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On the Controllability Result for a Class of Nonlocal Fractional Systems with $\psi-$Caputo Fractional Derivatives

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ABSTRACT: This work presents new controllability results for nonlocal fractional differential systems of order $\beta \in (1, 2)$ in infinite-dimensional Banach spaces. By using some fixed point theorems and certain properties of compact evolution operators, we establish sufficient conditions ensuring the controllability result. Finally, we provide a nontrivial example to illustrate the practical implications of our theoretical findings and demonstrate the application of the developed theory.

Keywords: controllability, fractional differential systems, fractional cosine family, nonlocal conditions, mild solutions.

Contents

1. Introduction

In recent years, fractional differential systems have become increasingly important in many fields, including engineering,physics, biology, chemistry and finance. This popularity is Owing to the memory properties inherent in fractional derivatives, sorceress generalize integer-order derivatives and make it possible to describe phenomena that classical derivatives cannot model. Fractional systems are therefore particularly useful for studying complex phenomena where the dynamics of the system depend on its history. The notion of controllability is fundamental in the study of dynamical systems, particularly for differential systems governed by fractional-order equations. In this context, the controllability of integer-order differential systems in infinite Banach spaces has been studied by many researchers, as shown in the review article [3]. Stochastic functional differential equations have also been tackled, notably in the work of [13,16]. However, the study of the controllability of fractional differential systems has become a particularly active area of research in recent years. Notable studies have been carried out on the controllability of fractional evolution systems. For example, Wang and Zhou [22] studied the complete controllability of fractional evolution systems without resorting to the compactness of the characteristic operator of solutions. They used fixed-point theorems and fractional calculus techniques to establish sufficient conditions for controllability. Other research, such as that of Sakthivel et al. [19], has addressed the approximate controllability of fractional nonlinear stochastic systems in Hilbert spaces, assuming that the associated linear system is approximately controllable. Research on fractional semilinear differential systems has been enhanced by contributions from Wang and Zhou [23], who established existence and controllability results for fractional semilinear differential inclusions. Yan [28] formulated sufficient conditions for the approximate controllability of control systems described by state-dependent delay fractional functional differential equations in abstract spaces. Moreover, Wang et al. [25] addressed the optimal controllability of fractional infinite-delay differential integration systems in Banach spaces.Non-local conditions, which are a generalization of classical initial conditions, have also been introduced in the study of semilinear differential systems. These non-local conditions have shown better results in practical applications than traditional initial conditions. For example, Wang et al. [24] studied the non-local controllability of

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fractional evolution systems. In addition, Sakthivel and Ren [18] established new sufficient con ditions for the controllability of nonlinear fractional systems governed by abstract neutral fractional differential equations, using fixed point theorems and extending the results to systems with non-local conditions.The reader can consult articles as well [10,15,7,9,8] and the references therein for more details on fractional differential equations.

This work focuses on deriving sufficient conditions for the controllability of fractional differential systems of order $\beta \in (1,2]$ with nonlocal constraints in infinite Banach spaces. To this end, we make use of Sadovskii's fixed-point theorem and vector operator theory. Our approach extends existing results by introducing sophisticated mathematical frameworks, such as semigroup operators, fractional calculus, and advanced fixed-point techniques, to better address the complexities of fractional systems.

Most of the previously cited works on fractional differential systems focus on fractional derivatives of order between zero and one. This study aims to establish sufficient conditions for the controllability of fractional differential inclusions of order $\beta \in (1,2)$, as expressed in (1.1). When $\beta = 2$, the problem reduces to second-order differential systems, thereby extending classical results. In this paper, we investigate a specific class of fractional differential systems:

$$\begin{cases} {}^{C}D_t^{\beta,\psi} z(t) = Az(t) + Bu(t) + G(t, z(t)), \quad t \in \mathfrak{I} = [0, b], \\ z(0) + h(z) = z_0, \qquad z'(0) = y_0, \end{cases} \tag{1.1}$$

In this setting, ${}^{C}D_t^{\beta,\psi}$ is the ψ-Caputo fractional derivative for $\beta \in (1,2]$. The operator A serves as the infinitesimal generator for a strongly continuous β-order cosine family $\{\mathfrak{C}_\beta(t)\}_{t\geq 0}$ on the Banach space $\mathcal{Y}$. The state $z(\cdot)$ lies in $\mathcal{Y}$, and the control function $u(t)$ is an element of $L^2(\mathfrak{I}; U)$, with U denoting a Banach space of admissible control functions. The bounded linear operator B maps U into $\mathcal{Y}$. The functions $G : \mathfrak{I} \times \mathcal{Y} \to \mathcal{Y}$ and $h : C(\mathfrak{I}; \mathcal{Y}) \to \mathcal{Y}$ are defined, and the initial conditions $z_0, y_0 \in \mathcal{Y}$ are given.

