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# Interpolation Error with the PCD Method on 3D-Composite Grid

Ahmed Tahiri

ABSTRACT: In this article, we present the interpolation error (the discretization error) with the PCD method in 3D order on composite grid. As in 2D order, for all function/distribution that is locally $H^2$, the discretization error (interpolation error) has an $O(h)$-convergence rate independently of the presence or not of the local mesh refinement. Here, we prove that the present method has the same $O(h)$-convergence rate on 3D-composite grid. In addition, its properties still valid in 3D order, namely the discrete versions of the Friedrichs inequalities and the trace inequality.

Keywords: Boundary value problem, discretization technique, PCD method, compact schemes, most sparse stiffness matrix, multilevel local refinement, computational costs, $O(h)$-convergence rate.

## Contents



## 1. Introduction

The PCD (piecewise constant distributions) method is a discretization technique of the boundary value problems in which the unknown distribution and its derivatives are represented by piecewise constant distributions but on distinct meshes. It has the advantage of producing the most sparse stiffness matrix resulting from the approximate problem. The aim of the PCD method is to produce a best accuracy with lower computational cost. With this method we can introduce a local mesh refinement without the use of the slave nodes. In this way, no interpolation is performed between the nodes of the interface boundary (the interface boundary is the intersection between the refined zone and the remaining part of the domain). Avoiding such interpolation, the resolution of the resulting linear system by iterative methods, even if it is very large, is not difficult because we use a suitable preconditioning technique for this discretization. The local refinement gives a better precision, locally and globally, with lower computational costs particularly if the considered problem has an anomaly. For diffusion equations see for example [4,5,6,16]. Using the PCD method we refer to [10,11,12,13,14,15].

For the boundary value problems in the case of the diffusion equation, the error analysis and the error estimate are usually based on the use of the Bramble-Hilbert Lemma. Introducing local mesh refinement, some authors use the Bramble-Hilbert Lemma and an inequality, known by Il'in's inequality, in order to give an error estimate on a strip around the interface boundary, see for example R. D. Lazarov et al. [6,8].

With the PCD method, the error estimate is similar to that proposed by the other methods with the same regularity conditions (the smoothness of the needed solution). Our error analysis is based on the use of Sobolev imbedding theorems and the use of local approximations of functions.

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The PCD method has the advantage, compared with other discretization methods, of producing the most compact discrete stencil. Essentially, it does not make use of the so-called slave nodes that appear in some finite element discretizations with local mesh refinement. Particularly, the graph of the discrete matrix turns out to be the grid itself of the mesh used for the unknown distribution.

In this work, we present the discretization method in 3D order on composite grid, we recall that a composite grid is a grid with local refined zone. The manner in which the interface boundary will be manipulated and the discrete properties are also considered. Next, we study the discretization error (the interpolation error), with the PCD method, and the convergence of the presented operators. In subsequent paper we will discuss the discretization of the diffusion equations on 3D-composite grid and its convergence.

## 2. The PCD Discretization

### 2.1. Discretization of the domain

We consider the domain of study a bounded open set of $\mathbb{R}^3$ denoted by $\Omega$. We denote by $\Gamma = \partial\Omega$ the boundary of the domain $\Omega$. The principle of the PCD method is given in three steps. First, we split $\Omega$ into $M$ rectangular parallelepiped elements $\Omega_\ell$ such that:

$$\overline{\Omega} = \bigcup_{\ell=1}^{M} \overline{\Omega}_\ell \ , \qquad \Omega_k \cap \Omega_\ell = \emptyset \quad \text{if} \quad k \neq \ell \ .$$

Second, we define different sub-meshes on each element $\Omega_\ell$ to represent elements of $H^1(\Omega)$ and their derivatives. We denote the representation of $v \in H^1(\Omega)$ by $v_h$ and the representation of its partial derivatives $\partial_i\, v$ $(i = 1,\, 2,\, 3)$ by $\partial_{hi}\, v_h$ $(i = 1,\, 2,\, 3)$.

Third, we require that the discrete representations $v_h$ of elements $v$ in $H^1(\Omega)$ must be continuous across the elements boundaries.

