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Some Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeros

Idrees Qasim and Sivaramakrishnan Chinnakali

ABSTRACT: In this paper, we establish some results concerning the polar derivative of a polynomial whose zeros all lie inside or outside a circle of radius k, where $k \leq 1$. Our results not only refine certain prior inequalities related to polar derivatives of polynomials but also generalize several well-known polynomial inequalities.

Keywords: Polar derivative, complex Polynomials, inequalities, zeros.

Contents

1. Introduction

Let F_n denote the space of complex polynomials $f(z) := \sum_{j=0}^{n} a_j z^j$ of degree $n \geq 1$. If $f \in F_n$, then concerning the estimate of $|f'(z)|$ on the unit circle, we have the following well-known result:

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)|. \tag{1.1}$$

Inequality (1.1) was found by Bernstein [3] (see also [18]). If we consider the class of polynomials $f \in F_n$ having no zero in $|z| < 1$, then the bound in inequality (1.1) can be considerably improved. In fact, Erdös conjectured and later Lax [9] verified that if $f \in F_n$ and $f(z) \neq 0$ for $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{2} \max_{|z|=1} |f(z)|. \tag{1.2}$$

Inequality (1.2) was further improved by Jain [7] who proved that, if $f \in F_n$ and $f(z) \neq 0$ in $|z| < 1$, then for every complex number β, $|\beta| \leq 1$,

$$\left| zf'(z) + n\frac{\beta}{2} f(z) \right| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \max_{|z|=1} |f(z)|. \tag{1.3}$$

Inequality (1.1) can be seen as a special case of following inequality which is also due to Bernstein [4].

Theorem A. *Let $F \in F_n$ having all zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$,* then

$$|f'(z)| \leq |F'(z)| \quad \text{for} \quad |z| \geq 1. \tag{1.4}$$

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In particular, if $p(z)$ does not vanish in $|z| < 1$, then

$$|p'(z)| \leq |q'(z)| \quad \text{for} \quad |z| \geq 1, \tag{1.5}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.
Inequality (1.4) is sharp and equality holds if $f(z) = e^{i\gamma}F(z)$ for some $\gamma \in \mathbb{R}$.
Inequality (1.1) can be obtained from inequality (1.4) by taking $F(z) = Mz^n$, where $M = \max\limits_{|z|=1}|f(z)|$.
Malik and Vong [13] generalized inequality (1.4) and with the same hypothesis as in Theorem A, they proved

$$\left|\frac{zf'(z)}{n} + \beta\frac{f(z)}{2}\right| \leq \left|z\frac{F'(z)}{n} + \beta\frac{F(z)}{2}\right| \tag{1.6}$$

for $|z| = 1$ and for every β satisfying $|\beta| \leq 1$, whereas Jain [6] proved that for $|z| = 1$,

$$\left|\frac{f'(z)}{n}\right| + \left|\frac{F(z)}{2}\right| \leq \left|\frac{F'(z)}{n}\right| + \left|\frac{f(z)}{2}\right|.$$

An inequality for minimum modulus was obtained by Aziz and Dawood [1]. It states:
If $p(z)$ is a polynomial of degree n which has all zeros in $|z| \leq 1$, then

$$\min_{|z|=1}|p'(z)| \geq n\min_{|z|=1}|p(z)|. \tag{1.7}$$

For any complex number α consider the operator D_α which maps a polynomial $f(z)$ of degree n into

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z).$$

The operator $D_\alpha f(z)$ is known as polar derivative of $f(z)$ with respect to α. $D_\alpha f(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha f(z)}{\alpha} = f'(z).$$

Now for a polynomial $f(z)$ of degree n and for complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$, $0 < t < n$, we construct a sequence of polar derivatives

$$\begin{aligned} D_{\alpha_1}f(z) &= nf(z) + (\alpha_1 - z)f'(z) \\ D_{\alpha_2}D_{\alpha_1}f(z) &= (n-1)D_{\alpha_1}f(z) + (\alpha_2 - z)(D_{\alpha_1}f(z))' \\ D_{\alpha_t}D_{\alpha_{t-1}}...D_{\alpha_1}f(z) &= (n-t+1)D_{\alpha_{t-1}}...D_{\alpha_1}f(z) + (\alpha_t - z)(D_{\alpha_{t-1}}...D_{\alpha_1}f(z))'. \end{aligned}$$

The points $\alpha_1, \alpha_2, ..., \alpha_t$; $t = 1, 2, ..., n-1$ may be equal or unequal. The t^{th} polar derivative $f_t(z) := D_{\alpha_t}D_{\alpha_{t-1}}...D_{\alpha_1}f(z)$ of $f(z)$ is a polynomial of degree at most $n-t$ with $f_0(z) = f(z)$.

