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Modification of Optimal Homotopy Asymptotic Method for Fractional Heat Equation

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ABSTRACT: In the present paper,Optimal Homotopy Asymptotic Method (OHAM) is used to solve fractional order heat equations.These equation,Which are essential in various scientific and engineering fields,, can present significant challenges due to complexity. Solutions to the fractional-order heat equation in series form are obtained by applying the OHAM. Numerical examples are provided in order to help understand the suggested method's operation.These examples serve as practical illustrations to help readers grasp how the method operates and the results it can yield. It has been demonstrated using the OHAM that other nonlinear problems may be readily solved with a high rate of convergence and a small volume of calculations. The fractional order is evaluated using the Caputo and Caputo-fabrizio operators. Thus, the OHAM is regarded as one of the best analytical methods for resolving linear and nonlinear equations of fractional order, especially fractional-order heat equations.

Key Words: Caputo Operator, FOHAM, fractional derivative operator, nonlinear PDE, heat equation.

Contents

1. Introduction

The significance of fractional calculus has a great introduction! In various scientific fields indeed, fractional calculus has emerged as a powerful tool for modeling complex phenomena. Viscoelasticity, thermal stresses, electromagnetism, porous electrodes, energy transmission, relaxation vibrations, and thermo elasticity are examples of fractional calculus's versatility. Fractional differential equations (FDEs) [1] for understanding these complex dynamics provide a mathematical framework. Entropy was first introduced by Clausius and Boltzmann in the study of thermodynamics in 1862, and Jaynes (1957) and Shannon (1948) later used it in the information theory.

More recently, in many type of complex systems more general entropy have been proposed for usage exceptional of additives axiom,s relaxation [2]. Using entropy's idea one may examine behavior of multi particle systems with fractional order dynamics and integer. The rate of entropy production was taken into consideration for the fractional diffusion process in [3]. Anomalous diffusion's fractional order modal may use total spectral entropy as a measure of the data comparable [4]. Entropy analysis is solve Fractional calculus based entropies integers and fractional dynamical systems [5-6].

The 3rd order dispersive in convexity, entropy and nonlinear partial differential equations [7-8]. Bifurcation of memristive circuits and recurrent analysis [9]. Density analysis of multi scroll chaotic systems and multi wing numerical integration in a conservative systems simulation [10]. The dynamics of a national soccer league of fractional derivative advection-diffusion and two-dimensional semi-conductor systems [11]. The exact solution to differential equations (DEs) including fractional order and mixed-partial derivatives, the space fractional diffusion equation, and tsallis-relative entropy [12]. In 1822 the heat equation was developed by Joseph Fourier, for understanding heat transfer and diffusion that was

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the foundation set by Joseph Fourier. Robert Brown discovery of irregular motion of particles in a fluid were a crucial discovery which helped in understanding the underlying mechanisms of heat transfer.

Particles exchange heat, which is their dynamic energy. From increased surrendering to a cooler scheme when emotional energy if transferred, Faster moving atoms in the surrounding collision with the scheme's dividers give some of this energy to the system's atoms, causing them to travel quickly [13-14]. Numerous mathematical writers have expressed interest in the solution of heat equations, like as OHAM [15-17], Modified Adomian Decomposition Method (MADM) [17] Variational Iteration Method (VIM) [18,19] Differential Transform Method, (DTM) [20], Homotopy Perturbation Method (HPM) [21] Adomian Decomposition Method (ADM) [22-25]. Bernstein polynomials with operational matrix using the Yang-Laplace transform fractional heat equation to aid in the variational iteration method [26] In current paper, the following kinds of partial differential equation (FPDEs) are being solved using optimal homotopy asymptotic method (OHAM)

$$\frac{\partial^{\gamma}\upsilon}{\partial t^{\gamma}} = g(a,b,c)\upsilon_{x,x} + h(a,b,c)\upsilon_{y,y} + k(a,b,c)\upsilon_{z,z} \tag{1.1}$$

with initial condition

$$\upsilon(a,b,c,0) = u(a,b,c), \upsilon_t(a,b,c,0) = p(a,b,c) \tag{1.2}$$

The study's principal goal is to apply OHAM to the time-fractional telegraph problem's analytical solution. The rest of the essay covers some main concepts and an overview of OHAM, as well as a study of two cases and a comparison of the results using graphics.

