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Application of Nevanlinna's Method in Approximation of functions

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ABSTRACT: In the recent past, while a galaxy of researchers were contributing immensely towards the degree of approximation of functions, quite a few significant results had been published by another galaxy of researchers using Nevanlinna's method. Looking at the growing interest of approximation of functions, we in this paper have studied the degree of approximation of Fourier series of functions in Besov Space using Nevanlinna's mean.

Key Words: Degree of approximation, Minkowski's inequality, Hölder's inequality, Besov space, Fourier series, Nevanlinna mean.

Contents

1. Introduction

The problem of approximation of a function is to select a function from a well defined class that approximates the target function. The concept of approximating a function was first introduced by K. Weierstrass and later it was developed by P. L. Chebyshev and others. The theory of approximation emerged as the potential field of research when the scientists can visualize the wide applications of it in both Mathematical Sciences (eg: Constructive approximation of functions, Solution of integral equations etc) and Engineering Sciences (eg: Signal analysis, Computer-aided geometric design, Digital image processing etc). From the literature survey, we found the applications of Summability Theory, Numerical Methods, Mathematical Statistics, Probability Theory, etc. in approximating the functions. A handful of results on degree of approximation of functions in Besov Space using different summability means established by various researchers (see [1], [3], [5], [7], [8], [9], [10] and [14]), left a significant impact in the field of Science and Engineering. In 2008, Pani [12] established a result on degree of approximation of Fourier series in the Hölder metric using the Nevanlinna's mean. This motivated us to study the degree of approximation of functions associated with Fourier series in the Besov space using Nevanlinna's mean.

The rest part of our paper is organized as follows. In section 2, we have stated all the definitions (that are required for the proof of the theorem) and mentioned all the notations relevent to this article. Section 3 is about the results and discussion which consists of the known theorem, the new theorem that we have prposed, lemmas required along with their proofs, and the concluding remarks. In section 4, we have acknowledged every individual for their contribution to develop this article. Finally, we have given the list of all refereces in the reference section.

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2. Definitions and Preliminariess

Let λ be periodic in $[0, 2\pi]$ such that $\int_0^{2\pi} |\lambda(u)|^\nu du < \infty$.
We denote

$$L_\nu[0, 2\pi] = \Big\{\lambda : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |\lambda(u)|^\nu du < \infty\Big\},\ \nu \geq 1.$$

The Fourier series of $\lambda(u)$ is given by

$$\sum_{j=0}^{\infty} \chi_j(u) = \frac{p_0}{2} + \sum_{j=1}^{\infty} \Big(p_j \cos ju + q_j \sin ju\Big). \tag{2.1}$$

When $0 < \nu < 1$, we can still regard (2.1) as the Fourier series of λ by further assuming $\lambda(u) \cos ju$ and $\lambda(u) \sin ju$ are integrable (see p.9 [16]). Here p_j, q_j are the Fourier coefficients.
The m^{th} order modulus of smoothness of a function $\lambda : B \to \mathbb{R}$ is defined by (see [15])

$$w_m(\lambda, x) = \sup_{0<\eta\leq x} = \{\sup |\Delta_\eta^m(\lambda, v)| : v, v + m\eta \in B\}, x \geq 0, \tag{2.2}$$

where

$$\Delta_\eta^m(\lambda, v) = \sum_{j=0}^{m} (-1)^{m-1} \binom{m}{j}\ \lambda(v + j\eta),\ m \in \mathbb{N}$$

and

$$B = \mathbb{R}, \mathbb{R}_+[c, d] \subset \mathbb{R}.$$

The m^{th} order modulus of smoothness of $\lambda \in L_\nu(B), 0 < \nu \leq \infty$ is defined by

$$w_m(\lambda, x)_\nu = \sup_{0<\eta\leq x} \|\Delta_\eta^m(\lambda, .)\|_\nu,\ x \geq 0. \tag{2.3}$$

Suppose $m = [\gamma] + 1$ and $\gamma > 0$. Then for $\nu > 0$ and $\mu \leq \infty$, the Besov space $B_\mu^\gamma(L_\nu)$ can be defined as (see [13])

$$B_\mu^\gamma(L_\nu) = \{\lambda \in L_\nu : |\lambda|_{B_\mu^\gamma(L_\nu)} = \|w_m(\lambda, .)\|_{(\gamma,\ \mu)} \text{ is finite}\}, \tag{2.4}$$

where

$$\|w_m(\lambda, .)\|_{(\gamma,\mu)} = \Big\{\int_0^\infty (x^{-\gamma} w_m(\lambda, x)_\nu)^\mu \frac{dx}{x}\Big\}^{\frac{1}{\mu}}, \text{for } 0 < \mu < \infty$$

and

$$\|w_m(\lambda, .)\|_{(\gamma,\mu)} = \sup_{x>0} x^{-\gamma} w_m(\lambda, x)_\nu, \text{for } \mu = \infty.$$

Clearly, $\|w_m(\lambda, .)\|_{(\gamma,\ \mu)}$ is a seminorm (see [13]) if $1 \leq \nu,\ \mu \leq \infty$.
The Besov norm for $B_\mu^\gamma(L_\nu)$ is

$$\|\lambda\|_{B_\mu^\gamma(L_\nu)} = \|\lambda\|_\nu + \|w_m(\lambda, .)\|_{(\gamma,\ \mu)}. \tag{2.5}$$