This paper is organized as follows: Section 2 provides the necessary notation and background information, followed by Section 3, which details the controllability analysis of the system (1.1).The paper concludes with an example that illustrates the primary findings.

2. Preliminaries

This section introduces the fundamental notation and definitions that will be referenced throughout the paper.

- Let $\mathcal{Y}$ be a Banach space equipped with the norm $\|\cdot\|$.
- The space of all bounded linear operators on $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{Y})$.
- The space of $\mathcal{Y}$-valued Bochner integrable functions $g : \mathfrak{I} \to \mathcal{Y}$ is denoted by $L^q(\mathfrak{I}; \mathcal{Y})$, and it is equipped with the norm:

$$\|g\|_{L^q(\mathfrak{I};\mathcal{Y})} = \left(\int_{\mathfrak{I}} \|g(t)\|^q \, dt \right)^{1/q}, \quad \text{for } 1 \leq q < \infty.$$

- The spaces of continuous functions $g : \mathfrak{I} \to \mathcal{Y}$ and continuously differentiable functions $g : \mathfrak{I} \to \mathcal{Y}$ are denoted by $C(\mathfrak{I}; \mathcal{Y})$ and $C^1(\mathfrak{I}; \mathcal{Y})$, respectively. These are Banach spaces with the following norms:

$$\|g\|_C = \sup_{t\in\mathfrak{I}} \|g(t)\|_{\mathcal{Y}}, \quad \|g\|_{C^1} = \sup_{t\in\mathfrak{I}} \sum_{k=0}^{1} \|g^{(k)}(t)\|_{\mathcal{Y}}.$$

- The resolvent operator of a linear operator A on $\mathcal{Y}$ is denoted by:

$$R(\lambda, A) = (\lambda I - A)^{-1},$$

where I is the identity operator on $\mathcal{Y}$ and λ is a scalar.

Definition 1 *[1] For a function z, the ψ-Riemann-Liouville fractional integral of order $\beta > 0$ is defined by the following expression:*

$$J_t^{\beta,\psi} z(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s) \mathcal{D}_{t,s}^{\beta-1} z(s) ds, \tag{2.1}$$

where $z(t) \in L^1(\mathfrak{J}; \mathcal{Y})$, $\mathcal{D}_{t,s} = \psi(t) - \psi(s)$ and $\psi \in C^n(\mathfrak{J}, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \mathfrak{J}$.

Remark 1 *The choice of $\psi(t) = t$ in equation (2.1) leads to the Riemann-Liouville fractional integral, whereas $\psi(t) = \log(t)$ results in the Hadamard fractional integral.*

Definition 2 *[4]. For $z(t) \in L^1(\mathfrak{J}; \mathcal{Y})$ and $D_t^\beta z(t) \in L^1(\mathfrak{J}; \mathcal{Y})$ the Riemann-Liouville fractional derivative of order $\beta \in (1, 2)$ is devenby*

$$D_t^\beta z(t) = \frac{d^2}{dt^2} J_t^{2-\beta} z(t).$$

Definition 3 *[1] Let $\beta > 0$, $g \in C^{n-1}(\mathfrak{J}, \mathbb{R})$ and $\psi \in C^n(\mathfrak{J}, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The $\psi-$Caputo fractional derivative at order β of the function g is given by*

$${}^C D_t^{\beta,\psi} z(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \psi'(s) \mathcal{D}_{t,s}^{n-\beta-1} z_\psi^{[n]}(s) ds,$$

where

$$z_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n z(s) \quad \text{and} \quad n = [\beta] + 1,$$

and $[\beta]$ denotes the integer part of the real number β.

Remark 2 *Particularly, it's worth noting that when $\psi(t) = t$ and $\psi(t) = \log(t)$, equation (3) simplifies to the Caputo fractional derivative and the Caputo-Hadamard fractional derivative, respectively.*

Proposition 1 *[1] Let $g \in C^{n-1}(\mathfrak{J}, \mathbb{R})$ and $\beta > 0$, we obtain:*

1) ${}^C D_t^{\beta,\psi} J_t^{\beta,\psi} g(t) = g(t)$.

2) $J_t^{\beta,\psi}$ *is linear and bounded, with its domain and codomain being* $C(\mathfrak{J}, \mathbb{R})$.