2. Preliminaries

Definition 1.Riemann-Liouville Fractional Integral is define as [27-29]

$T\epsilon C_{\mu}$ of order $\alpha \geq 0$

$$L_t^a T(t) = T(t), if a = 0, \tag{2.1}$$

$$= \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} T(s) ds, if a > 0 \tag{2.2}$$

where the gamma function is represent as Γ

Definition 2. The fractional derivative of $T \in C\mu$ has order $a \geq 0$ in the caputo-sence which is defined as [27-29]

$$D_t^{m-a} T(t) = L_t^{m-a} D_t T(t), if a = 0, \tag{2.3}$$

$$= \frac{1}{\Gamma(m-a)} \int_0^t (t-s)^{m-a-1} T^m(s) ds, if a > 0 \tag{2.4}$$

$$L_t^a D_t^a \xi(\varepsilon, T) = Y_0(\varepsilon, T) \Sigma_{i=1}^1 D_i^a L_t^a \xi_i(\varepsilon, T) \tag{2.5}$$

Definition 3. If the order of the Caputo-Fabrizio fractional derivative of $T \in C\mu$ has $a > 1$ then as [30,31]

$$D_t^a T(t) = L_t^{m-a} D_t^m T(t), a = 0, \tag{2.6}$$

$$D_t^a T(t) = \frac{1}{(m-a)} \int_0^t -exp \frac{a(t-s)^{m-a-1}}{1-a} T^m(s) ds, a > 0 \tag{2.7}$$

where a normalization function $B(a) > 0$ satisfies B(0)=B(1)=1

Definition 4 The Caputo (Caputo 1969) give the Caputo derivative of Laplace Transform;also et al.(2006)

as [30,31]

$$L[D_t^a T(t) = S^a L[T(t)] - \sum_{k=0}^{n-1} S^{a-k-1} T^{(k)}(0), n-1 < a < n = 0, \tag{2.8}$$

3. OHAM Alogorithm

The boundary value problem

$$W(u(\beta,\varphi) + z(\beta,\varphi) = 0, \beta \in \Omega, \varphi \geq 0, \beta \in \Gamma \tag{3.1}$$

$$B(u_0(\beta,\varphi), \frac{\partial u_0(\beta,\varphi)}{\partial \varphi}) = 0 \tag{3.2}$$

The variable in this case are the diffefrential operator W, the boundary value B, the source solution $u(x,t)$,Γ the border of Ω and z(x,t) the known analytic function,and the independent variables of space and time ,x and t (12),respectively.Consequently,W divided into L and N differential operater.

$$L(u(\beta,\varphi) + N(u(\beta,\varphi)) + z(\beta,\varphi) = 0, \beta \in \Omega \tag{3.3}$$

its easy to determine exact solution for L,which is the most simple and linear component of equation(12) .The difficult and nonlinear component of(12) is known as N which may be exact solution.

now $u_0(\beta,\varphi) : \Omega \to R$ are the solution of

$$L(u_0(\beta,\varphi) + z(\beta,\varphi) = 0, \tag{3.4}$$

$$B(u_0(\beta,\varphi), \frac{\partial u_0(\beta,\varphi)}{\partial \varphi}) \tag{3.5}$$

and functions continuously. And the continuous solution to the above equation be let as. $u(\beta,\varphi) : \Omega \to R$.The homotopy defined as $M(\beta,\varphi,q); \Omega \times [0,1] \to R$ which satisfies the specified technique is

$$(1-q)L(M(\beta,\varphi,q)) + z(\beta,\varphi) - H(q)W(M(\beta,\varphi,q) + z(\beta\varphi) = 0 \tag{3.6}$$

Where parameter embedded is $\beta \in \Omega$ and $q \in [0,1]$and the auxiliary function for (12) is

$H(q) \neq 0$,for all the q and h(0)=0. Clearly,

Equation 16 becames as $L(u_0(\beta,\varphi)) + z(\beta,\varphi) = 0$

by putting q=0 and By puttting q=1 Eq.(16) becomes as

$$N(u(\beta,\varphi)) + L(u(\beta,\varphi)) + z(\beta,\varphi) = 0 \tag{3.7}$$

Now by the given method

$$M(\beta,\varphi,q)=u_0(\beta,\varphi), at \text{ q=0},$$

As q goes from 0 to 1, $M(\beta,\varphi,q)$ Varies from $u_0(\beta,\varphi)$ to $u(\beta,\varphi)$; by merging equations (12), and (17). the equation (03) become as$u_0(\beta,\varphi)$ derived at q=0 This method states that, we use the differential equation's auxiliary function H(q) as

$$H(q)=C_1q+C_2q^2+C_3q^3+....+C_kq^k+..... \tag{3.8}$$

Where the constants which is denoted by $c_1,c_2,c_3,......c_k,.....$ that have to be determined
In order to find the approximate solution to (12) on expanding $M(x,t;q,c_1,c_2,c_3,.....)$
in the Taylor,s series with respect to

$$M(\beta,\varphi,q,c_1,c_2,c_3,.....)=u_0(\beta,\varphi)+\sum_{k=1}^{\infty}u_k(\beta,\varphi,c_1,c_2,c_3,.....)q^k \tag{3.9}$$