Clearly, for fixed γ and ν,

$$B_\mu^\gamma(L_\nu) \subset B_{\mu_1}^\gamma(L_\nu),\ \mu < \mu_1.$$

For fixed ν and μ,

$$B_\mu^\gamma(L_\nu) \subset B_\mu^\delta(L_\nu), \delta < \gamma,$$

and for fixed γ and μ,

$$B^{\gamma}_{\mu}(L_{\nu}) \subset B^{\gamma}_{\mu}(L_{\nu_1}),\ \nu_1 < \nu.$$

Let $\sum \chi_j$ be an infinite series with sequence of partial sums $\{\xi(j)\}$. The Nevanlinna mean (N_Γ -mean) of $\{\xi(j)\}$ is given by (see [6])

$$N_\Gamma\ \xi(j) = \int_0^1 \Gamma(\theta)\ \xi(j\theta)\ d\theta,$$

where $\Gamma(\theta)$ is such that

$$(i)\ \Gamma(\theta) \text{ is non-negative and monotonically increasing for } 0 < \theta < 1, \tag{2.6}$$

$$(ii)\ \int_0^1 \Gamma(\theta)\ d\theta = 1,\ \text{ and} \tag{2.7}$$

$$(iii)\ \lim_{\epsilon\to 0} \int_0^{1-\epsilon} \Gamma(\theta)\ \log \frac{1}{1-\theta}\ d\theta \text{ exists.} \tag{2.8}$$

If

$$\lim_{j\to\infty} N_\Gamma\ \xi(j) = s,$$

we say that $\xi(j)$ is N_Γ summable to s (see [4], [6]). In short we write

$$\lim_{j\to\infty} \xi(j) = s(N_\Gamma).$$

It is known that N_Γ method is regular.(see [6])
Let $S_q\Big(\lambda; u\Big)$ denotes the q^{th} partial sum of the Fourier series (2.1). It is known that (see [16])

$$S_q\Big(\lambda; u\Big) - \lambda(x) = \frac{1}{\pi}\int_0^{\pi} \varphi(u,x)\ \frac{\sin\Big(q+\frac{1}{2}\Big)x}{2\sin\frac{x}{2}}\ dx\ , \tag{2.9}$$

where

$$\varphi(u,x) = \lambda(u+x) + \lambda(u-x) - \lambda(u)\ . \tag{2.10}$$

Let $\sigma_q(\lambda; u)$ be the N_Γ mean of the Fourier series (2.1) then

$$\sigma_q(\lambda; u) = \int_0^1 \Gamma(\theta)\ S_q(\lambda; u)\ d\theta. \tag{2.11}$$

We know that (see [16])

$$l_q(u) = \sigma_q(\lambda; u) - \lambda(u) = \frac{1}{\pi}\int_0^{\pi} \varphi(u,x)\ \kappa_q(x)\ dx, \tag{2.12}$$

where

$$\kappa_q(x) = \int_0^1 \Gamma(\theta)\ \frac{\sin\Big(q\theta+\frac{1}{2}\Big)x}{2\sin\frac{x}{2}}\ d\theta. \tag{2.13}$$

Following additional notations are also used in the rest part of our present article.

$$\Psi(u,x,y) = \varphi(u+x,y) - \varphi(u,y), \text{ for } 0<\gamma<1 \tag{2.14}$$

and

$$\Psi(u,x,y) = \varphi(u+x,y) + \varphi(u-x,y) - 2\varphi(u,y), \text{ for } 1\le\gamma<2 \ . \tag{2.15}$$

For $m=[\gamma]+1, \nu\ge 1$; we have

$$w_m(\lambda,x)_\nu = w_1(\lambda,x)_\nu, \text{ for } 0<\gamma<1 \tag{2.16}$$

and

$$w_m(\lambda,x)_\nu = w_2(\lambda,x)_\nu, \text{ for } 1\le\gamma<2 \ . \tag{2.17}$$

We write

$$L_q(u,x) = l_q(u+x) - l_q(u), \text{ for } 0<\gamma<1 \tag{2.18}$$

and

$$L_q(u,x) = l_q(u+x) + l_q(u-x) - 2l_q(u), \text{ for } 1\le\gamma<2 \ . \tag{2.19}$$

By using (2.12), (2.14) and (2.15), we have

$$L_q(u,x) = \frac{1}{\pi}\int_0^\pi \Psi(u,x,y)\ \kappa_q(y)\ dy \ . \tag{2.20}$$

Using the definition of $w_m(\lambda,x)_\nu$, (2.18) and (2.19), we have

$$w_m(l_q,x)_\nu = \|L_q(.,x)\|_\nu \ . \tag{2.21}$$

3. Results and Discussion

Using Nevanlinna mean in Hölder metric, Pani established the following result (see [12]).

Theorem 3.1 *Let $\Gamma(\theta)$ be a function satisfying all the conditions given in (2.6), (2.7) and (2.8) respectively.*
If $0<\nu<1$ and $\lambda\in H(\gamma,\nu)$, $0\le\delta<\gamma\le 1$ then

$$\|N_\Gamma S_q(\lambda) - \lambda(\pi/j)\|_{\delta,\nu} = O(1)\frac{1}{q^{\gamma-\delta-\nu+1}}\int_{1/q}^1 \frac{[Q(h)]^\nu}{h^{\gamma-\delta-\nu+2}}\ dh,$$

where

$$Q(h) = \int_{1-h}^1 \Gamma(\theta)\ d\theta \quad \textit{with } h = \frac{\pi}{qh}\le 1,$$

$N_\Gamma S_q(\lambda;u)$ *denote the N_Γ transform of the Fourier series (2.1).*

We have generalized the above result and proved the following Theorem.