3) $J_t^{\beta,\psi}\, {}^C D_t^{\beta,\psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_\psi^{[k]}(0)}{k!} \mathcal{D}_{t,0}^k$.

Proposition 2 *[1] Let $t \in \mathfrak{J}$ and $\mathfrak{e} > \nu > 0$, we obtain:*

1) $J_t^{\mathfrak{e},\psi} \mathcal{D}_{t,0}^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\mathfrak{e}+\nu)} \mathcal{D}_{t,0}^{\mathfrak{e}+\nu-1}$.

2) $D_t^{\mathfrak{e},\psi} \mathcal{D}_{t,0}^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mathfrak{e})} \mathcal{D}_{t,0}^{\nu-\mathfrak{e}-1}$.

3) $D_t^{\mathfrak{e},\psi} \mathcal{D}_{t,0}^k = 0, \quad \forall k < n \in \mathbb{N}$.

Definition 4 *[11] The generalized Laplace transform of z is defined by:*

$$\mathcal{L}_{\psi}\{z(t)\}(\lambda) := \widehat{z}(\lambda) = \int_0^{\infty} \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} z(t) dt,$$

for all $\lambda > 0$, where $z : \mathfrak{I} \to \mathbb{R}$ be real valued function.

Lemma 1 *[11] Let z be a piecewise continuous function on each interval $\mathfrak{I}$ and $\psi(t)$-exponential order and $\beta > 0$ and. Then we obtain:*

1. $\mathcal{L}_{\psi}\{J_t^{\beta,\psi} z(t)\}(\lambda) = \dfrac{\widehat{z}(\lambda)}{\lambda^{\beta}}.$
2. $\mathcal{L}_{\psi}\{{}^{C}D_t^{\beta,\psi} z(t)\}(\lambda) = \lambda^{\beta}\left[\mathcal{L}_{\psi}\{z(t)\} - \sum_{k=0}^{n-1} \lambda^{-k-1} z^{(k)}(0)\right],$
   where $n = [\beta] + 1$.

Definition 5 *[4]. The ψ-Caputo fractional derivative of order $\beta \in (1, 2)$ is defined by*

$$^{C}D_t^{\beta,\psi} z(t) = D_t^{\beta,\psi}\left(z(t) - z(0) - \mathcal{D}_{t,0} z'(0)\right), \tag{2.1}$$

where $z(t) \in L^1(\mathfrak{I}; \mathcal{Y}) \cap C^1(\mathfrak{I}; \mathcal{Y})$, $D_t^{\beta,\psi} z(t) \in L^1(\mathfrak{I}; \mathcal{Y})$.

Let $A : D(A) \subset \mathcal{F} \to \mathcal{F}$ is closed and densely defined operator in $\mathcal{F}$.

Examine the linear system with a fractional order $\beta \in (1, 2]$,given by:

$$\begin{cases} {}^{C}D_t^{\beta,\psi} z(t) = Az(t), \\ z'(0) = 0, \\ z(0) = \xi, \end{cases} \tag{2.2}$$

By applying $J_t^{\beta,\psi}$ to both sides of (2.2), we obtain:

$$z(t) = \xi + \frac{1}{\Gamma(\beta)} \int_0^t \mathcal{D}_{b,s}^{\beta-1} \psi'(s) A z(s) \mathrm{d}s. \tag{2.3}$$

Definition 6 *[4] We introduce a family $\{\mathfrak{C}_{\beta}(t)\}_{t\geq 0} \subset \mathbb{L}(\mathcal{F})$ as a solution operator (or the continuously strong β-order fractional cosine family) for equation (2.2), where $\beta \in (1, 2]$ and A is the infinitesimal generator of $\mathfrak{C}_{\beta}(t)$, under the following conditions:*

(i) *$\mathfrak{C}_{\beta}(0) = I$ and $\mathfrak{C}_{\beta}(t)$ is continuously strong for $t \geq 0$,*

(ii) *for all $t \geq 0$, $\xi \in D(A)$: $A\mathfrak{C}_{\beta}(t)\xi = \mathfrak{C}_{\beta}(t)A\xi$ and $\mathfrak{C}_{\beta}(t)D(A) \subset D(A)$,*

(iii)*$\mathfrak{C}_{\beta}(t)\xi$ satisfies equation (2.3) for all $\xi \in D(A)$.*