Thus, if an element $\Omega_\ell$ has a common side with another element $\Omega_k$, one must have that $v_h|_{\Omega_\ell} = v_h|_{\Omega_k}$ for all vertices of this side.

In each regular element (rectangular parallelepiped elements), $v_h$ is a piecewise constant distribution with 8 values which correspond to the nodal values of $v_h$ at the vertices of this element. The operators $\partial_{hi}$ $(i = 1,\, 2,\, 3)$ will be finite difference quotients taken along the edges of the element faces.

The values of $\partial_{hi}\, v_h$ $(i = 1,\, 2,\, 3)$ are obtained by a finite difference quotient between the value of $v_h$ at a vertex and the value of $v_h$ at the opposite vertex in the $i$-direction of this element.

We have shown in A. Tahiri [13] that the optimal local refinement rate is 2, in the case of 2D discretization. Then, we introduce a local mesh refinement by subdividing the elements of the zone to be refined by the ratio 2 in each direction i. e. each coarse element is subdivided by 8 fine elements. The process defined previously on the regular elements still valid for the fine elements.

With the PCD method we can introduce a local mesh refinement without the use of the slave nodes. In this way, no interpolation is performed between the nodes of the interface boundary (the intersection between the coarse zone and refined zone). To avoid this interpolation which will create the slave nodes we neglect the existence of the irregular nodes in the direction where we will approach the derivative.

At the interface boundary, such local mesh refinement produce irregular elements, that are coarse elements with an irregular (refined) face. Figure 1 illustrates an example of $(x_2,\, x_3)$–irregular element, i. e. element having a refined face in $(x_2,\, x_3)$–plan. On the same figure the dashed lines represent the zones where $v_h$ has a constant values.

In each irregular element, $v_h$ is a piecewise constant distribution with 13 values (denoted $v_{hi}$ $\; i = 1, .., 13$) including 9 on the irregular (refined) face of this element and the 4 remaining are the values at the regular face (the opposite of the refined face) of this element. The values of $\partial_{hi} v_h$ $(i = 1,\, 2,\, 3)$ are obtained by a finite difference quotient between these values and the neighboring value in the $i$-direction.

If in one direction, we do not find nearest neighbor of a node (both are located on the same grid line), we will neglect this node that has no neighbor in the desired direction, in the aim to avoid the slave nodes.

The principle of local mesh refinement with the PCD method is that we neglect the existence of the irregular nodes in the direction where we will approach the derivative.

Note that the interface boundary is reduced to a face of refined zone at $(x_i,\, x_j)$–plan, then the problem will be posed for the approximation of the derivative in $k$-direction, where $i,\, j,\, k \in \{1,\, 2,\, 3\}$ and $i \neq j \neq k$.

In this work, since the derivatives and their analyzes are similar, we limit the presentations and the proofs only for the 1-direction.

More precisely, in the case of the shown example on figure 1, an $(x_2,\, x_3)$–irregular element. The irregular nodes are located in the face $(x_2,\, x_3)$. All its 13 nodal values contribute to get the values of $\partial_{hi} v_h$ $(i = 2,\, 3)$. But in the $x_1$–direction only the values of $v_h$ at the vertices 1, 2, 4, 5, 11, 12, 9 and 13 that contribute to get the $\partial_{h1} v_h$ values. We neglect the existence of the irregular nodes, here the values at the nodes 3, 7, 10, 6 and 8.

The representation $\partial_{h1} v_h$ of the partial derivative $\partial_1 v$ is defined by the following values, according the example illustrated on figure 1.

$$(\partial_{h1}\, v_h\,)_1 = \frac{v_{h2} - v_{h1}}{h_1} \quad , \quad (\partial_{h1}\, v_h\,)_2 = \frac{v_{h4} - v_{h5}}{h_1}$$

$$(\partial_{h1}\, v_h\,)_3 = \frac{v_{h9} - v_{h13}}{h_1} \quad , \quad (\partial_{h1}\, v_h\,)_4 = \frac{v_{h11} - v_{h12}}{h_1}$$