Theorem 3.2 *Let $0\le\delta<\gamma<2$. If $\lambda\in B_\mu^\gamma(L_\nu)$, $\nu\ge 1$, then*

$$\|l_q(.)\|_{B_\mu^\delta(L_\nu)} = O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big)\Big\{\int_{\frac{1}{q}}^1\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{2}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}};\quad \textit{for } 1<\mu<\infty, \tag{3.1}$$

and

$$\|l_q(.)\|_{B_\mu^\delta(L_\nu)} = O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\int_{\frac{1}{q}}^1 \frac{Q(t)}{t^{\gamma-\delta+1}}\ dt;\quad \textit{for } \mu=\infty \ .$$

We require the following lemmas to prove Theorem 3.2.

Lemma 3.1 *Let $1 \leq \nu \leq \infty$ and $0 < \gamma < 2$. If $\lambda \in L_\nu[0, 2\pi]$, then for $x > 0,\ y \leq \pi$,*

$$(i)\|\Psi(., x, y)\|_\nu \leq 4w_m(\lambda, x)_\nu\ ,$$
$$(ii)\|\Psi(., x, y)\|_\nu \leq 4w_m(\lambda, y)_\nu\ ,$$
$$(iii)\|\Psi(x)\|_\nu \leq 2w_m(\lambda, x)_\nu\ .$$

Lemma 3.2 *Let $0 < \gamma < 2$. Suppose that $0 \leq \delta < \gamma$. If $\lambda \in B^\gamma_\mu(L_\nu), \nu \geq 1, 1 < \mu < \infty$, then*

$$(i) \int_0^\pi |\kappa_q(y)|\Big(\int_0^y \frac{\|\Psi(., x, y)\|_\nu^\mu}{x^{\delta\mu}}\frac{dx}{x}\Big)^{\frac{1}{\mu}} dy = O(1)\Big\{\int_0^\pi \Big(y^{\gamma-\delta}|\kappa_q(y)|\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}}$$

and

$$(ii) \int_0^\pi |\kappa_q(y)|\Big(\int_0^y \frac{\|\Psi(., x, y)\|_\nu^\mu}{x^{\delta\mu}}\frac{dx}{x}\Big)^{\frac{1}{\mu}} dy = O(1)\Big\{\int_0^\pi \Big(y^{\gamma-\delta+\frac{1}{\mu}}|\kappa_q(y)|\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}}\ .$$

Lemma 3.3 *Let $0 < \gamma < 2$. Suppose that $0 \leq \delta < \gamma$. If $\lambda \in B^\gamma_\mu(L_\nu), \nu \geq 1$ and $\mu = \infty$ then*

$$\sup_{0<x\leq y\leq\pi} x^{-\delta}\|\Psi(., x, y)\|_\nu = O\Big(y^{\gamma-\delta}\Big).$$

Lemma 3.4 *([4],Lemma 5.1) Suppose that $\Gamma(\theta)$ satisfies all the three conditions given in (2.6), (2.7) and (2.8) respectively. Let*

$$Q(h) = \int_{1-h}^1 \Gamma(\theta)d\theta \quad where \quad h = \frac{\pi}{qy} \leq 1. \tag{3.2}$$

Then

$$\Big|\int_0^1 \sin qy\theta \quad \Gamma(\theta)d\theta\Big| \leq Q(h) \tag{3.3}$$

and

$$\Big|\int_0^1 \cos qy\theta \quad \Gamma(\theta)d\theta\Big| \leq Q(h). \tag{3.4}$$

Lemma 3.5 *Let the Nevanlinna kernel of the Fourier series is as defined in (2.13), then for $0 < y \leq \pi$,*

$$(i)|\kappa_q(y)| = O(q)\ for\ 0 \leq y \leq \frac{\pi}{q},$$
$$(ii)|\kappa_q(y)| = O\Big(\frac{1}{y}\Big)\ for\ 0 \leq y \leq \frac{\pi}{q},\ and$$
$$(ii)|\kappa_q(y)| = O\Big(\frac{1}{y}Q(\pi/qy)\Big)\ if\ y \geq \frac{\pi}{q}.$$

Proof of Lemma 3.1

For $0 < \gamma < 1$, $m = [\gamma] + 1 = 1$.
By virtue of Eq.(2.14),

$$\Psi(u, x, y) = \varphi(u + x, y) - \varphi(u, y)$$

can be written as

$$\begin{aligned}\Psi(u, x, y) = &\{\lambda(u + x + y) - \lambda(u + y)\}\\ &+\{\lambda(u + x - y) - \lambda(u - y)\} - 2\{\lambda(u + x) - \lambda(u)\}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\Psi(u,x,y) = &\{\lambda(u+x+y)-\lambda(u+x)\}+\{\lambda(u-y+x)-\lambda(u+x)\}\\ &-\{\lambda(u+y)-\lambda(u)\}-\{\lambda(u-y)-\lambda(u)\}.\end{aligned} \tag{3.6}$$

Applying Minkowski's inequality to (3.5) and (3.6), we get

$$\|\Psi(.,x,y)\|_\nu \leq 4w_m(\lambda,x)_\nu, \text{ for } \nu \geq 1.$$

Which completes the proof of (i).
Again, For $1<\gamma<2$, $m=[\gamma]+1=2$.
By virtue of Eq.(2.15),

$$\Psi(u,x,y)=\varphi(u+x,y)+\varphi(u-x,y)-2\varphi(u,y)$$

can be written as

$$\begin{aligned}\Psi(u,x,y) = &\{\lambda(u+x+y)+\lambda(u+x-y)-2\lambda(u+x)\}\\ &+\{\lambda(u-x+y)+\lambda(u-x-y)-2\lambda(u-x)\}\\ &-2\{\lambda(u+y)+\lambda(u-y)-2\lambda(u)\}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\Psi(u,x,y) = &\{\lambda(u+x+y)+\lambda(u-x+y)-2\lambda(u+y)\}\\ &+\{\lambda(u+x-y)+\lambda(u-x-y)-2\lambda(u-y)\}\\ &-2\{\lambda(u+x)+\lambda(u-x)-2\lambda(u)\}.\end{aligned} \tag{3.8}$$

If we apply Minkowski's inequality to (3.7) and (3.8), we get

$$\|\Psi(.,x,y)\|_\nu \leq 4w_m(\lambda,y)_\nu.$$

Which completes the proof of (ii).
We have omitted the proof of (iii) as it is trivial.