Definition 7 *[2]Associated with $\mathfrak{C}_{\beta}$, the fractional sine family $\mathfrak{S}_{\beta} : \mathbb{R}_+ \to \mathcal{B}(\mathcal{Y})$ is given by:*

$$\mathfrak{S}_{\beta}(t) = \int_0^t \mathfrak{C}_{\beta}(s) ds, \quad t \geq 0. \tag{2.4}$$

Definition 8 *[2]Associated with $\mathfrak{C}_{\beta}$, the fractional Riemann-Liouville family $\mathcal{P}_{\beta,\psi} : \mathbb{R}_+ \to \mathcal{B}(\mathcal{Y})$ is given by:*

$$\mathcal{P}_{\beta,\psi}(t) = J_t^{\beta-1,\psi} \mathfrak{C}_{\beta}(t). \tag{2.5}$$

Definition 9 *[2] We say that the β-order cosine family $\mathfrak{C}_\beta(t)$ is exponentially bounded if there are constants $\mathcal{K} \geq 1$ and $\omega \geq 0$ satisfying:*

$$\|\mathfrak{C}_\beta(t)\| \leq \mathcal{K} e^{\omega t}, \quad t \geq 0. \tag{2.6}$$

We say that an operator A belongs to $\mathcal{C}^\beta(\mathcal{K}, \omega)$ if there exists a β-order cosine family $\mathfrak{C}_\beta(t)$ solving problem (2.2) and fulfilling (2.6).

Next, we establish the definition of mild solutions for (1.1). For $A \in \mathcal{C}^\beta(\mathcal{K}, \omega)$ and its associated β-order cosine family $\mathfrak{C}_\beta(t)$, the following holds (see [4], (2.1)):

$$\lambda^{\beta-1} R\left(\lambda^\beta, A\right) \eta = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} \mathfrak{C}_\beta(t) \eta dt, \quad \eta \in \mathcal{Y}, \quad \operatorname{Re} \lambda > \omega. \tag{2.7}$$

By (2.4), (2.7), we obtain

$$\lambda^{\beta-2} R\left(\lambda^\beta, A\right) \eta = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} \mathfrak{S}_\beta(t) \eta dt, \quad \eta \in \mathcal{Y}, \quad \operatorname{Re} \lambda > \omega. \tag{2.8}$$

By (2.5), (2.7), we obtain

$$R\left(\lambda^\beta, A\right) \eta = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} \mathcal{P}_{\beta,\psi}(t) \eta dt, \quad \eta \in \mathcal{Y}, \quad \operatorname{Re} \lambda > \omega. \tag{2.9}$$

By lemma 1For the ψ-Caputo derivative, the Laplace transform is defined as: :

$$\mathcal{L}_\psi \left\{ {}^C D_t^{\beta,\psi} z(t) \right\} = \lambda^\beta \widehat{z}(\lambda) - z(0) \lambda^{\beta-1} - z'(0) \lambda^{\beta-2}. \tag{2.10}$$

By utilizing the generalized Laplace transforms to (1.1) and (2.10) and utilizing Lemma 1,we obtain

$$\lambda^\beta \widehat{z}(\lambda) - (z_0 - h(z)) \lambda^{\beta-1} - y_0 \lambda^{\beta-2} = A \widehat{z}(\lambda) + B \widehat{u}(\lambda) + \widehat{G}(\lambda).$$

where

$$\widehat{z}(\lambda) = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} z(t) dt,$$

and

$$\widehat{\mathfrak{u}}(\lambda) = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} \mathfrak{u}(t) dt,$$

and

$$\widehat{G}(\lambda) = \int_0^\infty \psi'(t) e^{-\lambda \mathcal{D}_{t,0}} g(t, z_t) dt,$$

Consequently,

$$\widehat{z}(\lambda) = \lambda^{\beta-1} R\left(\lambda^\beta, A\right) (z_0 - h(z)) + \lambda^{\beta-2} R\left(\lambda^\beta, A\right) y_0 + R\left(\lambda^\beta, A\right) B \widehat{u}(\lambda) + R\left(\lambda^\beta, A\right) \widehat{G}(\lambda)$$

.

Based on the properties of Laplace transforms and (2.7),(2.8), (2.9),

$$z(t) = \mathfrak{C}_\beta(t) (z_0 - h(z)) + \mathfrak{S}_\beta(t) y_0 + \int_0^t \mathcal{P}_{\beta,\psi}(t-s)(Bu(s) + G(s, z(s))) ds.$$

Definition 10 *[2,20] A function $z \in C(\mathfrak{I}; \mathcal{Y})$ is called a mild solution of (1.1) if z satisfies*

$$z(t) = \mathfrak{C}_\beta(t) (z_0 - h(z)) + \mathfrak{S}_\beta(t) y_0 + \int_0^t \mathcal{P}_{\beta,\psi}(t-s)(Bu(s) + G(s, z(s))) ds, \quad t \in \mathfrak{I}$$

.

