_1,x)\equiv 2 \pmod 3$ and 1 to other vertices. Now, applying the induction hypothesis on T' follows that

$$\begin{aligned}\gamma_{oi\{2\}}(T)&\le\gamma_{oi\{2\}}(T')+4t+2\\&\le\frac{3\alpha(T')+4s(T')-3\ell(T')}{2}+4t+2\\&\le\frac{3(\alpha(T)-3t-1)+4(s(T)-1)-3(\ell(T)-1)}{2}+4t+2\\&\le\frac{3\alpha(T)+4s(T)-3\ell(T)}{2},\end{aligned}$$

as desired. Moreover, if $v_1\in A_5$ and $vv_1v_2\ldots v_{6t}$ be a pendant path in T for some integer $t\ge 1$, then consider the tree $T'=T-T_{v_1}$. It is clear in this case that $\alpha(T')\le\alpha(T)-3t$, $s(T')=s(T)-1$ and $\ell(T')=\ell(T)-1$. Also, any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $4t+1$ by assigning 0 to every vertex $x\ne v_1$ with $d(v_1,x)\equiv 0 \pmod 3$, and 1 to any other vertex of T_{v_1}. Using the induction hypothesis on T' with the fact $t\ge 1$, it follows that

$$\begin{aligned}\gamma_{oi\{2\}}(T)&\le\gamma_{oi\{2\}}(T')+4t+1\\&\le\frac{3\alpha(T')+4s(T')-3\ell(T')}{2}+4t+1\\&\le\frac{3(\alpha(T)-3t)+4(s(T)-1)-3(\ell(T)-1)}{2}+4t+1\\&\le\frac{3\alpha(T)+4s(T)-3\ell(T)}{2},\end{aligned}$$

as desired. Hence, we can assume that $A_1 \cup A_5 = \emptyset$. Now, assume that $v_1 \in A_3$ and let $vv_1v_2 \dots v_{6t+4}$ be a pendant path in T for some integer $t \geq 1$. Consider the tree $T' = T - T_{v_1}$. Clearly, $\alpha(T') \leq \alpha(T) - 3t - 2$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Also, any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $4t+4$ by assigning 2 to v_1, 0 to every vertex x with $d(v_1, x) \equiv 1 \pmod 3$ and 1 to any other vertex of T_{v_1}. Using the induction hypothesis on T' with the condition that $t \geq 1$, we obtain

$$\begin{aligned}
\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 4t + 4 \\
&\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 4t + 4 \\
&\leq \frac{3(\alpha(T) - 3t - 2) + 4(s(T) - 1) - 3(\ell(T) - 1)}{2} + 4t + 4 \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2},
\end{aligned}$$

as desired. Likewise, if we assume that $v_1 \in A_4$ and considering a pendant path $vv_1v_2 \dots v_{6t+5}$ in T for some integer $t \geq 1$, then for $T' = T - T_{v_4}$, we will get $\alpha(T') \leq \alpha(T) - 3t - 2$, $s(T') = s(T) - 1, \ell(T') = \ell(T) - 1$, and $\gamma_{oi\{2\}}(T) \leq \gamma_{oi\{2\}}(T') + 4t + 4$. By applying the induction hypothesis on T' with the condition that $t \geq 1$, we obtain the desired result, that is, $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Consequently, if $u \in A_i$ for $i \in \{3, 4\}$, then we can assume that depth $(u) = i$.

In the next, we assume that $v_1 \in A_2$. Consider a pendant path $vv_1v_2 \dots v_{6t+3}$ in T such that $t \geq 2$, and let $T' = T - T_{v_1}$. Clearly, $\alpha(T') \leq \alpha(T) - 3t - 1$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Also, any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $4t + 3$ by assigning 0 to every vertex $x \neq v_1$ with $d(v_1, x) \equiv 0 \pmod 3$ and 1 to other vertices. Using the induction hypothesis on T' with the condition that $t \geq 2$, we obtain

$$\begin{aligned}
\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 4t + 3 \\
&\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 4t + 3 \\
&\leq \frac{3(\alpha(T) - 3t - 1) + 4(s(T) - 1) - 3(\ell(T) - 1)}{2} + 4t + 3 \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2},
\end{aligned}$$

Therefore, in the sequel, if $u \in A_2$, then we can assume that depth $(u) \in \{2, 8\}$. Hence if $A_2 \neq \emptyset$, let b_1 and b_2 be the number of vertices in A_2 with depths 2 and 8, respectively.