Proof of Lemma 3.2

For the proof of (i), applying Lemma 3.1(i), we have

$$\begin{aligned}&\int_0^\pi |\kappa_q(y)|\Big(\int_0^y \frac{\|\Psi(.,x,y)\|_\nu^\mu}{x^{\delta\mu}}\frac{dx}{x}\Big)^{\frac{1}{\mu}}\,dy\\ &=O(1)\int_0^\pi |\kappa_q(y)|\Big\{\int_0^y\Big(\frac{w_m(\lambda,x)_\nu}{x^\gamma}\Big)^\mu x^{(\gamma-\delta)\mu}\frac{dx}{x}\Big\}^{\frac{1}{\mu}}\,dy\\ &=O(1)\int_0^\pi |\kappa_q(y)|\ y^{\gamma-\delta}\Big\{\int_0^y \frac{w_m(\lambda,x)_\nu}{x^\gamma}\frac{dx}{x}\Big\}^{\frac{1}{\mu}}\,dy\\ &=O(1)\int_0^\pi |\kappa_q(y)|\ y^{\gamma-\delta}\,dy\\ &\text{(By definition of Besov space and 2nd mean value theorem)}\\ &=O(1)\Big\{\int_0^\pi\Big(|\kappa_q(y)|\ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}}\,dy\Big\}^{1-\frac{1}{\mu}}\Big\{\int_0^\pi dy\Big\}^{\frac{1}{\mu}}\\ &\qquad\qquad\qquad\text{(By applying Hölder 's inequality)}\\ &=O(1)\Big\{\int_0^\pi\Big(|\kappa_q(y)|\ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}}\,dy\Big\}^{1-\frac{1}{\mu}}.\end{aligned}$$

Which completes the proof of (i).

For the proof of (ii), applying Lemma 3.1(ii), we have

$$\begin{aligned}
&\int_0^{\pi} |\kappa_q(y)| \Big(\int_0^{y} \frac{\|\Psi(.,x,y)\|_{\nu}^{\mu}}{x^{\delta\mu}} \frac{dx}{x} \Big)^{\frac{1}{\mu}} \ dy \\
&= O(1) \int_0^{\pi} |\kappa_q(y)| \ w_m(\lambda,y)_{\nu} \Big\{ \int_0^{y} \frac{dx}{x^{\delta\mu+1}} \Big\}^{\frac{1}{\mu}} \ dy \\
&= O(1) \int_0^{\pi} |K_q(y)| \ w_m(\lambda,y)_{\nu} \ y^{-\delta} \ dy \\
&= O(1) \int_0^{\pi} |\kappa_q(y)| \ y^{\gamma-\delta-\frac{1}{\mu}} \Big\{ \frac{w_m(\lambda,y)_{\nu}}{y^{\gamma+\frac{1}{\mu}}} \Big\} \ dy \\
&= O(1) \Big\{ \int_0^{\pi} \Big(\frac{w_m(\lambda,y)_{\nu}}{y^{\gamma}} \Big)^{\mu} \ \frac{dy}{y} \Big\}^{\frac{1}{\mu}} \Big\{ \int_0^{\pi} \Big(|\kappa_q(y)| \ y^{\gamma-\delta-\frac{1}{\mu}} \ dy \Big) \Big\}^{1-\frac{1}{\mu}} \\
&\qquad\qquad\qquad\qquad \text{(By using Hölder 's inequality)} \\
&= O(1) \Big\{ \int_0^{\pi} \Big(|\kappa_q(y)| \ y^{\gamma-\beta-\frac{1}{\mu}} \ dy \Big) \Big\}^{1-\frac{1}{\mu}}. \\
&\qquad\qquad\qquad\qquad \text{(By using the definition of Besov space)}
\end{aligned}$$

This completes the proof of (ii).

Proof of Lemma 3.3

For $0 < x \leq y \leq \pi$, using Lemma 3.1(i), we have

$$\begin{aligned}
\sup_{0<x\leq y\leq\pi} x^{-\delta} \|\Psi(.,x,y)\|_{\nu} &= \sup_{0<x\leq y\leq\pi} x^{\gamma-\delta} \Big\{ x^{-\gamma} \|\Psi(.,x,y)\|_{\nu} \Big\} \\
&\leq 4y^{\gamma-\delta} \sup_{x} \Big(x^{-\gamma} w_m(\lambda,x)_{\nu} \Big) \\
&= O\Big(y^{\gamma-\delta} \Big). \text{ (by the hypothesis)}
\end{aligned}$$

Again, for $0 < y \leq x \leq \pi$, using Lemma 3.1(ii), we have

$$\begin{aligned}
\sup_{0<y\leq x\leq\pi} x^{-\delta} \|\Psi(.,x,y)\|_{\nu} &\leq 4w_m(\lambda,y)_{\nu} \sup_{0<y\leq x\leq\pi} x^{-\delta} \\
&\leq 4y^{\gamma-\delta} \sup_{y} \Big(y^{-\gamma} w_m(\lambda,y)_{\nu} \Big) \\
&= O\Big(y^{\gamma-\delta} \Big). \text{ (by the hypothesis)}
\end{aligned}$$

This completes the proof of the Lemma 3.3.