Definition 11 *Controllability of the system (1.1) on $\mathfrak{I}$ means that, for every $z_0, y_0, z_1 \in \mathcal{Y}$, there is a control $u \in L^2(\mathfrak{I}; U)$ ensuring that a mild solution z of (1.1) satisfies $z(b) + h(z) = z_1$.*

Lemma 2 *(Sadovskii fixed point theorem).*
Let N be a condensing operator on a Banach space $\mathcal{Y}$, where N is continuous and maps bounded sets to bounded sets. Suppose that for every bounded set D with $\gamma(D) > 0$, the inequality $\gamma(N(D)) < \gamma(D)$ holds, where $\gamma(\cdot)$ denotes the Kuratowski measure of noncompactness. If $N(S) \subset S$ for a convex, closed, and bounded set $S \subset \mathcal{Y}$, then N has a fixed point in S.

3. Main results

In order to establish the main results, we introduce the following assumptions:

($\mathbf{A}_1$) The operator A acts as the infinitesimal generator for a β-order cosine family $\mathfrak{C}_\beta(t)$ defined on $\mathcal{Y}$, with a constant $\mathcal{K} \geq 1$ such that:

$$\|\mathfrak{C}_\beta(t)\| \leq \mathcal{K}.$$

($\mathbf{A}_2$) Let $\mathcal{O} : L^2(\mathfrak{I}; U) \to \mathcal{Y}$ be the linear operator given by

$$\mathcal{O}u = \int_0^b \mathcal{P}_{\beta,\psi}(b-s)Bu(s)\,ds.$$

The inverse operator $\mathcal{O}^{-1}$ maps into the quotient space formed by $L^2(\mathfrak{I}; U)$ modulo the kernel of $\mathcal{O}$, and there are constants $\mathcal{K}_1$ and $\mathcal{K}_2$ for which

$$\|B\| \leq \mathcal{K}_1 \quad \text{and} \quad \left\|\mathcal{O}^{-1}\right\| \leq \mathcal{K}_2.$$

($\mathbf{A}_3$) $G : \mathfrak{I} \times \mathcal{Y} \to \mathcal{Y}$ satisfied the Carathéodory condition:
- $G(\cdot, z)$ is measurable for all $z \in \mathcal{Y}$,
- $G(t, \cdot)$ is continuous for a.e. $t \in \mathfrak{I}$.

($\mathbf{A}_4$) A function G exists, defined as $G : \mathfrak{I} \times \mathcal{Y} \to \mathcal{Y}$, and $\mathcal{Y}$ is a compact set.

($\mathbf{A}_5$) A function $L_G(\cdot)$ exists in the space of integrable functions $L^1(\mathfrak{I}; \mathbb{R}_+)$ satisfying :

$$\|G(t, z_2) - G(t, z_2)\| \leq L_G(t)\|z_1 - z_2\|, \quad \text{for all } z_1, z_2 \in \mathcal{Y}.$$

(A_6) There exists a constant L_h such that

$$\|h(\mathfrak{e}) - h(\nu)\| \leq L_h\|\mathfrak{e} - \nu\|, \quad \mathfrak{e}, \nu \in C(\mathfrak{I}; \mathcal{Y}).$$

Theorem 1 *If assumptions $(A_1) - (A_6)$ are satisfied, then the system (1.1) is controllable on $\mathfrak{I}$, under the condition that:*

$$\left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\left(L_h + \frac{\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\|L_G\|_{L^1}\right) < 1. \tag{3.1}$$

Proof 1 *Using the condition (A_2), we define the control corresponding to an arbitrary function $z(\cdot) \in C(\mathfrak{I}; \mathcal{Y})$ as:*

$$\begin{aligned} u_z(t) =& \mathcal{O}^{-1}\left\{z_1 - h(z) - \mathfrak{C}_\beta(b)\left(z_0 - h(z)\right) - \mathfrak{S}_\beta(b)y_0 \right.\\ &\left. - \int_0^b \mathcal{P}_{\beta,\psi}(b-s)G(s, z(s))ds\right\}(t), \quad t \in \mathfrak{I}. \end{aligned} \tag{3.2}$$