But for the derivatives in $x_2$ and $x_3$ directions all values contribute to define the approximate derivative. for example, the values of the approximate derivative in $x_2$ direction are given as follow :

$$(\partial_{h2}\, v_h\,)_1 = \frac{v_{h5} - v_{h1}}{h_2} \quad , \quad (\partial_{h2}\, v_h\,)_2 = \frac{v_{h12} - v_{h13}}{h_2}$$

$$(\partial_{h2}\, v_h)_3 = \frac{v_{h3} - v_{h2}}{0.5 * h_2} \;,\; (\partial_{h2}\, v_h)_4 = \frac{v_{h4} - v_{h3}}{0.5 * h_2} \;,\; (\partial_{h2}\, v_h)_5 = \frac{v_{h7} - v_{h6}}{0.5 * h_2}$$

$$(\partial_{h2}\, v_h)_6 = \frac{v_{h8} - v_{h7}}{0.5 * h_2} \;,\; (\partial_{h2}\, v_h)_7 = \frac{v_{h10} - v_{h9}}{0.5 * h_2} \;,\; (\partial_{h2}\, v_h)_8 = \frac{v_{h11} - v_{h10}}{0.5 * h_2}$$

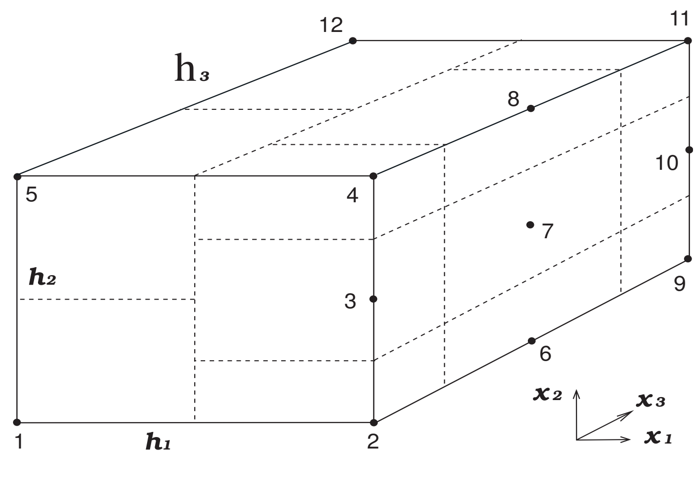


Figure 1: Irregular element having a refined face in $(x_2,\, x_3)$–plan

### 2.2. The PCD Spaces

The further handling and analysis of our discretization method require appropriate notation for the various spaces involved, which we introduce now.

We denote by $H_{h0}$ (respectively $H_{hi}$ $(i = 1,\, 2,\, 3)$) the subspace of $L^2(\Omega)$ of the piecewise constant distributions used to define globally $v_h$ the approximation of an element $v$ of $H^1(\Omega)$ (respectively $\partial_{hi}\, v_h$, $i = 1,\, 2,\, 3$ the approximation of its derivative).

The meshes used to define the above approximations ($v_h$ and $\partial_{hi}\, v_h$ , $i = 1, 2, 3$) define themselves cells in which these approximations are constants.
We denote the cells of these meshes by $\Omega_{\ell i}$ , $\ell \in J_i$ , $i = 0, 1, 2, 3$ respectively. The measures of these cells will be denoted by $|\,\Omega_{\ell i}\,|$ , $i = 0, 1, 2, 3$.
The elements are denoted by $\Omega_\ell$, $\ell \in J = \{1, \dots, M\}$ ($M$ is the number of elements). We similarly denote the cells of the $H_{h0}$-mesh by $\Omega_{\ell 0}$ with $\ell \in J_0 = \{1, \dots, N\}$ where $N$ is the number of the grid nodes. It is important to note that each node of the mesh may be uniquely associated with a cell of $H_{h0}$-mesh. We therefore denote them by $N_\ell$, $\ell \in J_0$.
We further denote by $H_h$ the space $H_{h0}$ equipped with the inner product:

$$( v_h \, , \, w_h \, )_h = ( v_h \, , \, w_h \, ) \; + \; \sum_{i=1}^{3} ( \, \partial_{hi}\, v_h \, , \, \partial_{hi}\, w_h \, ) \; , \tag{2.1}$$

and its associate norm is denoted $\|\,.\,\|_h$ , and $\|\,.\,\|_{0,\Omega}$ (or simply $\|\,.\,\|$ if there is no confusion) denotes the norm of $L^2(\Omega)$. In this paper, we use the standard notation for Sobolev spaces [1] for norms and semi-norms.