Aziz and Shah [2] extended inequality (1) to the j^{th} polar derivative of a polynomial and obtained several sharp inequalities concerning the maximum modulus of the polar derivative of the polynomial. In fact they proved:

Theorem B. *If $f(z)$ is a polynomial of degree n. Then for complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| \geq 1$, $1 \leq i \leq t < n$, we have*

$$|f_t(z)| \leq n(n-1)(n-2)...(n-t+1)|\alpha_1\alpha_2\ldots\alpha_t||z|^{n-1}\max_{|z|=1}|f(z)| \ \textit{for} \ |z| \geq 1.$$

The inequality is sharp and equality holds for $f(z) = Mz^n$, where $M = \max\limits_{|z|=1}|f(z)|$.

There have been many recent papers concerning the polar derivative of polynomials; see [11,14,15,16,17] and the references therein.

NOTATIONS
From now onward we will use the following notations:

$$\begin{aligned} S_0^t f(z) &:= f(z). \\ S_1^t f(z) &:= D_{\alpha_1} f(z) + D_{\alpha_2} f(z) + ... + D_{\alpha_t} f(z) = \sum_{i=1}^{t} D_{\alpha_i} f(z). \\ S_2^t f(z) &:= D_{\alpha_1} D_{\alpha_2} f(z) + D_{\alpha_2} D_{\alpha_3} f(z) + ... \qquad = \sum_{1=i<j}^{t} D_{\alpha_i} D_{\alpha_j} f(z). \\ S_3^t f(z) &:= \sum_{1=i<j<k}^{t} D_{\alpha_i} D_{\alpha_j} D_{\alpha_k} f(z). \\ &\vdots \\ S_t^t f(z) &:= D_{\alpha_t} D_{\alpha_{t-1}} ... D_{\alpha_1} f(z) = f_t(z). \\ n_t &:= n(n-1)(n-2)...(n-t+1), \ n_0 = 1, \\ \xi &:= \alpha_1 \alpha_2 \ldots \alpha_t \end{aligned}$$

and

$$A_{\alpha_t} := (|\alpha_1| - k)(|\alpha_2| - k)...(|\alpha_t| - k).$$

In this paper, we prove some results concerning to the polar derivative of a polynomial. Some results are deduced which provide refinement of inequalities (1.5) and (1.7).

2. Main Results

In this section we state our main results. Our first result is central to this paper.

Theorem 2.1 *Let $F \in F_n$ having all zeros in $|z| \leq k$, where $k \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$. If $|f(z)| \leq |F(z)|$ for $|z| = k$, then for all real or complex numbers β with $|\beta| < 1$ and α_i with $|\alpha_i| \geq k, \ 1 \leq i \leq t < n$, and $|z| \geq k$*

$$\left| z^t \sum_{i=0}^{t} (n-m)_{t-i} S_i^t f(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} f(z) \right| \leq \left| z^t F_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} F(z) \right|. \tag{2.1}$$

The result is sharp and equality holds for $f(z) = e^{i\gamma} F(z), \ \gamma \in \mathbb{R}$.

As a generalization of inequality (1.4) to the t^{th} polar derivative one would expect an inequality of the form

$$|f_t(z)| \leq |F_t(z)| \quad \text{for} \quad |z| \geq 1.$$

However, from Theorem 1 we get more refined inequality. In fact from inequality (2.1) for $\beta = 0$ and $k = 1$, we get

$$\left| \sum_{i=0}^{t} (n-m)_{t-i} S_i^t f(z) \right| \leq \left| F_t(z) \right| \quad \text{for} \quad |z| \geq 1.$$

If we take $F(z) = M(z/k)^n$, where $M = \max_{|z|=k} |f(z)|$ in Theorem 1, we get the following result:

Corollary 2.1 *If $f(z)$ is a polynomial of degree m, then for real or complex numbers α_i with $|\alpha_i| \geq k$, $1 \leq i \leq t < n$, $k \leq 1$, $m \leq n$ and β with $|\beta| < 1$,*

$$\left| z^t \sum_{i=0}^{t} (n-m)_{t-i} S_i^t f(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} f(z) \right|$$

$$\leq n_t \frac{|z|^n}{k^n} \left| \xi + \frac{\beta}{(1+k)^t} A_{\alpha_t} \right| \max_{|z|=k} |f(z)| \text{ for } |z| \geq k. \tag{2.2}$$

The result is sharp and equality holds for $f(z) = M(z/k)^n$, where $M = \max_{|z|=k} |f(z)|$.

Remark 2.1 If $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then on dividing both sides of inequality (2.2) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get a generalization of a result due to Jain [8].

If $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then on dividing both sides of inequality (2.1) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following generalization of inequality (1.6).