The value of C in equation (20) tells us that we must extend equation (16) and compare the coefficient of the same power of the problem having zero order, which is specified in equation (16). The problems with first and second order are defined by

$$L(u_k(\beta,\varphi,))=C_1N_0(u_0(\beta,\varphi,))$$

$$B(u_1(\beta,\varphi),\frac{\partial u_1(\beta,\varphi)}{\partial\varphi})=0 \tag{3.10}$$

and $L(u_2(\beta,\varphi))=C_2N_0(u_0(\beta,\varphi,))+C_1N_1u_0(\beta,\varphi),(u_1(\beta,\varphi,))+(1+C_1)L(u_1(\beta,\varphi,)),$

$$B(u_2(\beta,\varphi),\frac{\partial u_2(\beta,\varphi)}{\partial\varphi})=0 \tag{3.11}$$

respectively We now apply the Caputo and Caputo-Fabrizio operators, along with any other operators mentioned in above definition to the ordered solution. Additionally, for the analytical solution of kth order problems $s_k(\lambda,y)$ are

$$L(u_k(\beta,\varphi))=L(u_{k-1}(\beta,\varphi))+C_kN_0(u_0(\beta,\varphi))+\textstyle\sum_{j=1}^{k-1}C_j[L(u_{k-j}(\beta,\varphi))$$

$$+N_{k-j}(u_0(\beta,\varphi)(u_1(\beta,\varphi),.....,u_{k-j}(\beta,\varphi))],B(u_k(\beta,\varphi,)\frac{\partial u_2(\beta,\varphi)}{\partial\varphi})=0 \tag{3.12}$$

where $N_{k-j}(u_0(\beta,\varphi)(u_1(\beta,\varphi),.....,u_{k-j}(\beta,\varphi))]$ be coefficient of q^{k-j}in expansion of $N(M(\beta,\varphi,q))$ with regard to embedded parameter q.

$$N(M(\beta,\varphi,q,c_1,c_2,c_3,....))=N_0(u_0(\beta,\varphi))+\sum_{k=1}^{\infty}N_k(u_0,u_1,u_2,....u_k)q^k \tag{3.13}$$

The boundary conditions and the solution $u_k(\beta,\varphi)$ are linear equation of (12),is simple to solve .Because of the given homotopy the series (19)converges at q=1 .so

$$\tilde{u}(\beta,\varphi,c_1,c_2,c_3,..)=u_0(\beta,\varphi))+\sum_{k=1}^{\infty}u_k(\beta,\varphi,c_1,c_2,c_3,....,c_k) \tag{3.14}$$

The approximate solution of equation (25) can be found as

$$\tilde{u}(\beta,\varphi,c_1,c_2,c_3,..) = u_0(\beta,\varphi)) + \sum_{k=1}^{\infty} u_k(\beta,\varphi,c_1,c_2,c_3,....,c_k) \tag{3.15}$$

By putting eq.(25)into (12) ,we get the residual of (12) which is

$$R(\beta,\varphi,c_1,c_2,c_3,..) = T(\tilde{u}(\beta,\varphi,c_1,c_2,c_3,..) + k(\beta,\varphi). \tag{3.16}$$

The exact solution is U if a residual R exists (12). It never occurs, especially when it comes to nonlinear DE. For finding optimal constants values various techniques are available, including the Ritz Method, Collocation Method, Least Squares Method, and Galerkin's Method.

In our method, we apply the Least Squares technique.

$$J(c_1,c_2,c_3,..) = \int_0^1 \int_0 R^2(x,t;c_1,c_2,c_3,..)dxdt, \tag{3.17}$$

as well as ideal constant values $c_1, c_2, c_3, ..$ can be computed.