We denote by $h$ the mesh size defined by $h = \max(h_\ell)$, $\ell \in J$, where $h_\ell = \text{diam}\ (\Omega_\ell)$ , $\ell \in J$ and we denote by $h_{\ell 1}$, $h_{\ell 2}$ and $h_{\ell 3}$, the length, the width and the height of the element $\Omega_\ell$.
We note that, with the PCD discretization, for any pair of nodes of the mesh we can find a path connecting these nodes (succession of mesh grid segments). Then, that $\partial_{hi}\, v_h$ has the following property:

$$\int_P^Q \partial_{hi}\, v_h \, dx_i \; = \; v_h(Q) \; - \; v_h(P) \qquad\qquad (i = 1, 2, 3) \tag{2.2}$$

for any pair of nodes $\{P,\; Q\}$ of the mesh.

We sometimes need to assume that the discretization is *regular*. We hereby mean that there exist positive constants $C_1$ , $C_2$ $> 0$ independent of $h$ such that:

$$C_1\, h \;\le\; h_{\ell 1} \; , \; h_{\ell 2}, \; h_{\ell 3} \;\le\; C_2\, h \qquad \text{for all} \;\; \ell \in J \tag{2.3}$$

$$C_1\, h^3 \;\le\; |\,\Omega_{\ell\, 0}\,|, \; |\,\Omega_{\ell 1}\,|, \; |\,\Omega_{\ell 2}\,|, \; |\,\Omega_{\ell 3}\,| \;\le\; C_2\, h^3 \, ; \quad \ell \in J_i \, , \; i = 0, 1, 2, 3 \tag{2.4}$$

The notation $C$ is used throughout this work to denote a generic positive constant independent of the mesh size.

## 3. Properties and Interpolation

### 3.1. Discrete Properties of the PCD Discretization

The PCD discretization has properties which represent a discrete version of the Friedrichs inequalities and the trace inequality. Using property (2.2), we have prove these properties in the case of 2D discretization, the process used in the proof still valid in the case of 3D, see A. Tahiri [12,14]. Then, we can write:

**Lemma 3.1** *Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^3$. Then, there exists a constant $C > 0$, independent of $h$ such that:*

$$\|\, v_h \,\|_h \le C \left( \|\, \partial_{h1}\, v_h \,\|^2 + \|\, \partial_{h2}\, v_h \,\|^2 + \|\, \partial_{h3}\, v_h \,\|^2 + \|\, v_h \,\|_\Gamma^2 \, \right)^{1/2} \quad \forall \; v_h \in H_h \; . \tag{3.1}$$

**Lemma 3.2** *Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^3$. Then, there exists a constant $C > 0$, independent of $h$ such that:*

$$\int_\Gamma v_h(x)^2 \, ds \;\; = \;\; \|\, v_h \,\|_\Gamma^2 \;\; \le \;\; C \;\|\, v_h \,\|_h^2 \qquad\qquad \forall \; v_h \in H_h \; . \tag{3.2}$$

We note that lemmas 3.1 and 3.2 are important for proving the uniform coercivity of the approximate boundary value problems. This uniform coercivity implies that the resulting linear system has a positive definite stiffness matrix.