Corollary 2.2 *Let $F \in F_n$ having all zeros in $|z| \leq k$, $k \leq 1$ and $f(z)$ be a polynomial of degree $m (\leq n)$. If $|f(z)| \leq |F(z)|$ for $|z| = k$, then for all real or complex number β with $|\beta| < 1$ and $1 \leq t < n$, we have*

$$\left| z^t f^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t f(z) \right| \leq \left| z^t F^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t F(z) \right| \text{ for } |z| \geq k.$$

From Theorem 1, we also get the following result:

Corollary 2.3 *Let $p \in F_n$, $p(z) \neq 0$ in $|z| < k$, $k \leq 1$ and $q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$, then for real or complex numbers α_i with $|\alpha_i| \geq k$, $1 \leq i \leq t < n$ and β with $|\beta| < 1$, we have*

$$\left| z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z) \right| \leq \left| z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z) \right| \text{ for } |z| \geq k.$$

Inasmuch as, $f(z)$ is a polynomial of degree $m \leq n$, taking in particular $f(z) = (z/k)^n \min_{|z|=k} |F(z)|$ in Theorem 1, we get the following corollary:

Corollary 2.4 *If $F(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$ then for complex numbers α_i, β with $|\alpha_i| \geq k$, $1 \leq i \leq t < n$, $|\beta| < 1$ and $|z| \geq k$,*

$$\left| z^t F_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} F(z) \right| \geq n_t \frac{|z|^n}{k^n} \left| \xi + \frac{\beta}{(1+k)^t} A_{\alpha_t} \right| \min_{|z|=k} |F(z)|. \tag{2.3}$$

If $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then on dividing both sides of inequality (2.3) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following generalization of a result due to Dewan and Hans [5].

Corollary 2.5 *If $F(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, then for any real or complex number β with $|\beta| < 1$,*

$$\left| z^t F^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t F(z) \right| \geq n_t \frac{|z|^n}{k^n} \left| 1 + \frac{\beta}{(1+k)^t} \right| \min_{|z|=k} |F(z)| \text{ for } |z| \geq k.$$

From Corollary 3 and Lemma 4 (to be mentioned later), we obtain the following generalization of inequality (1.3) to the t^{th} polar derivative of a polynomial and to the polynomials which does not vanish in $|z| < k$, $k \leq 1$.

Corollary 2.6 *If $p \in F_n$ and $p(z) \neq 0$ in $|z| < k$, $k \leq 1$ then for complex numbers α_i, β with $|\alpha_i| \geq k$, $1 \leq i \leq t < n$, $|\beta| < 1$ and for $|z| = k$,*

$$\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| \leq \frac{n_t}{2}\left\{\left|\xi + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right| + \left|z^t + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right|\right\}\max_{|z|=k}|p(z)|. \tag{2.4}$$

The following result can also be deduced from Theorem 1:

Corollary 2.7 *Let $F \in F_n$ having all zeros in $|z| \leq k$, where $k \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$. If $|f(z)| \leq |F(z)|$ for $|z| = k$, then for all real or complex numbers β, α_i with $|\beta| < 1$, $|\alpha_i| \geq k$, $1 \leq i \leq t < n$ and for $|z| \geq k$,*

$$\left|\frac{z^t}{n_t}\sum_{i=0}^{t}(n-m)_{t-i}S_i^t f(z)\right| + \left|\frac{A_{\alpha_t}}{(1+k)^t}F(z)\right| \leq \left|\frac{z^t}{n_t}F_t(z)\right| + \left|\frac{A_{\alpha_t}}{(1+k)^t}f(z)\right|.$$

We next prove the following result:

Theorem 2.2 *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$, then for complex numbers β, α_i with $|\beta| < 1$, $|\alpha_i| \geq k$, $1 \leq i \leq t < n$ and $|z| \geq k$,*

$$\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| + n_t\left\{\frac{|z|^n}{k^n}\left|\xi + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right| - \left|z^t + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right|\right\}\min_{|z|=k}|p(z)| \leq \left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z)\right|, \tag{2.5}$$

where $q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$.

The inequality is sharp and equality holds for $f(z) = e^{i\gamma}F(z)$, where $\gamma \in \mathbb{R}$.

Using Lemma 4 in Theorem 2, we get the following result:

Corollary 2.8 *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$, then for complex numbers β, α_i with $|\beta| < 1$, $|\alpha_i| \geq k$, $1 \leq i \leq t < n$ and $|z| \geq k$,*

$$\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| \leq \frac{n_t}{2}\left[\left\{\frac{|z|^n}{k^n}\left|\xi + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right| + \left|z^t + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right|\right\}\max_{|z|=k}|p(z)| - \left\{\frac{|z|^n}{k^n}\left|\xi + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right| - \left|z^t + \frac{\beta}{(1+k)^t}A_{\alpha_t}\right|\right\}\min_{|z|=k}|p(z)|\right]. \tag{2.6}$$

If $t = 1$, Corollary 8 gives refinement of a result due to Liman et al. [12].