Theorem; if $n-1 < a < N, n \in N, \xi(\varepsilon,T), nCk\xi\beta, \varphi\varepsilon T \in k \geq 0$,then

$$(1-\delta)L_t^a(K(\beta,\varphi,\delta)) + z(\beta,\varphi) - A(\delta)U(K(\beta,\varphi,q) + J(\beta,\varphi)) \tag{3.18}$$

$(K(\beta,\varphi,q) : \Omega \times [01] \to R$ is the nonlinear PDE, with Linear plus nonlinear term which is A and with analytic function $G(\beta,\varphi)$.
Let $G(\beta,\varphi) :\to R$ be the solution of Eq.(28) is be a continuous function. Considering the Homotopy $K(\beta,\varphi : q) : \Omega \times [01] \to R$ boundary of the given PDE is Ω
It satisfies
$(1-\delta)[L_t^a(K(\beta,\varphi,\delta)) + z(\beta,\varphi) - A(\delta)U(K(\beta,\varphi,q) + J(\beta,\varphi))]$ Where the embedded parameter is $\delta \in [0,1]$.
Since δ varies from 0 to 1,

$$K(\varepsilon,T;\delta,\lambda_1,\lambda_2.....) = \xi_0(\beta,\varphi) + \textstyle\sum_{i=1}^{\infty} \xi_1(\beta,\varphi,\lambda_1,\lambda_2.....\lambda_i)\delta,$$

varies from
$L_t^a(\xi_i(\beta,\varphi,) = L(\xi_{i=1}(\beta,\varphi)) + \delta_i N_0 \xi_0(\beta,\varphi) + \sum_{j=1}^{i=1} \delta_0[L(\xi_{i=j}(\beta,\varphi)$
$+ N_{i=j}(\xi_0(\beta,\varphi), (\xi_1(\beta,\varphi), ..., (\xi_{i=j}(\beta,\varphi)$ to $(\xi(\beta,\varphi)$ as δ varies from 0 to 1 , $K(\beta,\varphi : \delta)$ varies from $\xi_0(\beta,\varphi) to \xi(\beta,\varphi)$ where the solution of Eq.(28) is $(\xi_0(\beta,\varphi)$ at $\delta = 0$ In the Caputo sense,this leads to the solution as follows:

$$D_t^a L_t^a(\xi(\beta,\varphi) = (\xi(\beta,\varphi)\lambda \tag{3.19}$$

On the other hand, utilising the auxiliary function of the approximate solution to Equation (28) $A(\delta) = \lambda_1\delta + \lambda_2\delta^2 + \lambda_3\delta^3 +, ...\lambda_i\delta^i + ...$ by expanding $K(\beta,\varphi,\delta,\lambda_1,\lambda_2...)$ in Taylor's series, w.r.t δ as

$$K(\beta,\varphi,\delta,\lambda_1,\lambda_2...) = \xi_0(\beta,\varphi) + \Sigma_{i=1}^{\infty}\xi_i(\beta,\varphi,\delta,\lambda_1,\lambda_2...\lambda_i)\delta \tag{3.20}$$

First order, second order, zero order, etc. can be obtained by expanding Eq. (30), comparing the coefficient of the same power of δ, where $\lambda_1, \lambda_2 ... \lambda_i$ are the optimal constants

$$L_t^a \xi_i(\beta, \varphi,) = L\xi_{i=1}(\beta, \varphi)) + \delta_i N_0(\xi_0(\beta, \varphi)) + \sum_{j=1}^{i=1} \delta_0 [L(\xi_{i=j}(\beta, \varphi) + N_{i=j}(\xi_0(\beta, \varphi), (\xi_1(\beta, \varphi), ..., (\xi_{i=j}(\beta, \varphi))] \tag{3.21}$$

Where $N(K(\beta, \varphi, \delta, \lambda_1, \lambda_2 ...) = N_0(\xi_0(\beta, \varphi)) + \Sigma_{i=1}^{\infty} N_i(\varepsilon_0, \varepsilon_1, \varepsilon_2 .... \varepsilon_i)\delta$
To solve equation (31) having the given boundary condition, we obtained

$$w(\phi, 0) = 3 \sec h^2 (\frac{\phi - 15}{2}) \tag{3.22}$$

Equation (32) is used with the Caputo operator we obtain

$$L_t^a D_t^a \xi(\beta, \varphi) = \gamma_0(\beta, \varphi) + \Sigma_{i=1}^{1} D_t^a L_t^a \xi_i(\beta, \varphi), \tag{3.23}$$

The approximate solution can be obtained by using the auxiliary/optimal constants.