### 3.2. Interpolation Error

As usual, the interpolation error (the discretization error) that can be obtained depends on the regularity of $v$, an element of $H^1(\Omega)$. Here, we assume that $v \in H^2(\Omega)$. In this case, $v$ is continuous on $\overline{\Omega}$ and its nodal values are well defined. We denote by $v_\ell$ the nodal value of $v$ at the grid point $x_\ell = (x_{\ell 1}, x_{\ell 2}, x_{\ell 3})$. We can then define its interpolant $v_I$ in $H_h$ through the nodal values:

$$v_I(N_\ell) = v(N_\ell) \quad \text{for all nodes } N_\ell\,,\ \ell \in J_0\,. \tag{3.3}$$

It means that $v_I|_{\Omega_{\ell 0}} = v(N_\ell)$ for all nodes $N_\ell\,,\ \ell \in J_0\,.$

since the derivatives and their analyzes are similar, we limit the presentations and the proofs only for the 1-direction.
By $x_{\ell_E}$ we denote the right neighbor in 1-direction of $x_\ell$ if it exists.
By $r_{hi}$ we denote the $L^2$–orthogonal projection from $L^2(\Omega)$ onto $H_{hi}$ $(i = 0, 1, 2, 3)$.
For any $v \in H^2(\Omega)$, we let $r_h\, v$ denote the element of $H_h$ determined by $r_{h0}\, v \in H_{h0}$. Its values in $H_{hi}$ $(i = 1, 2, 3)$ are defined by $\partial_{hi}\,(r_{h0} v)$ $(i = 1, 2, 3)$.
We call $r_h$ the $L^2$–orthogonal projection on $H_h$, which is $H_{h0}$ with the norm $\|\,.\,\|_h$.

For all $v \in H^2(\Omega)$, the interpolant $v_I$ in $H_{h0}$ is defined by (3.3).
For all $v \in H^2(\Omega)$, see for example Brezzi and Fortin [3] and Douglas et al. [7], we have

$$\|\, v - r_{h0}\, v \,\| \;\leq\; C\, h\, |\, v\,|_1 \;\leq\; C\, h \|\, v\,\|_{2,\Omega} \tag{3.4}$$

and,

$$\|\, \partial_i\, v - r_{hi}\,(\partial_i\, v)\,\| \;\leq\; C\, h\, |\, \partial_i\, v\,|_{1,\Omega} \;\leq\; C\, h\, \|\, v\,\|_{2,\Omega} \qquad (i = 1, 2, 3) \tag{3.5}$$

where $r_{hi}$ is the $L^2$–orthogonal projection onto $H_{hi}$, $(i = 1, 2, 3)$.
$r_{h0}\, v$ is defined by :

$$(\, r_{h0} v\,)_{\,\ell} = \frac{1}{|\,\Omega_{\ell 0}\,|} \int_{\Omega_{\ell 0}} v(\,x\,)\, dx \quad , \qquad \forall\, \ell \in J_0 \tag{3.6}$$

Similarly, $r_{hi}(\partial_i\, v)$ are defined by :

$$(\, r_{hi}(\partial_i\, v)\,)_{\,\ell i} = \frac{1}{|\,\Omega_{\ell i}\,|} \int_{\Omega_{\ell i}} \partial_i\, v(\,x\,)\, dx \quad , \quad \forall\, \ell \in J_i \qquad (i = 1, 2, 3) \tag{3.7}$$

**Lemma 3.3** *Let $\Omega$ be a rectangular parallelepiped bounded open in $\mathbb{R}^3$, for all $v$ in $H^2(\Omega)$ and for all $d$ $(3 < d \leq 6)$, there exists a constant $C > 0$ ( independent of $h$), such that:*

$$\alpha_{\,\ell} = |\,(r_{h0}\, v\,)_{\,\ell} - v_\ell\,| \leq Ch\,|\,\Omega_{\ell 0}\,|^{-1/d}\, \|\, v\,\|_{W^{\,1,d}(\Omega_{\ell 0})} \quad , \qquad \forall\, \Omega_{\ell 0}\,,\ \ell \in J_0 \tag{3.8}$$

$$\theta_{\ell 1} = \left| (r_{h1}\partial_1\, v)_{\ell 1} - \frac{(v_{\ell_E} - v_\ell)}{h_{\ell 1}} \right| \leq Ch\,|\,\Omega_{\ell 1}\,|^{-1/2} \|\, v\,\|_{2,\,\Omega_{\ell 1}}\,,\ \forall\, \Omega_{\ell 1}\,,\ \ell \in J_1\ . \tag{3.9}$$

*where $v_\ell$, respectively $v_{\ell_E}$, the value of $v$ at the node $x_\ell$, respectively $x_{\ell_E}$.*