If $\alpha_1 = \alpha_2 = \ldots = \alpha_t = \alpha$, then on dividing both sides of inequality (2.8) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following:

Corollary 2.9 *If* $p \in F_n$ *such that* $p(z) \neq 0$ *for* $|z| < k$, $k \leq 1$, *then for any* β *with* $|\beta| < 1$ *and* $|z| \geq k$,

$$\left|z^t p^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t p(z)\right| \leq \frac{n_t}{2}\left[\left\{\frac{|z|^n}{k^n}\left|1 + \frac{\beta}{(1+k)^t}\right| + \left|\frac{\beta}{(1+k)^t}\right|\right\}\max_{|z|=k}|p(z)| - \left\{\frac{|z|^n}{k^n}\left|1 + \frac{\beta}{(1+k)^t}\right| - \left|\frac{\beta}{(1+k)^t}\right|\right\}\min_{|z|=k}|p(z)|\right].$$

Remark 2.2 For $t = 1$, Corollary 9 gives generalization of a result due to Dewan and Hans [5].

Taking $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then on dividing both sides of inequality (2.5) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following result:

Corollary 2.10 *If* $p \in F_n$ *and* $p(z) \neq 0$ *in* $|z| < k,\ k \leq 1$, *then for complex number* β *with* $|\beta| < 1$ *and* $|z| \geq k$,

$$\left|z^t p^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t p(z)\right| + n_t\left\{\frac{|z|^n}{k^n}\left|1 + \frac{\beta}{(1+k)^t}\right| - \left|\frac{\beta}{(1+k)^t}\right|\right\}\min_{|z|=k}|p(z)| \leq \left|z^t q^{(t)}(z) + \frac{\beta}{(1+k)^t} n_t q(z)\right|,$$

where $q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$.

Taking $\beta = 0,\ t = 1$ in Corollary 10, we obtain the following refinement of inequality (1.5).

Corollary 2.11 *If* $p \in F_n$ *and* $p(z) \neq 0$ *in* $|z| < k,\ k \leq 1$, *then for* $|z| \geq k$,

$$|zp'(z)| + n\frac{|z|^n}{k^n}\min_{|z|=k}|p(z)| \leq |zq'(z)|. \tag{2.7}$$

In particular for $|z| = k$, *we get*

$$|p'(z)| + \frac{n}{k}\min_{|z|=k}|p(z)| \leq |q'(z)|.$$

We end this section by noting that, it is natural to ask as to what happens to Theorem 2 if we let all the zeros to be in $|z| \leq k$. Our next Theorem answers the same question:

Theorem 2.3 *If* $p(z)$ *is a polynomial of degree* n *having all zeros in* $|z| \leq k,\ k \leq 1$, *then for complex numbers* $\beta,\ \alpha_i$ *with* $|\beta| < 1,\ |\alpha_i| \geq k,\ 1 \leq i \leq t < n$ *and* $|z| \geq k$,

$$\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| - \left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z)\right| \geq n_t\left\{\frac{|z|^n}{k^n}\left|\xi + \frac{\beta}{(1+k)^t} A_{\alpha_t}\right| - \left|z^t + \frac{\beta}{(1+k)^t} A_{\alpha_t}\right|\right\}\min_{|z|=k}|p(z)|,$$

where $q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$.

For $\beta = 0,\ t = 1$, we get the following result:

Corollary 2.12 *If* $p(z)$ *is a polynomial of degree* n *having all zeros in* $|z| \leq k,\ k \leq 1$, *then for complex number* α *with* $|\alpha| \geq k$ *and* $|z| = k$,

$$|D_\alpha p(z)| - |D_\alpha q(z)| \geq \frac{n}{k}(|\alpha| - k)\min_{|z|=k}|p(z)|. \tag{2.8}$$

Remark 2.3 If we divide both sides of inequality (2.8) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following inequality.

$$|p'(z)| - |q'(z)| \geq \frac{n}{k} \min_{|z|=k} |p(z)|. \tag{2.9}$$

The above inequality provides generalized refinement of inequality (1.7).

3. Lemmas

For the proofs of these theorems we need the following lemmas. The first Lemma is due to Ahmad Zireh [19].

Lemma 3.1 *If $p(z)$ is a polynomial of degree n having all zeros in $|z| \leq k$, $k \leq 1$, then for all α_i with $|\alpha_i| \geq k$, $1 \leq i \leq t < n$, we have*

$$|p_t(z)| \geq \frac{n_t}{(1+k)^t} A_{\alpha_t} |p(z)| \quad for \quad |z| = 1.$$

The next lemma is due to Laguerre [10, p.38].