Corollary: Convergence Theorem: As mentioned in section 2, the ideal/auxiliary constants are obtained using the Least Squares method.

$$L_t^a D_t^a \xi(\beta, \varphi) = \gamma_0(\beta, \varphi) + \Sigma_{i=1}^{1} D_t^a L_t^a \xi_i(\beta, \varphi), \tag{3.24}$$

the residual of the modal is

$$R(\beta, \varphi, \lambda_1, \lambda_2 ...) = T(\xi(\beta, \varphi, \lambda_1, \lambda_2 ....) + k(\beta, \varphi) \tag{3.25}$$

$\xi(\beta, \varphi, \lambda_1, \lambda_2 ....)$ be the exact solution of Eq.(28),If $R(\beta, \varphi, \lambda_1, \lambda_2 ...)$, but it doesn't always happen, though, especially when dealing with nonlinear equations. Therefore, unknown/optimal constants are found as

$$\partial_{\lambda_1} K = 0, \partial_{\lambda_2} K = 0, ...\partial_{\lambda_i} K = 0, \tag{3.26}$$

By using the list of values for λ_s , we were able to determine the approximate value

$$\tilde{\xi}(\beta, \varphi) = \xi_0(\beta, \varphi) + \sum_{i=1}^{1} \xi_i(\beta, \varphi), \tag{3.27}$$

For optimal constants, the approximate solution to Eq. (34) is well-known. Rapid convergence of the auxiliary function $A(\delta)'s$ and effective error minimization both depend on the optimal

Numerical Problem-1: Let the One Dimensional Fractional Heat Equation

$$\frac{\partial^Y u}{\partial t^Y} - \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2} = 0, 0 < Y \leq 1, t \geq 0 \tag{3.28}$$

Having the initial condition

$$u(x, 0) = x^2 \tag{3.29}$$

To obtained the zero order $1st, 2nd, 3rd, etc$ OHAM formulation can be used .The zero order is

$$u_0[x, t] = x^2 \tag{3.30}$$

By applying the provided method, the first order approximation equation is generated.

$$(u_1)^{(0,1)}[x,t] = (u_0)^{(0,1)}[x,t] + c_1(u_1)^{(0,1)}[x,t] + \frac{1}{2}x^2c_1(u_1)^{(2,0)}[x,t] \tag{3.31}$$

After using the first order approximation, the outcome is

$$(u_1)^{(0,1)}[x,t] = x^2c_1; \tag{3.32}$$

Using the Caputo-Fabrizio operator, the answer may be represented as follows:

$$u_1[x,t] = \frac{t^a x^2 c_1}{aGamma[1-a]}; \tag{3.33}$$

The second order approximation is obtained as

$$(u_2)^{(0,1)}[x,t] = c_2(u_0)^{(0,1)}[x,t] + \frac{1}{2}x^2c_2(u_0)^{(2,0)}[x,t] + (u_1)^{(0,1)}[x,t] + c_1(u_1)^{(0,1)}[x,t] + \frac{1}{2}x^2c_1(u_1)^{(2,0)}[x,t] \tag{3.34}$$

The second order approximation's solution might be provided as

$$u_2[x,t] = \frac{t^{-1+a}x^2c_1}{Gamma[1-a]} + \frac{t^{-1+a}x^2c_1^2}{Gamma[1-a]} + \frac{t^a x^2 c_1^2}{aGamma[1-a]} + x^2c_2 \tag{3.35}$$

The caputo-Fabrizio operator may be used to solve the second order approximation, and the result is

$$u_2[x,t] = \frac{t^{-1+a}x^2(t^a c_1(a+(a+t)c_1) + atGamma[1-a]c_2)}{a^2Gamma[1-a]^2}; \tag{3.36}$$

$$u[x,t] = u_0[x,t] + u_1[x,t] + u_2[x,t] \tag{3.37}$$

$$u[x,t] = x^2 + \frac{t^a x^2 c_1}{aGamma[1-a]} - \frac{t^{-1+a}x^2(t^a c_1(a+(a+t)c_1) + atGamma[1-a]c_2)}{a^2Gamma[1-a]^2}; \tag{3.38}$$

The residual of the problem is

$$R = \frac{1}{Gamma[1-a]} \int_0^1 -(t-r)^{a-1}\Big(\frac{t^{-1+a}x^2c_1}{Gamma[1-a]} - \frac{t^{-1+a}x^2(t^a c_1^2 + at^{-1+a}c_1(a+(a+t)c_1) + aGamma[1-a]c_2)}{a^2Gamma[1-a]^2}$$
$$\frac{(-1+a)t^{-2+a}x^2(t^a c_1(a+(a+t)c_1) + atGamma[1-a]c_2)}{a^2Gamma[1-a]^2}\Big)dr - \frac{x^2}{2}\partial_{x,x}u[x,t] \tag{3.39}$$

There are a few ways to get the optimal constant in the above approximation equation; in this instance, the least squares approach-discussed in section is employed.