**Proof:** First note that by Sobolev imbedding theorem, if $v \in H^2(\Omega)$ then also $v \in W^{1,\,d}(\Omega)$ for all $d$ such that $(3 < d \leq 6)$, then (3.8) is well defined.
Since, $v \in H^2(\Omega)$, the restriction of $v$ to $\Omega_{\ell 0}$, respectively $\Omega_{\ell 1}$, belongs to $H^2(\Omega_{\ell 0})$, respectively $H^2(\Omega_{\ell 1})$. The space $C^2(\overline{\Omega_{\ell 0}})$, respectively $C^2(\overline{\Omega_{\ell 1}})$, is dense in $H^2(\Omega_{\ell 0})$, respectively $H^2(\Omega_{\ell 1})$, it is sufficient to prove (3.8) in $C^2(\overline{\Omega_{\ell 0}})$ and (3.9) in $C^2(\overline{\Omega_{\ell 1}})$.
Taylor expansion gives on a closed domain $K$, for all $x\,, x_0 \in K$:

$$v(x) = v(x_0) + \int_0^1 \nabla(v)(tx + (1-t)x_0)(x - x_0)\, dt \tag{3.10}$$

$$\begin{aligned} v(x) &= v(x_0) + \nabla v(x_0)\cdot(x-x_0) \\ &+ \int_0^1 H(v)(tx+(1-t)x_0)(x-x_0)\cdot(x-x_0)\,(1-t)\,dt \end{aligned} \tag{3.11}$$

where $H(v)(x)$ denotes the *Hessian* matrix of $v$ at point $x$.
Taylor expansion (3.10) gives for all $\overline{\Omega_{\ell 0}}$ with $x_0 = x_\ell$ and $\forall x \in \overline{\Omega_{\ell 0}}$ :

$$(r_h v)_\ell = v_\ell + \frac{1}{|\Omega_{\ell 0}|}\int_{\Omega_{\ell 0}}\int_0^1 \nabla(v)(tx+(1-t)x_\ell)(x-x_\ell)\,dt\,dx$$

Since $|x - x_\ell| \le h \quad \forall x \in \overline{\Omega_{\ell 0}}$

$$\alpha_\ell = |(r_h v)_\ell - v_\ell| \le C\frac{h}{|\Omega_{\ell 0}|}\int_{\Omega_{\ell 0}}\int_0^1 |\nabla(v)(tx+(1-t)x_\ell)|\,dt\,dx$$

By using the change of variable $z = tx + (1-t)x_\ell$ , then $dz = t^3 dx$ .
We denote by $\Omega_{\ell 0}^{\,t}$ the transformation of $\Omega_{\ell 0}$ by such a change of variable, since this change of variable is convex, then $\Omega_{\ell 0}^{\,t} \subset \Omega_{\ell 0}$ and $|\Omega_{\ell 0}^{\,t}| \le t^3|\Omega_{\ell 0}|$ . Then

$$\alpha_\ell \le C\frac{h}{|\Omega_{\ell 0}|}\int_{\Omega_{\ell 0}}\int_0^1 |\nabla(v)(z)|\,\chi_{\Omega_{\ell 0}^{\,t}}\,t^{-3}\,dt\,dz$$

where $\chi_{\Omega_{\ell 0}^{\,t}}$ is the characteristic function of $\Omega_{\ell 0}^{\,t}$ .
*Hölder's* inequality for $d$ such that $(3 < d \le 6)$ and $d'$ , $(\frac{1}{d} + \frac{1}{d'} = 1)$ gives:

$$\begin{aligned} \alpha_\ell &\le \frac{C\,h}{|\Omega_{\ell 0}|}\int_0^1 \left(\int_{\Omega_{\ell 0}} |\nabla(v)(z)|^d\,dz\right)^{1/d}\left(\int_{\Omega_{\ell 0}} \chi_{\Omega_{\ell 0}^{\,t}}\,t^{-3d'}\,dz\right)^{1/d'} dt \\ &\le \frac{C\,h}{|\Omega_{\ell 0}|}\,\|v\|_{W^{1,d}(\Omega_{\ell 0})}\,|\Omega_{\ell 0}|^{1/d'}\int_0^1 t^{3/d'-3}\,dt \\ &\le C\,h\,|\Omega_{\ell 0}|^{-1/d}\,\|v\|_{W^{1,d}(\Omega_{\ell 0})}\int_0^1 t^{-3/d}\,dt \\ &\le C\,h\,|\Omega_{\ell 0}|^{-1/d}\,\|v\|_{W^{1,d}(\Omega_{\ell 0})} \qquad \text{since} \quad d > 3 \quad . \end{aligned}$$