Lemma 3.2 *If all the zeros of an nth degree polynomial $f(z)$ lie in a circular region C and ω is any zero of $D_\alpha f(z)$, then at most one of the points ω and α may lie outside C.*

Lemma 3.3 *If $F \in F_n$, $f(z)$ is a polynomial of degree $m \leq n$ and $H(z) = \mu f(z) - F(z)$, where μ is an arbitrary number, then*

$$H_t(z) = \mu \sum_{i=0}^{t} (n-m)_{t-i} S_i^t f(z) - F_t(z).$$

Proof:

In order to prove the result we use mathematical induction on t. Since $F(z)$ is a polynomial of degree n and $f(z)$ is a polynomial of degree $m (m \leq n)$, therefore, $\mu f(z) - F(z)$ is a polynomial of degree n and hence by definition of polar derivative, we have for any real or complex number α_1,

$$\begin{aligned}
D_{\alpha_1} H(z) &= D_{\alpha_1}\{\mu f(z) - F(z)\} \\
&= n\{\mu f(z) - F(z)\} + (\alpha_1 - z)\{\mu f(z) - F(z)\}' \\
&= \mu\{(n-m) f(z) + m f(z) + (\alpha_1 - z) f'(z)\} - \{nF(z) + (\alpha_1 - z) F'(z)\} \\
&= \mu\{(n-m) f(z) + D_{\alpha_1} f(z)\} - D_{\alpha_1} F(z). \\
&= \mu \sum_{i=0}^{1} (n-m)_{1-i} S_i^1 f(z) - F_1(z).
\end{aligned}$$

Hence the result is true for $t = 1$. Except for one value α_1' (say) of α_1, $D_{\alpha_1} H(z)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\neq \alpha_1')$ and fix it up. Thus $D_{\alpha_1} H(z)$ is a polynomial of degree $(n-1)$, so that for any arbitrary number α_2, we have

$$\begin{aligned}
&D_{\alpha_2}D_{\alpha_1}H(z)\\
&= D_{\alpha_2}[\mu\{(n-m)f(z)+D_{\alpha_1}f(z)\}-D_{\alpha_1}F(z)]\\
&= (n-1)[\mu\{(n-m)f(z)+D_{\alpha_1}f(z)\}-D_{\alpha_1}F(z)]\\
&\qquad\qquad\qquad + (\alpha_2-z)[\mu\{(n-m)f(z)+D_{\alpha_1}f(z)\}-D_{\alpha_1}F(z)]'\\
&= \mu[(n-m)\{(n-1)f(z)+(\alpha_2-z)f'(z)\}+(n-1)D_{\alpha_1}f(z)+(\alpha_2-z)(D_{\alpha_1}f(z))']\\
&\qquad\qquad\qquad\qquad\qquad - [(n-1)D_{\alpha_1}F(z)+(\alpha_2-z)(D_{\alpha_1}F(z))']\\
&= \mu[(n-m)(n-m-1)f(z)+(n-m)\{D_{\alpha_1}f(z)+D_{\alpha_2}f(z)\}+D_{\alpha_2}D_{\alpha_1}f(z)]\\
&\qquad\qquad\qquad\qquad\qquad - D_{\alpha_2}D_{\alpha_1}F(z).\\
&= \mu\sum_{i=0}^{2}(n-m)_{2-i}S_i^2 f(z)-F_2(z).
\end{aligned}$$

Hence the result is true for $t=2$.

Assume that the result is true for $k<t$, we prove the result for $k+1$. Except for one value α'_k (say) of α_k, $D_{\alpha_k}D_{\alpha_{k-1}}...D_{\alpha_1}H(z)$ will be a polynomial of degree $(n-k)$. Let us take any $\alpha_k(\neq\alpha'_k)$ and fix it up. Thus $D_{\alpha_k}D_{\alpha_{k-1}}...D_{\alpha_1}H(z)$ is a polynomial of degree $(n-k)$, so that for any arbitrary number α_{k+1}, we have