$$J = \int_0^1 \int_0^1 R^2 dxdt \tag{3.40}$$

Thus the optimal Constants are

$C_1 = 0.3319003366389685; C_2 = 0.967091424359809;$

The above example Exact solution is

$$u[x,t] = x^2 e^t; \tag{3.41}$$

Table 1: Description of OHAM sol. Via Caputo at $\alpha = 0.1$ of example 1.

x	OHAM	Close form	Abs error OHAM
0.0	0.	0.	0.
0.1	0.0099699450	0.100010005	0.0000403104
0.2	0.0398787782	0.40040020	0.0001612417
0.3	0.0897272510	0.0900090045	0.0003627939
0.4	0.1595151130	0.160160080	0.0006449670
0.5	0.2492423640	0.250250125	0.0010077609
0.6	0.3589090043	0.360360180	0.0014511757
0.7	0.4885150336	0.490490245	0.0019752114
0.8	0.6380604520	0.640640320	0.0025798680
0.9	0.8075452596	0.180810405	0.0032651454
1.0	0.9969694563	1.001000500	0.0040310437

The following figures display the solutions for the suggested model.
The graphs that follow illustrate the exact and OHAM solutions of example 1 at $\alpha = 0.1$ in Figure 1. It is evident from the figure that follows that the exact and OHAM outcomes are closely related to one another.

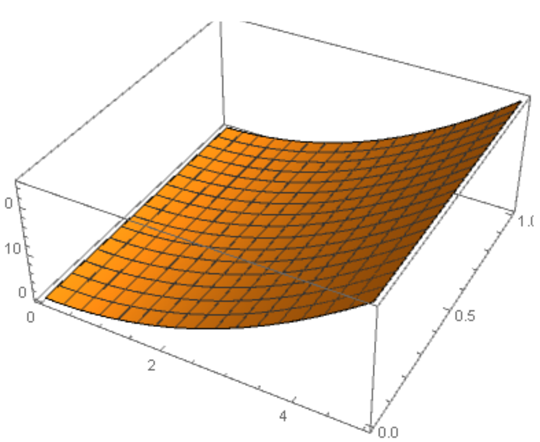

Fig.1:3D OHAM solution at value of $\tau = 0.0001$.

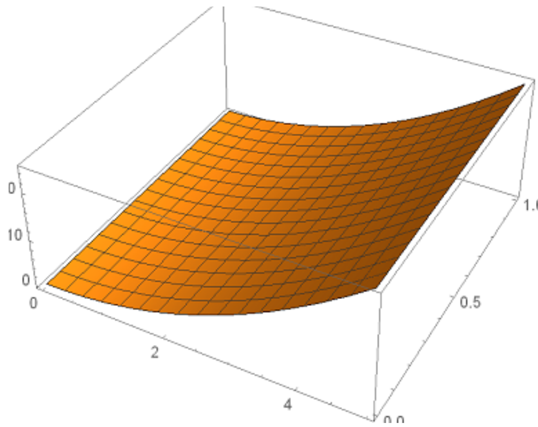

Fig.2:3D Exact solution at value of $\tau = 0.0001$.

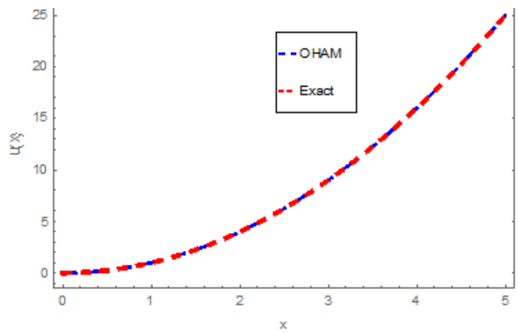

Fig.3:2D approximate and exact solution at $\tau = 0.001 and \varepsilon \in [0, 2]$

Numerical Problem-2: Let the Two Dimensional Fractional Heat Equation [19]:

$$\frac{\partial^Y u}{\partial t^Y} - \frac{y^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{x^2}{2}\frac{\partial^2 u}{\partial y^2} = 0, 0 < Y \leq 1, \tag{3.42}$$

Having initial condition

$$u(x, y, 0) = y^2 \tag{3.43}$$

the OHAM formulation can be used to obtained the zero order, first, second, third, etc. approximations. The zero order is

$$u_0(x, y, t) = y^2 \tag{3.44}$$

The first order approximation equation is generated by using the given method.