Proof of Lemma 3.5

For $0 \leq y \leq \frac{\pi}{q}$,

$$
\begin{aligned}
(i)|\kappa_q(y)| &= \left| \int_0^1 \Gamma(\theta)\ \frac{\sin\ \left(q\theta + \frac{1}{2}\right)y}{2\sin\ \frac{y}{2}}\ d\theta \right| \\
&= \frac{1}{|2\sin\ \frac{y}{2}|}\left| \int_0^1 \Gamma(\theta)\ \sin\left(q\theta + \frac{1}{2}\right)y\ d\theta \right| \\
&\leq \frac{\pi}{2y}\ \int_0^1 \Gamma(\theta)\ \left|\sin\left(q\theta + \frac{1}{2}\right)y\right|\ d\theta \\
&\leq \frac{\pi}{2y}\ \int_0^1 \theta\Gamma(\theta)\ d\theta \\
&= O(q).\ (\text{since } \theta < 1 \text{ and } \int_0^1 \Gamma(\theta)\ d\theta = 1)
\end{aligned}
$$

$$
\begin{aligned}
(ii)|\kappa_q(y)| &= \frac{\pi}{2y}\ \int_0^1 \Gamma(\theta)\left|\sin\left(q\theta + \frac{1}{2}\right)y\right|\ d\theta \\
&\leq \frac{\pi}{2y}\int_0^1 \Gamma(\theta)\ d\theta \\
&= O\left(\frac{1}{y}\right).
\end{aligned}
$$

$$
\begin{aligned}
(iii)|\kappa_q(y)| &= \frac{1}{|2\sin\ \frac{y}{2}|}\left| \int_0^1 \Gamma(\theta)\ \sin\left(q\theta + \frac{1}{2}\right)y\ d\theta \right| \\
&= \frac{1}{|2\sin\ \frac{y}{2}|}\left| \int_0^1 \Gamma(\theta)\ [\sin qy\theta \cos\frac{y}{2} + \cos qy\theta \sin\frac{y}{2}]\ d\theta \right| \\
&\leq \frac{1}{|2\tan\ \frac{y}{2}|}\left| \int_0^1 \Gamma(\theta)\ \sin qy\theta\ d\theta \right| + \frac{1}{2}\left| \int_0^1 \Gamma(\theta) \cos qy\theta\ d\theta \right| \\
&\leq \frac{1}{y}Q\left(\frac{\pi}{qy}\right) + \frac{1}{2}Q\left(\frac{\pi}{qy}\right) \qquad (by\ Lemma\ 3.4) \\
&= O\left(\frac{1}{y}\ \ Q\left(\frac{\pi}{qy}\right)\right).
\end{aligned}
$$

Hence, the proof of Lemma 3.5 is completed.

Proof of Theorem 3.2

We first consider the case $1 < \mu < \infty$, we have for $\nu \geq 1$, $0 \leq \delta < \gamma < 2$,

$$
\|l_q(.)\|_{B^{\delta}_{\mu}L(\nu)} = \|l_q(.)\|_{\nu} + \|w_m(l_q, .)\|_{(\delta,\mu)}. \tag{3.9}
$$

Applying Lemma 3.1(iii) in (2.12), we get

$$
\begin{aligned}
\|l_q(.)\|_{\nu} &\leq \frac{1}{\pi}\int_0^{\pi} \|\Psi_.(y)\|_{\nu}|\kappa_q(y)|dy \\
\Rightarrow \|l_q(.)\|_{\nu} &\leq \frac{2}{\pi}\int_0^{\pi} |\kappa_q(y)|w_m(\lambda, y)_{\nu}\ dy.
\end{aligned} \tag{3.10}
$$

Applying Hölder's inequality, we have

$$
\begin{aligned}
\|l_q(.)\|_\nu &\le \frac{2}{\pi}\Big\{\int_0^\pi \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\Big\{\int_0^\pi \Big(\frac{w_m(\lambda,y)_\nu}{y^{\gamma+\frac{1}{\mu}}}\Big)^{\mu}\ dy\Big\}^{\frac{1}{\mu}}\\
&= O(1)\Big\{\int_0^\pi \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\text{(by defn.of Besov space)}\\
&= O(1)\Big[\Big\{\int_0^{\frac{\pi}{q}} \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}+\Big\{\int_{\frac{\pi}{q}}^\pi \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\Big]\\
&= O(1)[I+J],\ \text{(say)}.
\end{aligned}
\tag{3.11}
$$

Now,

$$
\begin{aligned}
I &= \Big\{\int_0^{\frac{\pi}{q}} \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(q)\Big\{\int_0^{\frac{\pi}{q}} \Big(y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\\
&= O\Big(\frac{1}{q^\gamma}\Big).
\end{aligned}
\tag{3.12}
$$