$$\begin{aligned}
H_{k+1}(z) &= D_{\alpha_{k+1}}H_k(z)\\
&= (n-k)H_k(z)+(\alpha_{k+1}-z)H'_k(z)\\
&= (n-k)\left(\mu\sum_{i=0}^{k}(n-m)_{k-i}S_i^k f(z)-F_k(z)\right)\\
&\qquad\qquad +(\alpha_{k+1}-z)\left(\mu\sum_{i=0}^{k}(n-m)_{k-i}S_i^k f(z)-F_k(z)\right)'\\
&= \mu\left(\sum_{i=0}^{k}(n-m)_{k-i}(n-m-k+i)S_i^k f(z)+\sum_{i=0}^{k}(n-m)_{k-i}\left((m-i)S_i^k f(z)\right)\right)\\
&\quad +\mu\left((\alpha_{k+1}-z)\sum_{i=0}^{k}(n-m)_{k-i}(S_i^k f)'(z)\right)-F_{k+1}(z)\\
&= \mu\left(\sum_{i=0}^{k}(n-m)_{k+1-i}S_i^k f(z)+\sum_{i=0}^{k}(n-m)_{k-i}D_{\alpha_{k+1}}S_i^k f(z)\right)-F_{k+1}(z)\\
&= \mu\left(\sum_{i=0}^{k}(n-m)_{k+1-i}S_i^k f(z)+\sum_{i=1}^{k+1}(n-m)_{k-(i-1)}D_{\alpha_{k+1}}S_{i-1}^k f(z)\right)-F_{k+1}(z)\\
&= \mu\Bigg((n-m)_{k+1}f(z)+\sum_{i=1}^{k}(n-m)_{k+1-i)}\Bigg(S_i^k f(z)\\
&\quad +D_{\alpha_{k+1}}S_{i-1}^k f(z)\Bigg)+D_{\alpha_{k+1}}S_k^k f(z)\Bigg)-F_{k+1}(z)\\
&= \mu\left((n-m)_{k+1}f(z)+\sum_{i=1}^{k}(n-m)_{k+1-i}S_i^{k+1}f(z)+S_{k+1}^{k+1}f(z)\right)-F_{k+1}(z)\\
&= \mu\left(\sum_{i=0}^{k+1}(n-m)_{k+1-i}S_i^{k+1}f(z)\right)-F_{k+1}(z).
\end{aligned}$$

This proves that the result is true for $k+1$ also. Hence by the principal of mathematical induction we conclude that the result is true for all $t,\ 1\le t<n$. □

Lemma 3.4 *If $p(z)$ is a polynomial of degree n, then for complex numbers $\beta,\ \alpha_i$ with $|\beta|<1,\ |\alpha_i|\geq k,\ k\leq 1,\ 1\leq i\leq t<n$ and $|z|\geq k$,*

$$\left|z^tp_t(z)+\frac{\beta}{(1+k)^t}n_tA_{\alpha_t}p(z)\right|+\left|z^tq_t(z)+\frac{\beta}{(1+k)^t}n_tA_{\alpha_t}q(z)\right|$$

$$\leq n_t\left\{\frac{|z|^n}{k^n}\left|\xi+\frac{\beta}{(1+k)^t}A_{\alpha_t}\right|+\left|z^t+\frac{\beta}{(1+k)^t}A_{\alpha_t}\right|\right\}\max_{|z|=k}|p(z)|,$$

where $q(z)=(z/k)^n\overline{p(k^2/\overline{z})}$.

Proof:

Let λ be an arbitrary number with $|\lambda|>1$ and $M=\max_{|z|=k}|p(z)|$, by Rouche's Theorem all the zeros of the polynomial $g(z):=p(z)+\lambda M(z/k)^n$ lie in $|z|<k,\ k\leq 1$, so that $h(z):=(z/k)^n\overline{g(k^2/\overline{z})}=q(z)+\overline{\lambda}M$ will have all zeros in $|z|>k$ and $|h(z)|=|g(z)|$ for $|z|=k$. Therefore, for any arbitrary number δ with $|\delta|<1$, the polynomial $\delta h(z)+g(z)$ has all its zeros in $|z|<k$, $k\leq 1$ and hence by Lemma 1, for all α_i with $|\alpha_i|\geq k,\ i=1,2,...,t,(t<n)$ and for $|z|\geq 1$, we have

$$n_tA_{\alpha_t}|\delta h(z)+g(z)|\leq(1+k)^t|z^t(\delta h_t(z)+g_t(z))|.$$

Since, $|\alpha_i|\geq k,\ for\ i=1,2,...t$, by Lemma 2 the polynomial $\delta h_t(z)+g_t(z)$ has all its zeros in $|z|<k$. By Rouche's Theorem for any real or complex number β with $|\beta|<1$, the polynomial $(1+k)^tz^t(\delta h_t(z)+g_t(z))+n_tA_{\alpha_t}\beta(\delta h(z)+g(z))$ has all its zeros in $|z|<k$. Equivalently, for $|z|\geq k$

$$\delta\{(1+k)^tz^th_t(z)+n_tA_{\alpha_t}\beta h(z)\}+\{(1+k)^tz^tg_t(z)+n_tA_{\alpha_t}\beta g(z)\}\neq 0.$$

Note that $|\delta|<1$, hence we have for $|z|\geq k$,

$$\left|(1+k)^tz^th_t(z)+n_tA_{\alpha_t}\beta h(z)\right|\leq\left|(1+k)^tz^tg_t(z)+n_tA_{\alpha_t}\beta g(z)\right|. \tag{3.1}$$