Finally assume that $v_1 \in A_0$ and consider a pendant path $vv_1v_2 \dots v_{6t+1}$ in T such that $t \geq 3$. Let $T' = T - T_{v_1}$, and observe that $\alpha(T') \leq \alpha(T) - 3t$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Also, any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $4t + 2$ by assigning 2 to v_1, 0 to every vertex x with $d(v_1, x) \equiv 1 \pmod 3$ and 1 to any other vertex of T_{v_1}. Using the induction hypothesis on T' with the condition that $t \geq 3$, we obtain

$$\begin{aligned}
\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 4t + 2 \\
&\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 4t + 2 \\
&\leq \frac{3(\alpha(T) - 3t) + 4(s(T) - 1) - 3(\ell(T) - 1)}{2} + 4t + 2 \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.
\end{aligned}$$

Hence, $A_0 \neq \emptyset$, then for every $u \in A_0$, we can assume that depth $(u) \in \{0, 6, 12\}$. Note that A_0 has at most one vertex with depth 0, for otherwise v would be a strong stem, which contradicts our earlier assumption. Therefore, if $A_0 \neq 0$, then c_i be the number of vertices in A_0 with depth i for $i \in \{0, 6, 12\}$. In the following let $\eta = a_3 + a_4 + b_1 + b_2 + c_1 + c_2 + c_3$, and $\eta' = a_3 + a_4 + c_1 + c_2 + c_3$. Note that $\eta \geq 2$, since $\deg(v) \geq 3$.

Let $(v =)w_0w_1 \cdots w_k(= z)$ denote the vertices on the (v, z)-path in T, and consider the following two cases.

Case 1. $\deg(w_1) \geq 3$.
Let $T' = T - T_v$. It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3)$, $s(T') = s(T) - \eta$ and $\ell(T') = \ell(T) - \eta$. Moreover, it can be seen that any $\gamma_{oi\{2\}}(T')$-function can be extended to an OI{2}D-function of T of weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + t$, where $t = 1$ if $\eta' \geq 1$ and $t = 2$ if $\eta' = 0$. Now, applying the induction hypothesis and the fact $\eta \geq 2$, we have

$$\begin{aligned}
\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + t \\
&\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + t \\
&\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3)) + 4(s(T) - \eta) - 3(\ell(T) - \eta)}{2} \\
&+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + t \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 - 2t}{2} \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.
\end{aligned}$$

Case 2. $\deg(w_1) = 2$.
Let r be a smallest index in $\{1, \dots, k-1\}$ such that $\deg_T(w_{r+1}) \geq 3$. Observe that since $\deg_T(z) \geq 3$, such an index r exists. In this case, let $T' = T - T_{w_r}$ and consider a $\gamma_{oi\{2\}}(T')$-function f. Clearly, $s(T') = s(T) - \eta$ and $\ell(T') = \ell(T) - \eta$. We consider the following situations.

- $r = 6m$ for some integer $m \geq 1$.
  It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m)$, and f can be extended to a OI{2}D-function of T of weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + t$, where $t = 1$ if $\eta' \geq 1$ and $t = 2$ if $\eta' = 0$. Applying the induction hypothesis and the fact $\eta \geq 2$, we have

$$\begin{aligned}
\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + t \\
&\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + t \\
&\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m)) + 4(s(T) - \eta) - 3(\ell(T) - \eta)}{2} \\
&+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + t \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 2t}{2} \\
&\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.
\end{aligned}$$

- $r = 6m + 1$ for some integer $m \geq 0$.
  As before, one can see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m)$, and f can be extended to a OI{2}D-function of T of weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 1$.

Applying the induction hypothesis and the fact $\eta \geq 2$, we get

$$\begin{aligned}\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 1\\ &\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 1\\ &\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m)) + 4(s(T) - \eta) - 3(\ell(T) - \eta)}{2}\\ &+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 1\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 2}{2}\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.\end{aligned}$$

- $r = 6m + 2$ for some integer $m \geq 0$.
  It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 1)$, and f can be extended to an OI{2}D-function of T with weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 3$. Now, if $a_3 \geq 3$ or $\eta - a_3 \geq 1$ or $m \geq 1$, then applying the induction hypothesis and the fact $\eta \geq 2$, we get

$$\begin{aligned}\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 3\\ &\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 3\\ &\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 1)) + 4(s(T) - \eta) - 3(\ell(T) - \eta)}{2}\\ &+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 3\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 3}{2}\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.\end{aligned}$$

  Hence assume that $a_3 \leq 2$, $m = 0$, $\eta - a_3 = a_4 + b_1 + b_2 + c_1 + c_2 + c_3 = 0$. It follows that $a_3 = 2$ and $r = 2$. Let $vz_4^i z_3^i z_2^i z_1^i$ be a pendant path in T for $i = 1, 2$. Let $T' = T - T_{z_4^1}$ and consider a $\gamma_{oi\{2\}}(T')$-function f. Clearly we may assume that $f(v) \geq 1$ and so f can be extended to an OI{2}D-function of T by assigning the value 0 to z_3^1 and 1 to z_1^1, z_2^1, z_4^1. On the other hand, we have $\alpha(T') \leq \alpha(T) - 2$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Applying the induction hypothesis we get $\gamma_{oi\{2\}}(T) < \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$.