$$(u_1)^{(0,0,1)}[x, y, t] = (u_0)^{(0,0,1)}[x, y, t] + c_1(u_0)^{(0,0,1)}[x, y, t] + \frac{1}{2}y^2 c_1(u_0)^{(2,0,0)}[x, y, t] + \frac{1}{2}x^2 c_1(u_0)^{(0,2,0)}[x, y, t] \tag{3.45}$$

The result of the first order approximation is

$$u_1[x, t] = x^2 c_1 \tag{3.46}$$

The solution may be expressed as follows using the Caputo-Fabrizio operator:

$$u_1[x,y,t]=\frac{t^a x^2 c_1}{aGamma[1-a]}; \tag{3.47}$$

The approximate second order is derived as

$$\begin{aligned}(u_2)^{(0,0,1)}[x,y,t] &= c_2(u_0)^{(0,0,1)}[x,y,t]+\tfrac{1}{2}y^2c_2(u_0)^{(2,0,0)}[x,y,t]+\tfrac{1}{2}x^2c_1(u_0)^{(0,2,0)}[x,y,t]\\ &+\frac{1}{2}x^2c_2(u_0)^{(0,2,0)}[x,y,t]+(u_1)^{(0,0,1)}[x,y,t]+c_1(u_1)^{(0,0,1)}[x,y,t]+\frac{1}{2}y^2c_1(u_1)^{(2,0,0)}[x,y,t]\end{aligned} \tag{3.48}$$

Solution of the 2nd order approximation may be given as

$$u_2[x,y,t]=x^2c_1+\frac{t^{-1+a}x^2c_1}{Gamma[1-a]}+\frac{t^{-1+a}x^2c_1^2}{Gamma[1-a]}+\frac{t^a y^2 c_1^2}{aGamma[1-a]}+x^2c_2; \tag{3.49}$$

Solution of the second order approximation using the Caputo-Fabrizio operator may be given as

$$u_2[x,y,t]=\frac{t^a\left(-\frac{t^{-1+a}c_1(ax^2+(ax^2+ty^2)c_1)}{Gamma[1-a]}-ax^2(c_1+c_2)\right)}{a^2Gamma[1-a]}; \tag{3.50}$$

$$u[x,y,t]=u_0[x,y,t]+u_1[x,y,t]+u_2[x,y,t]; \tag{3.51}$$

$$u[x,y,t]=y^2+\frac{t^a x^2 c_1}{aGamma[1-a]}+\frac{t^a\left(-\frac{t^{-1+a}c_1(ax^2+(ax^2+ty^2)c_1)}{Gamma[1-a]}-ax^2(c_1+c_2)\right)}{a^2Gamma[1-a]}; \tag{3.52}$$

The problem's residual is

$$R=\frac{1}{Gamma[1-a]}\int_0^t -(t-r)^{a-1}(\partial_t u[x,t])dr-\frac{y^2}{2}u[x,y,t]\partial_{x,x}u[x,y,t]-\frac{x^2}{2}\partial_{y,y}u[x,y,t] \tag{3.53}$$

The optimal constant in the approximation equation above can be obtained in several different methods; in the present case, the least squares method which is discussed in section 2 is used.

$$J=\int_0^1\int_0^1 R^2 dxdt \tag{3.54}$$

Consequently, Caputo operator optimal constants are

$C_1=0.5676119980799766; C_2=32.10373315775143;$

Exact solution of the form

$$u[x,t]=x^2\sinh[t]+y^2\cosh[t]; \tag{3.55}$$

The proposed model's solutions are shown in the following figures.
The graphs that follow illustrate the exact and OHAM solutions of example 1 at $\alpha=0.01$ in Figure 1. It is evident from the figure that follows that the exact and OHAM results are closely related to one another.

Table 2: Description of OHAM sol. Via Caputo at $\alpha = 0.1$ of example 2.

x	OHAM	Closed form	Abs error OHAM
1.0	0.15967006068994324	0.16001000000800002	0.0003399393180569532
1.2	0.15952488739578646	0.16001440000800027	0.0004895126122138049
1.4	0.15935331895723756	0.16001960000800033	0.0006662810507627737
1.6	0.15915535537429654	0.16002560000800045	0.0008702446337039149
1.8	0.15893099664696336	0.16003240000800056	0.0011014033610372010
2.0	0.15868024277523800	0.16004000000800070	0.0013597572327626872
2.2	0.15840309375912057	0.16004840000800083	0.0016453026488802628
2.4	0.15809954959861097	0.16005760000800098	0.0019580504093900110
2.6	0.15776961029370923	0.16006760000800116	0.0022979897142919314
2.8	0.15741327584441536	0.16007840000800133	0.0026651241635859690
3.0	0.15703054625072935	1.16009000000800153	0.0030594537572721790