Replacing values of $h(z)$ and $g(z)$ in inequality (3.1) and then using Lemma 3, we get for $|z|\geq k$,

$$\left|(1+k)^tz^tq_t(z)+n_tA_{\alpha_t}\beta q(z)+\overline{\lambda}n_t\{(1+k)^tz^tM+A_{\alpha_t}\beta M\}\right|$$

$$\leq\left|(1+k)^tz^tp_t(z)+n_tA_{\alpha_t}\beta p(z)+\frac{\lambda}{k^n}n_t\{(1+k)^t\alpha_1\alpha_2...\alpha_tM+A_{\alpha_t}\beta M\}z^n\right|. \tag{3.2}$$

By choosing a suitable argument of λ on the right hand of equation (3.2) which is possible by Corollary 1, we get for $|z|\geq k$,

$$\left|(1+k)^tz^tq_t(z)+n_tA_{\alpha_t}\beta q(z)\right|-|\overline{\lambda}|n_t\left|(1+k)^tz^t+A_{\alpha_t}\beta\right|M$$

$$\leq\frac{|\lambda|}{k^n}n_t\left|(1+k)^t\alpha_1\alpha_2...\alpha_t+A_{\alpha_t}\beta\right|M|z^n|-\left|(1+k)^tz^tp_t(z)+n_tA_{\alpha_t}\beta p(z)\right|.$$

Letting $|\lambda|\to 1$, gives the required result.

□

4. Proofs of the Theorems

Proof: [Proof of Theorem 1] Since by hypothesis, $|f(z)| \leq |F(z)|$ for $|z| = k$, any zero of $F(z)$ that lies on $|z| = k$ is also zero of $f(z)$. Hence, for any real or complex number λ with $|\lambda| < 1$, it follows by Rouche's Theorem, that the polynomial $H(z) := \lambda f(z) - F(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$. Therefore, applying Lemma 1 to the polynomial $H(z)$, we have for $|z| \geq 1$

$$n_t A_{\alpha_t}|H(z)| \leq (1+k)^t|z^t H_t(z)|. \tag{4.1}$$

Since $H(z)$ has all its zeros in $|z| \leq k,\ k \leq 1$ and $|\alpha_i| \geq k,\ 1 \leq i \leq t < n$, by Lemma 2, the polynomial $H_t(z)$ has all its zeros in $|z| \leq k$. Hence, for any real or complex number β with $|\beta| < 1$, it follows from inequality (4.1) by direct application of Rouche's Theorem that the polynomial $G(z) := (1+k)^t z^t H_t(z) + \beta n_t A_{\alpha_t} H(z)$ has all its zeros in $|z| \leq k$. Applying Lemma 3 to the polynomial $G(z)$, it follows that

$$\begin{aligned} G(z) := \lambda\Big\{(1+k)^t z^t \sum_{i=0}^{t}(n-m)_{t-i}S_i f(z) + \beta n_t A_{\alpha_t} f(z)\Big\} \\ -\{(1+k)^t z^t F_t(z) + \beta n_t A_{\alpha_t} F(z)\} \neq 0 \end{aligned}$$

for $|z| \geq k$. Since $|\lambda| < 1$, we conclude that for $|z| \geq k$,

$$\left|z^t \sum_{i=0}^{t}(n-m)_{t-i}S_i f(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} f(z)\right| \leq \left|z^t F_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} F(z)\right|.$$

This completes the proof of Theorem 1.

□

Proof: [Proof of Corollary 7] Since all zeros of $F(z)$ lie in $|z| \leq k,\ k \leq 1$, by Lemma 1 for complex numbers α_i, β with $|\alpha_i| \geq k,\ 1 \leq i \leq t < n$, β with $|\beta| < 1$ and for $|z| \geq k$,

$$|z^t F_t(z)| > \frac{n_t}{(1+k)^t} A_{\alpha_t}|\beta||F(z)|.$$

Hence we can choose argument of β such that

$$\left|z^t F_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} F(z)\right| = \left|z^t F_t(z)\right| - \left|\frac{\beta}{(1+k)^t} n_t A_{\alpha_t} F(z)\right|. \tag{4.2}$$

Using triangle inequality on the left hand side and equation (4.2) on the right hand side of inequality (21), we obtain

$$\left|z^t \sum_{i=0}^{t}(n-m)_{t-i}S_i f(z)\right| - |\beta|\left|\frac{A_{\alpha_t}}{(1+k)^t} n_t f(z)\right| \leq \left|z^t F_t(z)\right| - |\beta|\left|\frac{A_{\alpha_t}}{(1+k)^t} n_t F(z)\right|.$$

Letting $|\beta| \to 1$, gives us the required result.