Using this control method, we will demonstrate that the operator $\mathcal{Q}$, defined as:

$$
\begin{aligned}
(\mathcal{Q}z)(t) =& \mathfrak{C}_\beta(t)\left(z_0 - h(z)\right) + \mathfrak{S}_\beta(t) y_0 \\
&+ \int_0^t \mathcal{P}_{\beta,\psi}(t-s)\left(G(s,z(s)) + Bu_z(s)\right) ds, \quad t \in \mathfrak{I}.
\end{aligned} \tag{3.3}
$$

possesses a fixed point. For all $\tau > 0$, take:

$$
B_\tau = \{z \in C(\mathfrak{I};\mathcal{Y}) : \|z\| \leq \tau\}. \tag{3.4}
$$

Clearly, B_τ is a bounded, closed, and convex subset of $C(\mathfrak{I};\mathcal{Y})$.Our goal is to prove the existence of a $\tau > 0$ satisfying $\mathcal{Q}(B_\tau) \subset B_\tau$.If this condition is not met, then for every $\tau > 0$, we can find a function $z^\tau \in B_\tau$ with $\mathcal{Q}(z^\tau) \notin B_\tau$, implying $\|\mathcal{Q}(z^\tau)(t)\| > \tau$ for some $t \in \mathfrak{I}$. From (A_1), (A_2), (A_3), (A_5), (A_6), *and equations (2.4), (2.5), we conclude:*

$$
\begin{aligned}
\tau <& \left\|\left(\mathcal{Q}z^{\tau}\right)(t)\right\|, \\
\leq& \mathcal{K}\left\|z_0\right\| + \mathcal{K}\left\|\mathcal{Q}\left(z^{\tau}\right)\right\| + \mathcal{K}b\left\|y_0\right\| + \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \left\|G\left(s, z^{\tau}(s)\right)\right\| ds \\
&+ \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\left(\left\|z_1\right\| + \mathcal{Q}\left(z^{\tau}\right) + \mathcal{K}\left\|z_0\right\| + \mathcal{K}\left\|\mathcal{Q}\left(z^{\tau}\right)\right\|\right. \\
&\left.+\mathcal{K}b\left\|y_0\right\| + \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \left\|G\left(s, z^{\tau}(s)\right)\right\| ds\right), \\
\leq& \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\left\|z_1\right\| + \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\left\|z_0\right\| \\
&+ \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}b\left\|y_0\right\| \\
&+ \mathcal{K}\left(L_h\left\|z^{\tau}\right\| + \|h(0)\|\right) + \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \left(L_1(s)\left\|z^{\tau}(s)\right\| + \|G(s,0)\|\right) ds \\
&+ \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\left(\left(\mathcal{K}+1\right)\left(L_h\left\|z^{\tau}\right\| + \|h(0)\|\right)\right. \\
&\left.+\frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \left(L_G(s)\left\|z^{\tau}(s)\right\| + \|G(s,0)\|\right) ds\right), \\
\leq& \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\left\|z_1\right\| + \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\left\|z_0\right\| \\
&+ \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}b\left\|y_0\right\| \\
&+ \left(1 + \frac{(\mathcal{K}+1)\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)M\|h(0)\| \\
&+ \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}L_h\left\|z^{\tau}\right\| \\
&+ \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\left\|L_G\right\|_{L^1}\left\|z^{\tau}\right\| \\
&+ \left(1 + \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \|G(s,0)\| ds.
\end{aligned} \tag{3.5}
$$

Based on (3.5) and from the inequality $\|z^\tau\| \leq \tau$, *we conclude that:*

$$
\begin{aligned}
\tau < & \left\|\left(\mathcal{Q}z^\tau\right)(t)\right\|, \\
\leq & \frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\|z_1\| + \left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\|z_0\|, \\
& + \left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}b\|y_0\|, \\
& + \left(1+\frac{(\mathcal{K}+1)\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)M\|h(0)\|, \\
& + \left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\int_0^b \|G(s,0)\|ds, \\
& + \left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\left(L_h + \frac{\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\|L_G\|_{L^1}\right)\tau.
\end{aligned} \tag{3.6}
$$

If we divide both sides of (3.6) by τ *and take the limit as* $\tau \to \infty$, *we find:*

$$
\left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2 b\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}\left(L_h + \frac{\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\|L_G\|_{L^1}\right) \geq 1. \tag{3.7}
$$

This stands in contradiction to (3.1). Thus, for some $\tau > 0$, *we have* $\mathcal{Q}(B_\tau) \subset B_\tau$.