- $r = 6m + 3$ for some integer $m \geq 0$.
  It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 1)$, and f can be extended to a OI{2}D-function of T with weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 2 + t$ where $t = 1$ if $\eta' \geq 1$ and $t = 2$ if $\eta' = 0$. Applying the induction hypothesis we have

$$\begin{aligned}\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 2 + t\\ &\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 2 + t\\ &\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 1)) + 4s(T) - 3\ell(T) - \eta}{2}\\ &+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 2 + t\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 1 - 2t}{2}. \quad (3.1)\end{aligned}$$

  Now, if $b_1 + b_2 \geq 2$, then clearly the above inequalities lead to $\gamma_{oi\{2\}}(T) < \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence, assume that $b_1 + b_2 \leq 1$. It follows from $\eta \geq 2$ that $\eta' = \eta - (b_1 + b_2) \geq 1$ and so $t = 1$. If $m \geq 1$

or $\eta - a_3 \geq 1$ or $a_3 \geq 2$, then (3.1) implies that $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence assume that $\eta = a_3 = 2$. Similar to previous item one can see that $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$.

- $r = 6m + 4$ for some integer $m \geq 0$.
It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 2)$, and f can be extended to a OI{2}D-function of T with weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 4$. Applying the induction hypothesis and the fact $\eta \geq 2$ we have

$$\begin{aligned}\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 4\\ &\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 4\\ &\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 2)) + 4s(T) - 3\ell(T) - \eta}{2}\\ &+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 4\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 2}{2}\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2}.\end{aligned}$$

- $r = 6m + 5$ for some integer $m \geq 0$.
It is easy to see that $\alpha(T') \leq \alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 2)$, and f can be extended to a OI{2}D-function of T of weight $\gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 5$. Applying the induction hypothesis and the fact $\eta \geq 2$ we have

$$\begin{aligned}\gamma_{oi\{2\}}(T) &\leq \gamma_{oi\{2\}}(T') + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 5\\ &\leq \frac{3\alpha(T') + 4s(T') - 3\ell(T')}{2} + 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 5\\ &\leq \frac{3(\alpha(T) - (2a_3 + 3a_4 + 2b_1 + 5b_2 + c_1 + 4c_2 + 7c_3 + 3m + 2)) + 4s(T) - 3\ell(T) - \eta}{2}\\ &+ 3a_3 + 4a_4 + 2b_1 + 6b_2 + c_1 + 5c_2 + 9c_3 + 4m + 5\\ &\leq \frac{3\alpha(T) + 4s(T) - 3\ell(T)}{2} - \frac{a_3 + 2a_4 + 3b_1 + 4b_2 + 2c_1 + 3c_2 + 4c_3 + m - 4}{2}. \qquad (3.2)\end{aligned}$$

If $b_1+b_2+c_2+c_3 \geq 1$, then it follows from (3.2) and $\eta \geq 2$ that $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Assume that $b_1+b_2+c_2+c_3 = 0$. If $a_3 \geq 2$, then similar as in Item 3 one can see that $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. If $c_1 + a_4 \geq 2$, then (3.2) leads to $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence assume that $a_3 \leq 1$ and that $c_1 + a_4 \leq 1$. It follows from $\eta \geq 2$ that either $a_3 = c_1 = 1$ or $a_3 = a_4 = 1$. If $m \geq 1$, then (3.2) leads to $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Hence let $m = 0$. First let $a_3 = c_1 = 1$. Then T_v is a path of order 6. say $P_6 = z_1z_2z_3z_4vz$. Let $T' = T - \{z_1, z_2, z_3, z_4\}$. It is easy to see that $\alpha(T') \leq \alpha(T) - 2$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. On the other hand, any $\gamma_{oi\{2\}}(T')$-function with $f(v) \geq 1$ (note that such a function exists by Observation 1.1), can be extended to an OI{2}D-function by assigning the value 1 to z_1, z_2, z_4 and 0 to z_3. Applying the induction hypothesis on T', it follows that $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. Henceforth, we assume that $a_3 = a_4 = 1$. Then T_v is a path of order 10, say $P_{10} = u_1u_2u_3u_4vz_1z_2z_3z_4z_5$. Let $T' = T - \{z_2, z_3, z_4, z_5\}$. It is easy to see that $\alpha(T') \leq \alpha(T) - 2$, $s(T') = s(T)$ and $\ell(T') = \ell(T)$. On the other hand, any $\gamma_{oi\{2\}}(T')$-function with $f(z_1) \geq 1$ (which actually exists), can be extended to an OI{2}D-function by assigning the value 1 to z_2, z_4, z_5 and 0 to z_3. Applying the induction hypothesis on T', we get $\gamma_{oi\{2\}}(T) \leq \frac{3\alpha(T)+4s(T)-3\ell(T)}{2}$. This completes the proof. □

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