Again,

$$
\begin{aligned}
J &= \Big\{\int_{\frac{\pi}{q}}^\pi \Big(|\kappa_q(y)|\ y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^\pi \Big(\frac{1}{y}\ Q(\pi/qy)y^{\gamma+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}\ dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^\pi \Big(y^{\gamma+\frac{1}{\mu}-1}\Big)^{\frac{\mu}{\mu-1}}\ [Q(\pi/qy)]^{\frac{\mu}{\mu-1}}\ dy\Big\}^{\frac{\mu-1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{1}{q}}^1 \Big(\Big(\frac{\pi}{qt}\Big)^{\gamma-\frac{1}{\mu}-1}\Big)^{\frac{\mu}{\mu-1}}[Q(t)]^{\frac{\mu}{\mu-1}}\ \frac{\pi}{qt^2}\ dt\Big\}^{\frac{\mu-1}{\mu}}\\
&= O\Big(\frac{1}{q^\gamma}\Big)\Big\{\int_{\frac{1}{q}}^1 \Big(\frac{Q(t)}{t^{\gamma-\frac{1}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}dt\Big\}^{\frac{\mu-1}{\mu}}.
\end{aligned}
\tag{3.13}
$$

From (3.11), (3.12) and (3.13), it is concluded that

$$
\|l_q(.)\|_\nu = O\Big(\frac{1}{q^\gamma}\Big)\Big\{\int_{\frac{1}{q}}^1 \Big(\frac{Q(t)}{t^{\gamma-\frac{1}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}dt\Big\}^{\frac{\mu-1}{\mu}}.
\tag{3.14}
$$

Now,

$$
\begin{aligned}
\|w_m(l_q,.)\|_{(\delta,\mu)} &= \Big\{\int_0^\pi \Big(\frac{w_m(l_q,x)_\nu}{x^\delta}\Big)^{\mu}\frac{dx}{x}\Big\}^{\frac{1}{\mu}}\\
&= \Big\{\int_0^\pi \Big(\frac{\|L_q(.,x)\|_\nu}{x^\delta}\Big)^{\mu}\frac{dx}{x}\Big\}^{\frac{1}{\mu}}\\
&= \Big\{\int_0^\pi \Big(\int_0^\pi |L_q(u,x)|^\nu du\Big)^{\frac{\mu}{\nu}}\ \frac{dx}{x^{\delta\mu+1}}\Big\}^{\frac{1}{\mu}}\\
&= \Big\{\int_0^\pi \Big(\int_0^\pi \Big|\frac{1}{\pi}\int_0^\pi \Psi(u,x,y)\ \kappa_q(y)\ dy\Big|^\nu du\Big)^{\frac{\mu}{\nu}}\ \frac{dx}{x^{\delta\mu+1}}\Big\}^{\frac{1}{\mu}}.
\end{aligned}
$$

By repeated application of generalized Minkowski's inequality, we have

$$
\begin{aligned}
\|w_m(l_q,.)\|_{(\delta,\mu)} &\leq \frac{1}{\pi}\Big[\int_0^\pi \Big\{\int_0^\pi \Big(\int_0^\pi |\Psi(u,x,y)|^\nu \ |\kappa_q(y)|^\nu \ du\Big)^{\frac{1}{\nu}} \ dy\Big\}^\mu \frac{dx}{x^{\delta\mu+1}}\Big]^{\frac{1}{\mu}} \\
&= \frac{1}{\pi}\Big[\int_0^\pi \Big\{\int_0^\pi |\kappa_q(y)| \ \|\Psi(.,x,y)\|_\nu \ dy\Big\}^\mu \frac{dx}{x^{\delta\mu+1}}\Big]^{\frac{1}{\mu}} \\
&\leq \frac{1}{\pi}\int_0^\pi \Big(\int_0^\pi |\kappa_q(y)|^\mu \ \|\Psi(.,x,y)\|_\nu^\mu \frac{dx}{x^{\delta\mu+1}}\Big)^{\frac{1}{\mu}} dy \\
&= \frac{1}{\pi}\int_0^\pi \Big\{\Big(\int_0^y + \int_y^\pi\Big)\frac{\|\Psi(.,x,y)\|_\nu^\mu \ dx}{x^{\delta\mu+1}}\Big\}|\kappa_q(y)| \ dy \\
&\leq \frac{1}{\pi}\int_0^\pi \Big\{\int_0^y \frac{\|\Psi(.,x,y)\|_\nu^\mu \ dx}{x^{\delta\mu+1}}\Big\}|\kappa_q(y)| \ dy \\
&\qquad + \frac{1}{\pi}\int_0^\pi \Big\{\int_y^\pi \frac{\|\Psi(.,x,y)\|_\nu^\mu \ dx}{x^{\delta\mu+1}}\Big\}|\kappa_q(y)| \ dy.
\end{aligned}
$$

Applying Lemma 3.2, we have

$$
\begin{aligned}
\|w_m(l_q,.)\|_{(\delta,\mu)} &= O(1)\Big[\Big\{\int_0^\pi \Big(|\kappa_q(y)| \ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} \\
&\qquad + \Big\{\int_0^\pi \Big(|\kappa_q(y)| \ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}}\Big] \\
&= O(1)[I^{'} + J^{'}], \ \text{(say)}.
\end{aligned} \tag{3.15}
$$

Now,

$$
\begin{aligned}
I^{'} &= \Big\{\int_0^\pi \Big(|\kappa_q(y)| \ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} \\
&\leq \Big\{\int_0^{\frac{\pi}{q}} \Big(|\kappa_q(y)| \ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} + \Big\{\int_{\frac{\pi}{q}}^\pi \Big(|\kappa_q(y)| \ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} \\
&= O(1)[I_1^{'} + I_2^{'}], \ \text{(say)}.
\end{aligned} \tag{3.16}
$$