Let us decompose the operator $\mathcal{Q}$ *into two operators,* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$, *where* $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, *where*

$$
\begin{aligned}
\left(\mathcal{Q}_1 z\right)(t) &= \mathfrak{C}_\beta(t)\left(z_0 - h(z)\right) + \mathfrak{S}_\beta(t)y_0 + \int_0^t \mathcal{P}_{\beta,\psi}(t-s)Bu_z(s)ds, \quad t \in \mathfrak{I}. \\
\left(\mathcal{Q}_2 z\right)(t) &= \int_0^t G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s)ds, \quad t \in \mathfrak{I}.
\end{aligned}
$$

We will prove that $\mathcal{Q}_1$ *is a contraction operator and that* $\mathcal{Q}_2$ *is a completely continuous operator.*

For $z, y \in B_\tau$, *the assumptions* $(A_1)-(A_6)$ *imply that*

$$
\begin{aligned}
& \left\|\left(\mathcal{Q}_1 z\right)(t) - \left(\mathcal{Q}_1 y\right)(t)\right\| \\
& \qquad \leq \mathcal{K}L_h\|z-y\| + \frac{\mathcal{K}^2\mathcal{K}_1\mathcal{K}_2\mathcal{D}_{b,0}^{\beta-1}L_h}{\Gamma(\beta)}\|z-y\|, \\
& \qquad \quad + \frac{\mathcal{K}^2\mathcal{K}_1\mathcal{K}_2\mathcal{D}_{b,0}^{2\beta-1}}{(\Gamma(\beta))^2}\|L_G\|_{L^1}\|z-y\|, \\
& \qquad \leq \left(\left(1+\frac{\mathcal{K}\mathcal{K}_1\mathcal{K}_2\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)}\right)\mathcal{K}L_h + \frac{\mathcal{K}^2\mathcal{K}_1\mathcal{K}_2\mathcal{D}_{b,0}^{2\beta-1}}{(\Gamma(\beta))^2}\|L_G\|_{L^1}\right)\|z-y\|.
\end{aligned}
$$

Thus, by (3.1), it is evident that $\mathcal{Q}_1$ *is a contraction operator.*

Next, we prove that $\mathcal{Q}_2$ *is completely continuous.*

Let $z_n \in B_\tau$ *with* $z_n \to z$ *in* B_τ.*From* (A_3) *and* (A_5), *it follows that for* $s \in \mathfrak{I}$,

$$
\begin{aligned}
& G\left(s, z_n(s)\right) \to G(s, z(s)), \quad n \to \infty, \\
& \left\|G\left(s, z_n(s) - G(s, z(s))\right)\right\| \leq 2\tau L_G(s).
\end{aligned}
$$

The continuity of $\mathcal{Q}_2$ on B_τ can be easily shown using the dominated convergence theorem. To confirm the compactness of $\mathcal{Q}_2$, we invoke the Ascoli-Arzela theorem, which requires demonstrating that $\mathcal{Q}_2(B_\tau) \subset C(\mathfrak{I};\mathcal{Y})$ is equicontinuous and that $\{\mathcal{Q}_2(B_\tau)(t)\}_{t\in\mathfrak{I}}$ is precompact. For any $z \in B_\tau$ and $h > 0$, we derive:

$$\begin{aligned}
&\|\mathcal{Q}_2 z(t+h) - \mathcal{Q}_2 z(t)\| \\
&\quad= \left\| \int_0^{t+h} G(s,z(s))\mathcal{P}_{\beta,\psi}(t+h-s)ds - \int_0^t G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s)ds \right\|, \\
&\quad\leq \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)} \int_t^{t+h} \|G(s,z(s))\|ds \\
&\qquad+ \int_0^t \|G(s,z(s))\mathcal{P}_{\beta,\psi}(t+h-s) - G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s)\|\, ds, \\
&\quad\leq \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}\tau}{\Gamma(\beta)} \int_t^{t+h} L_G(s)ds + \frac{\mathcal{K}\mathcal{D}_{b,0}^{\beta-1}}{\Gamma(\beta)} \int_t^{t+h} \|G(s,0)\|ds \\
&\qquad+ \int_0^t \|G(s,z(s)) - G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s)\|\, ds. \qquad (3.8)
\end{aligned}$$

Because $\mathcal{P}_{\beta,\psi}(t)$ is strongly continuous when $t \geq 0$, and F is compact, (3.8) implies that $\mathcal{Q}_2(B_\tau) \subset C(\mathfrak{I};\mathcal{Y})$ is equicontinuous. Additionally, the compactness of F ensures that the set $\{G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s) : t,s \in \mathfrak{I}, z \in B_\tau\}$ is precompact. Together with:

$$\mathcal{Q}_2(B_\tau)(t) \subset t\overline{\text{conv}}\{G(s,z(s))\mathcal{P}_{\beta,\psi}(t-s) : s,t \in \mathfrak{I}, \quad z \in B_\tau\}, \quad t \in \mathfrak{I}.$$

This establishes that $\mathcal{Q}_2(B_\tau)(t) \subset \mathcal{Y}$ is precompact.
Hence, $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ is a condensing operator on B_τ. From Lemma 2, $\mathcal{Q}$ has a fixed point z on B_τ. It is simple to demonstrate that z is a mild solution of the system (1.1) satisfying $z(b) + h(z) = z_1$. The proof is finished.