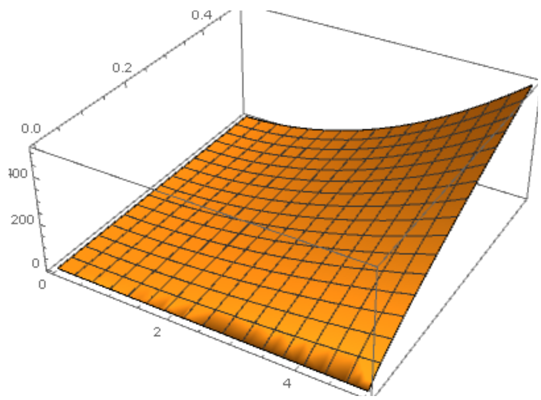

Fig.4:3D OHAM solution at $\tau = 0.0001$.

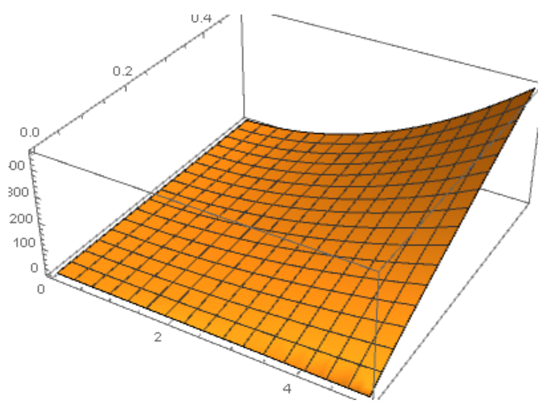

Fig.5:3D Exact solution at $\tau = 0.0001$.

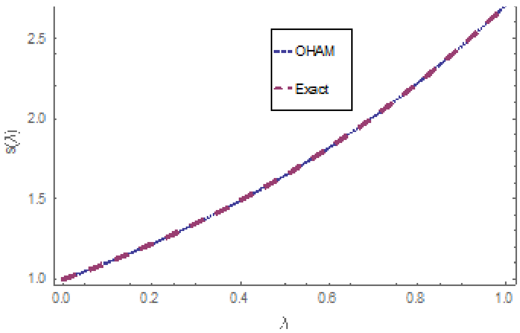

Fig.6:2D approximate and exact solution at $\tau = 0.0001.$ and $\varepsilon \in [0, 2]$

4. Result and Discussion:

In the present study, we utilize the least squares approach to find optimal constants and solve the equal wave equation using OHAM. We used Mathematica 11 for the computations. The OHAM solution in Caputo operators is represented by Tables 1. and 2. of Examples.1 and 2, respectively, for different values of α and $\epsilon \in [0, 2]$ This clearly showed the differences between the various Caputo derivative operators. We determine if there is a substantial agreement between the closed-form solution and the solution generated by the Caputo operator. It also shows how the absolute error is getting closer to the real answer as we get closer to the fractional number α. It shows that the proposed strategy approaches the actual Caputo solution rather rapidly. Figs. 1 and 2 display the exact and OHAM 3D solutions at $\tau = 0.01$ of example 1. Refer to Fig. 3 for the OHAM and exact 2D solution for the range of $\tau = 0.001$ and $\epsilon = [0, 5]$. Figs. 4 and 5 display the third order approximation and Exact 3D solutions at $\tau = 0.001$, of example 2. Figure 6 displays the OHAM and exact 2D solution for the ranges of $\tau = 0.001$ and $\epsilon = [0, 5]$. The graphical presentation and tables show how the current method's fast convergence analysis works. Overall, the study mentioned above highlights OHAM's potential for use in a variety of domains by demonstrating how well it works to solve the equal wave

5. Conclusion:

This work uses OHAM to derive the semi analytical solutions to fractional-order heat equations. For all problem, at both fractional and integer orders the OHAM solution are found. The greatest degree of agreement with the exact solutions to the problems was shown by the results. The validity of the proposed method has been demonstrated by the OHAM solutions for a few numerical cases. Additionally, as fractional order becomes closer to integer order, it is examined whether the fractional order solutions converge to the exact solution for the problems. When compared to integer order models, the fractional order mathematical model can accurately reflect any experimental data, as demonstrated by the application of OHAM to illustrative examples.

OHAM can be used in the future to various no linear if PDEs analytically, which is a common application in engineering and science.OHAM solution for fractional order problems will demonstrate a deeper comprehension of the real world problem that FPDE's represent.

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