□

Proof: [Proof of Theorem 2] Since $p(z)$ does not vanish in $|z| < k$, $q(z) = (z/k)^n\overline{p(k^2/\overline{z})}$ has all its zeros in $|z| \leq k$, hence applying Corollary 3, we have for all $\beta,\ \alpha_i$ with $|\beta| < 1,\ |\alpha_i| \geq k,\ 1 \leq i \leq t < n$ and $|z| \geq k$,

$$\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| \leq \left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z)\right|. \tag{4.3}$$

Let $m' = \min_{|z|=k} |p(z)|$. If $p(z)$ has all zeros in $|z| = k$, then $m' = 0$ and the result trivially follows form inequality (4.3). Therefore, we suppose that all the zeros of $p(z)$ lie in $|z| > k$, $k \leq 1$, so that $m' > 0$. Further let λ be an arbitrary number satisfying $|\lambda| < 1$. Consider the polynomial $g(z) := p(z) - \lambda m'$. Since $m' \leq |p(z)|$ for $|z| = k$, by Rouche's Theorem, $g(z)$ does not vanish in $|z| < k$, so that the polynomial $h(z) = (z/k)^n \overline{g(k^2/\overline{z})} = q(z) - \overline{\lambda} m'(z/k)^n$ has all zeros in $|z| < k$ and $|g(z)| = |h(z)|$ for $|z| = k$. Hence applying Theorem 1, we have for all $\beta,\ \alpha_i$ with $|\beta| < 1,\ |\alpha_i| \geq k,\ 1 \leq i \leq t < n$ and $|z| \geq k$,

$$\left|z^t g_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} g(z)\right| \leq \left|z^t h_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} h(z)\right|.$$

Substituting values of $h(z)$ and $g(z)$, we have for $|z| \geq k$,

$$\begin{aligned}&\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z) - \lambda m' n_t \left(z^t + \frac{\beta}{(1+k)^t} A_{\alpha_t}\right)\right| \\ &\qquad \leq \left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z) - \frac{\overline{\lambda} z^n m'}{k^n} n_t \left(\xi + \frac{\beta}{(1+k)^t} A_{\alpha_t}\right)\right|. \end{aligned} \tag{4.4}$$

Since $q(z)$ has all zeros in $|z| < k,\ k \leq 1$, choosing a suitable argument of λ in inequality (4.4), which is possible by Corollary 4, we have for $|z| \geq k$,

$$\begin{aligned}&\left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| - |\lambda| m' n_t \left|z^t + \frac{\beta}{(1+k)^t} A_{\alpha_t}\right| \\ &\qquad \leq \left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z)\right| - \frac{|\lambda||z^n| m'}{k^n} n_t \left|\xi + \frac{\beta}{(1+k)^t} A_{\alpha_t})\right|.\end{aligned}$$

Letting $|\lambda| \to 1$, we get the desired result.

□

Proof: [Proof of Theorem 3] Let $m' = \min_{|z|=k} |p(z)|$ and μ be any arbitrary number with $|\mu| < 1$, then by Roucher's Theorem the polynomial $T(z) = p(z) + \mu m'(z/k)^n$ has all zeros in $|z| \leq k$. Moreover, for $W(z) = (z/k)^n \overline{T(k^2/\overline{z})} = q(z) + \overline{\mu} m'$, where $q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$, $|W(z)| \leq |T(z)|$ for $|z| = k$. Hence using Theorem 1, we get

$$\left|z^t W_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} W(z)\right| \leq \left|z^t T_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} T(z)\right|.$$

Substituting the values of $T(z)$ and $W(z)$, we obtain

$$\begin{aligned}&\left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z) + \overline{\mu} n_t \left(z^t + \frac{\beta}{(1+k)^t} A_{\alpha t}\right) m'\right| \\ &\qquad \leq \left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z) + \frac{\mu n_t z^n}{k^n} \left(\xi + \frac{\beta}{(1+k)^t} A_{\alpha t}\right) m'\right|.\end{aligned} \tag{4.5}$$

Inasmuch as $p(z)$ has all zeros in $|z| \leq k$, choosing argument of μ on the right hand side of equation (4.5) which is possible by Corollary 4, we obtain

$$\begin{aligned}&\left|z^t q_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} q(z)\right| - |\mu| n_t \left|z^t + \frac{\beta}{(1+k)^t} A_{\alpha t}\right| m' \\ &\qquad \leq \left|z^t p_t(z) + \frac{\beta}{(1+k)^t} n_t A_{\alpha_t} p(z)\right| - \frac{|\mu| n_t |z|^n}{k^n} \left|\xi + \frac{\beta}{(1+k)^t} A_{\alpha t}\right| m'.\end{aligned}$$

Letting $|\mu| \to 1$, gives us the required result. □

5. Declaration

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Idrees Qasim,
Department of Mathematics,
National Institute of Technology Srinagar,
India.
E-mail address: `idreesf3@nitsri.ac.in`

and

Sivaramakrishnan Chinnakali,
Department of Mathematics,
National Institute of Technology Srinagar,
India.
E-mail address: `sivaramakrishnan@nitsri.ac.in`