Now, we consider the proof of (3.9):
For all defined $\Omega_{\ell 1}$ , using (3.11) for $(x = x_\ell\,, x_0 = x)$ and for $(x = x_{\ell E}\,, x_0 = x)$, $\forall x \in \overline{\Omega_{\ell 1}}$, subtracting one from the other:

$$(v(x_{\ell E}) - v(x_\ell)) = \nabla v(x)\cdot(x_{\ell E} - x_\ell) + \phi_{\ell E} - \phi_\ell$$

where

$$\phi_\ell = \int_0^1 H(v)(tx_\ell + (1-t)x)(x_\ell - x)\cdot(x_\ell - x)\,(1-t)\,dt$$

$$\phi_{\ell E} = \int_0^1 H(v)(tx_{\ell E} + (1-t)x)(x_{\ell E} - x)\cdot(x_{\ell E} - x)\,(1-t)\,dt$$

integrating over $\Omega_{\ell 1}$:

$$\left|\frac{1}{|\Omega_{\ell 1}|}\int_{\Omega_{\ell 1}} \nabla v(x)\,(x_{\ell E} - x_\ell)\,dx - (v_{\ell E} - v_\ell)\right| = \left|\frac{1}{|\Omega_{\ell 1}|}\int_{\Omega_{\ell 1}} (\phi_{\ell E} - \phi_\ell)\,dx\right|$$

since $(x_{\ell E} - x_\ell) = (h_{\ell 1}\,, 0\,, 0)$ then ,

$$\nabla v(x)\cdot(x_{\ell E} - x_\ell) = h_{\ell 1}\,\partial_1 v(x)$$

Therefore

$$\theta_{\ell 1} = \left| (\, r_{h1}\, \partial_1\, v\,)_{\,\ell 1} \; - \; \frac{(v_{\,\ell E} - v_{\,\ell})}{h_{\ell 1}} \right| \leq \frac{1}{h_{\ell 1}|\Omega_{\ell 1}|} \left( \int_{\Omega_{\ell 1}} |\,\phi_{\ell E}|\; dx \; + \int_{\Omega_\ell} |\,\phi_\ell|\; dx \right)$$

One may write since Then $|\, x - x_\ell\,| \, \leq h_{\ell 1} \quad \forall\, x \in \, \overline{\Omega_{\ell 1}}$

$$\int_{\Omega_{\ell 1}} |\,\phi_\ell|\; dx \;\leq\; C\, h_{\ell 1}^2 \int_{\Omega_{\ell 1}} \int_0^1 |H(v)(tx_\ell + (1-t)x)\,|\; (\,1 \; - \; t\,)\, dt\, dx$$

and

$$\int_{\Omega_{\ell 1}} |\,\phi_{\ell E}|\; dx \;\leq\; C\, h_{\ell 1}^2 \int_{\Omega_{\ell 1}} \int_0^1 |H(v)(tx_{\ell E} + (1-t)x)\,|\; (\,1 \; - \; t\,)\, dt\, dx$$