4. An illustrative example

Let us now examine the fractional differential system defined

$$\begin{cases}
{}^C D_t^{\beta,\psi} f(t,w) = \frac{\partial^2 f}{\partial z^2}(t,w) + h(t,f(t,w)) + Bu(t,w), \quad w \in (0,\pi),\ t \in \mathfrak{I}, \\
f(t,0) = f(t,\pi) = 0, \quad t \in \mathfrak{I}, \\
f(0,w) + g(f) = f_0(w), \quad \frac{\partial f}{\partial t}(0,w) = y_0(w), \quad w \in (0,\pi),
\end{cases} \qquad (4.1)$$

where $\beta \in (1,2)$.
Let $\mathcal{Y} = L^2(0,\pi)$ and $A : D(A) \subset \mathcal{Y} \to \mathcal{Y}$ given by

$$A\mathfrak{e} = \mathfrak{e}'',$$

where $D(A) = \{\mathfrak{e} : \mathfrak{e}$ and $\mathfrak{e}'$ are absolutely continuous, $\mathfrak{e}'' \in \mathcal{Y}, \mathfrak{e}(0) = \mathfrak{e}(\pi) = 0\}$. Then

$$A\mathfrak{e} = \sum_{n=1}^{\infty} -n^2 (\mathfrak{e}, \mathfrak{e}_n), \quad \mathfrak{e} \in D(A),$$

where $\mathfrak{e}_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt)$, $n = 1, 2, \ldots$, represents the orthonormal set of eigenvalues of A. It is simple to verify that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$,and

$$C(t)\mathfrak{e} = \sum_{n=1}^{\infty} \cos nt\, (\mathfrak{e}, \mathfrak{e}_n)\, \mathfrak{e}_n, \quad \mathfrak{e} \in \mathcal{Y}, \quad t \in \mathbb{R}.$$

For $\beta = 2$, set $f(t)(w) = f(t,w), \quad G(t,f(t))(w) = h(t,f(t,w)), \quad (Bu)(t)(w) = Bu(t,w).$

Thus, the problem 4.1 can be reformulated as:

$$\begin{cases} f''(t) = Bu(t) + Af(t) + G(t,f(t)), & t \in \mathfrak{I}, \\ f(0) + h(z) = f_0, \quad f'(0) = y_0. \end{cases} \tag{4.2}$$

Therefore, according to Theorem 1, the differential system (1.1) is controllable on $\mathfrak{I}$ provided that the conditions $(A_2) - (A_6)$ are satisfied.

For $\beta \in (1,2)$, since A generates a strongly continuous cosine family $C(t)$, the subordinate principle (see Theorem 1, [4]) implies that A also generates a strongly continuous and exponentially bounded fractional cosine family $\mathfrak{C}_\beta(t)$ with $\mathfrak{C}_\beta(0) = I$, and

$$\mathfrak{C}_\beta(t) = \int_0^\infty \varphi_{t,\beta/2}(s) C(s) ds, \quad t > 0,$$

where $\varphi_{t,\beta/2}(s) = t^{-\beta/2}\phi_{\beta/2}\left(st^{-\beta/2}\right)$, and

$$\phi_\gamma(w) = \sum_{n=0}^{\infty} \frac{(-w)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.$$

Set $f(t)(w) = f(t,w), \quad G(t,f(t))(w) = h(t,f(t,w)), \quad (Bu)(t)(w) = Bu(t,w).$

Therefore, problem (4.1) can be rewritten in the following form:

$$\begin{cases} {}^C D_t^{\beta,\psi} f(t) = Af(t) + Bu(t) + G(t,f(t)), & t \in \mathfrak{I}, \\ f(0) + h(z) = f_0, \quad f'(0) = y_0. \end{cases} \tag{4.3}$$

So, by Theorem 1, the fractional differential system (1.1) is controllable on $\mathfrak{I}$ under the conditions $(A_2) - (A_6)$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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