Applying Lemma 3.5 in $I_1^{'}$ and $I_2^{'}$, we have

$$
\begin{aligned}
I_1^{'} &= \Big\{\int_0^{\frac{\pi}{q}} \Big(|\kappa_q(y)| \ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} \\
&= O(q)\Big\{\int_0^{\frac{\pi}{q}} y^{(\gamma-\delta)\frac{\mu}{\mu-1}} dy\Big\}^{1-\frac{1}{\mu}} \\
&= O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big),
\end{aligned} \tag{3.17}
$$

and

$$\begin{aligned}
I_2^{'} &= \Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(|\kappa_q(y)|\ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(\frac{1}{y}\ Q(\pi/qy)\ y^{\gamma-\delta}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(y^{\gamma-\delta-1}Q(\pi/qy)\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{\pi}{qt}\Big)^{(\gamma-\delta-1)\frac{\mu}{\mu-1}}\frac{\pi}{qt^2}\ [Q(t)]^{\frac{\mu}{\mu-1}}dt\Big\}^{1-\frac{1}{\mu}}\\
&= O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{2}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}. \qquad (3.18)
\end{aligned}$$

Collecting the results from (3.16), (3.17) and (3.18), we have

$$I^{'} = O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{2}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}. \qquad (3.19)$$

Similarly,

$$\begin{aligned}
J^{'} &= \Big\{\int_{0}^{\pi}\Big(|\kappa_q(y)|\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&\le \Big\{\int_{0}^{\frac{\pi}{q}}\Big(|\kappa_q(y)|\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}} + \Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(|\kappa_q(y)|\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)[J_1^{'}+J_2^{'}],\ \text{(say)}. \qquad (3.20)
\end{aligned}$$

As before, if we apply Lemma 3.5 in $J_1^{'}$ and $J_2^{'}$ then we have

$$\begin{aligned}
J_1^{'} &= \Big\{\int_{0}^{\frac{\pi}{q}}\Big(|\kappa_q(y)|\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(q)\Big(\int_{0}^{\frac{\pi}{q}}y^{\frac{\mu}{\mu-1}(\gamma-\delta+1)-1}dy\Big)^{\frac{\mu-1}{\mu}}\\
&= O\Big(\frac{1}{q^{\gamma-\delta}}\Big), \qquad (3.21)
\end{aligned}$$

and

$$\begin{aligned}
J_2^{'} &= \Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(|\kappa_q(y)|\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(\frac{1}{y}\ Q(\pi/qy)\ y^{\gamma-\delta+\frac{1}{\mu}}\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{\pi}{q}}^{\pi}\Big(y^{\gamma-\delta+\frac{1}{\mu}-1}Q(\pi/qy)\Big)^{\frac{\mu}{\mu-1}}dy\Big\}^{1-\frac{1}{\mu}}\\
&= O(1)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{\pi}{qt}\Big)^{(\gamma-\delta+\frac{1}{\mu}-1)\frac{\mu}{\mu-1}}\frac{\pi}{qt^2}\ [Q(t)]^{\frac{\mu}{\mu-1}}dt\Big\}^{1-\frac{1}{\mu}}\\
&= O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{1}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}. \qquad (3.22)
\end{aligned}$$

From (3.20), (3.21) and (3.22),we get

$$J^{'}=O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{1}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}. \tag{3.23}$$

Using (3.15),(3.19) and (3.23), we get

$$\|w_m(l_q,.)\|_{(\delta,\mu)}=O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{2}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}. \tag{3.24}$$

From (3.9), (3.14) and (3.24), we get

$$\begin{aligned}&\|l_q(.)\|_{B^{\delta}_{\mu}}L(\nu)=\|l_q(.)\|_{\nu}+\|w_m(l_q,.)\|_{(\delta,\mu)}\\&=O\Big(\frac{1}{q^{\gamma-\delta-\frac{1}{\mu}}}\Big)\Big\{\int_{\frac{1}{q}}^{1}\Big(\frac{Q(t)}{t^{\gamma-\delta-\frac{2}{\mu}+1}}\Big)^{\frac{\mu}{\mu-1}}\ dt\Big\}^{1-\frac{1}{\mu}}.\end{aligned} \tag{3.25}$$

Now, we consider the case when $\mu=\infty$.
Clearly,

$$\begin{aligned}\|w_m(l_q,.)\|_{(\delta,\infty)}&=\sup_{x}\frac{\|L_q(.,x)\|_{\nu}}{x^{\delta}}\\&=\sup_{x>0}\frac{x^{-\delta}}{\pi}\Big[\int_0^{\pi}\Big|\int_0^{\pi}\Psi(u,x,y)\kappa_q(y)\ dy\Big|^{\nu}du\Big]^{\frac{1}{\nu}}.\end{aligned}$$

On application of generalized Minkowski's inequality, we get

$$\begin{aligned}\|w_m(l_q,.)\|_{(\delta,\infty)}&\leq\sup_{x>0}\frac{x^{-\delta}}{\pi}\int_0^{\pi}\Big\{\int_0^{\pi}|\Psi(u,x,y)|^{\nu}|\kappa_q(y)|^{\nu}\ du\Big\}^{\frac{1}{\nu}}\ dy\\&=\sup_{x>0}\frac{x^{-\delta}}{\pi}\int_0^{\pi}|\kappa_q(y)|\ \|\Psi(.,x,y)\|_{\nu}\ dy\\&\leq\frac{1}{\pi}\int_0^{\pi}|\kappa_q(y)|\ \Big\{\sup_{x>0}x^{-\delta}\|\Psi(.,x,y)\|_{\nu}\Big\}\ dy\\&=O(1)\int_0^{\pi}y^{\gamma-\delta}|\kappa_q(y)|\ dy\ \text{(by Lemma 3.5)}\\&=O(1)\Big[\int_0^{\frac{\pi}{q}}y^{\gamma-\delta}|\kappa_q(y)|\ dy+\int_{\frac{\pi}{q}}^{\pi}y^{\gamma-\delta}|\kappa_q(y)|\ dy\Big]\\&=O(1)[I^{"}+J^{"}],\ \text{(say)}.\end{aligned} \tag{3.26}$$

Again by usng Lemma 3.5 in $I^{"}$ and $J^{"}$, we have

$$\begin{aligned}I^{"}&=\int_0^{\frac{\pi}{q}}y^{\gamma-\delta}|\kappa_q(y)|\ dy\\&=O(q)\int_0^{\frac{\pi}{q}}y^{\gamma-\delta}\ dy\\&=O\Big(\frac{1}{q^{\gamma-\delta}}\Big),\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
J" &= \int_{\frac{\pi}{q}}^{\pi} y^{\gamma-\delta}|\kappa_q(y)|\ dy \\
&= O(1)\int_{\frac{\pi}{q}}^{\pi} \frac{1}{y}\ y^{\gamma-\delta}\ Q(\pi/qy)\ dy \\
&= O(1)\int_{\frac{\pi}{q}}^{\pi} y^{\gamma-\delta-1}\ Q(\pi/qy)\ dy \\
&= O(1)\int_{\frac{1}{q}}^{1} \Big(\frac{\pi}{qt}\Big)^{\gamma-\delta-1} \frac{\pi}{qt^2} Q(t)\ dt \\
&= O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\int_{\frac{1}{q}}^{1} \frac{Q(t)}{t^{\gamma-\delta+1}}\ dt. \qquad (3.28)
\end{aligned}$$

From (3.26), (3.27) and(3.28), we obtain

$$\|w_m(l_q,.)\|_{(\delta,\infty)} = O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\int_{\frac{1}{q}}^{1} \frac{Q(t)}{t^{\gamma-\delta+1}}\ dt. \qquad (3.29)$$

Also, for $\mu=\infty$,

$$\begin{aligned}
\|l_q(.)\|_{\nu} &\leq \frac{2}{\pi}\int_0^{\pi} |\kappa_q(y)| w_m(\lambda,y)_{\nu}\ dy \\
&= O(1)\int_0^{\pi} |\kappa_q(y)|\ y^{\gamma}\ dy \text{ (by the hypothesis)} \\
&= O(1)\Big[\int_0^{\frac{\pi}{q}} |\kappa_q(y)|\ y^{\gamma}\ dy + \int_{\frac{\pi}{q}}^{\pi} |\kappa_q(y)|\ y^{\gamma}\ dy\Big] \\
&= O[I_1" + J_1"],\ \text{(say)}. \qquad (3.30)
\end{aligned}$$

After splitting the integral as above and applying Lemma 3.5, it can be seen that

$$\begin{aligned}
I_1" &= \int_0^{\frac{\pi}{q}} |\kappa_q(y)|\ y^{\gamma}\ dy \\
&= O(q)\int_0^{\frac{\pi}{q}} y^{\gamma}\ dy \\
&= O\Big(\frac{1}{q^{\gamma}}\Big), \qquad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
J_1" &= \int_{\frac{\pi}{q}}^{\pi} |\kappa_q(y)|\ y^{\gamma}\ dy \\
&= O(1)\int_{\frac{\pi}{q}}^{\pi} \frac{1}{y}\ y^{\gamma}\ Q(\pi/qy)\ dy \\
&= O(1)\int_{\frac{\pi}{q}}^{\pi} y^{\gamma-1}\ Q(\pi/qy)\ dy \\
&= O(1)\int_{\frac{1}{q}}^{1} \Big(\frac{\pi}{qt}\Big)^{\gamma-1} \frac{\pi}{qt^2} Q(t)\ dt \\
&= O\Big(\frac{1}{q^{\gamma}}\Big)\int_{\frac{1}{q}}^{1} \frac{Q(t)}{t^{\gamma+1}}\ dt. \qquad (3.32)
\end{aligned}$$

From (3.30), (3.31) and (3.32), we get

$$\|l_q(.)\|_\nu = O\Big(\frac{1}{q^\gamma}\Big)\int_{\frac{1}{q}}^{1}\frac{Q(t)}{t^{\gamma+1}}\,dt. \tag{3.33}$$

From (3.29) and (3.33), we get

$$\begin{aligned}\|l_q(.)\|_{B^\delta_\infty L(\nu)} &= \|l_q(.)\|_\nu + \|w_m(l_q,.)\|_{(\delta,\infty)}\\ &= O\Big(\frac{1}{q^{\gamma-\delta}}\Big)\int_{\frac{1}{q}}^{1}\frac{Q(t)}{t^{\gamma-\delta+1}}\,dt.\end{aligned} \tag{3.34}$$

The proof of the above theorem follows from (3.14) and (3.34).

4. Concluding Remarks

Remark 4.1 *If $\Gamma(\theta) = \alpha(1-\theta)^{\alpha-1},\ 0<\alpha<1;\ 0<\theta<1,$ then N_Γ transformation reduces to (C,α) (Cesaro mean of order α) transformation. Hence our result generalizes the results of [1] and [12].*

Remark 4.2 *If $\Gamma(\theta) = 1$ then N_Γ transformation reduces to $(C,\ 1)$ transformation which is a specific case of Remark 4.1.*

5. Future Work

Present investigation can reach further heights by investigating the degree of approximation of conjuagate Fourier series and conjugate derived Fourier series by Nevanlinna's mean.

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