Using the change of variable $z = tx_\ell + (1-t)x$ , then $dz = (1-t)^3 dx$:

$$\int_{\Omega_{\ell 1}} |\,\phi_\ell|\; dx \;\leq\; C\, h_{\ell 1}^2 \int_{\Omega_{\ell 1}} \int_0^1 \; |H(v)(z)\,|\; \chi_{\Omega_{\ell 1}^t}\; (1-t)^{-2}\, dt\, dz$$

where, $\Omega_{\ell 1}^t$ is the transformation of $\Omega_{\ell 1}$ by this change of variable, and $\chi_{\Omega_{\ell 1}^t}$ is the characteristic function of $\Omega_{\ell 1}^t$.
With the $Cauchy - Schwarz$ inequality, and since $|\,\Omega_{\ell 1}^t\,| \leq (1-t)^3\,|\,\Omega_{\ell 1}\,|$, we obtain:

$$\int_{\Omega_{\ell 1}} |\phi_\ell|\; dx \;\leq\; C\, h_{\ell 1}^2 \int_0^1 \left( \int_{\Omega_{\ell 1}} |H(v)(z)\,|^2\; dz \right)^{1/2} \left( \int_{\Omega_{\ell 1}} \chi^2_{\Omega_{\ell 1}^t}\; dz \right)^{1/2} (1-t)^{-2}\, dt$$

$$\leq\; C\, h_{\ell 1}^2 \;\|\, v\,\|_{2,\Omega_{\ell 1}} \;\;|\,\Omega_{\ell 1}\,|^{1/2} \int_0^1 (1-t)^{(3/2)\,-\,2}\, dt \;\;\leq\; C\, h_{\ell 1}^2 \;\;|\,\Omega_{\ell 1}\,|^{1/2} \;\;\|\, v\,\|_{2,\Omega_{\ell 1}}$$

The same can be written for $\phi_{\ell E}$ , whence it follows that:

$$\theta_{\ell 1} = \left| \,(\, r_{h1}\, \partial_1\, v\,)_{\,\ell 1} \;\; - \;\; \frac{(v_{\,\ell E} - v_{\,\ell})}{h_{\ell 1}} \right| \;\leq\; C\, h\; |\,\Omega_{\ell 1}\,|^{-1/2} \;\|\, v\,\|_{2,\Omega_{\ell 1}} \;.$$

□

We can prove the following lemma that gives the global interpolation error (the discretization error).

**Lemma 3.4** *Let $\Omega$ be a rectangular parallelepiped bounded open of $\mathbb{R}^3$, for all $v \in \; H^2(\Omega)$ and for all $d$ $(3 \;<\; d \;\leq\; 6)$ , there exists a constant $C > 0$ (independent of $h$), such that:*

$$\|\, v \; - \; v_I\,\|^2 + \sum_{i=1}^{i=3} \|\, \partial_i\, v \; - \; \partial_{hi}\, v_I\,\|^2 \;\; \leq\; C\, h^2 \;\|\, v\,\|^2_{2,\,\Omega} \tag{3.12}$$

**Proof:** One may write for $r_{h0}v$ and $v_I$

$$\|\; r_{h0}\, v \; - \; v_I\;\|^2_{0,\,\Omega} \;\; = \; \sum_{J_0} \;\|\; r_{h0}\, v \; - \; v_I\;\|^2_{0,\,\Omega_{\ell 0}} \;\; = \; \sum_{J_0} \;|\,\Omega_{\ell 0}\,|\; \alpha_\ell^2$$

$$\leq\;\; C\, h^2 \sum_{J_0} \;\; |\,\Omega_{\ell 0}\,|\,|\,\Omega_{\ell 0}\,|^{-2/d} \;\;\|\, v\,\|^2_{W^{\,1,d}(\Omega_{\ell 0})}$$

$$\leq C\, h^2 \left( \sum_{J_0} \;|\,\Omega_{\ell 0}\,|^{(1-\,(2/d))q} \right)^{1/q} \left( \sum_{J_0} \;\|\, v\,\|^{2\,d/2}_{W^{\,1,d}\;\;(\Omega_{\ell 0})} \right)^{2/d}$$

using $H\ddot{o}lder's$ inequality for $d/2$ and $q$ with $\frac{2}{d} + \frac{1}{q} = 1 \;\Rightarrow\; q(1-(2/d)) = 1$

$$\| r_{h0}\, v \,-\, v_I \|_{0,\Omega}^2 \,\leq\, C\, h^2 \left( \sum_{J_0} |\, \Omega_{\ell 0}\,| \right)^{1-(2/d)} \left( \sum_{J_0} \| \,v\, \|_{W^{1,d}\ (\Omega_{\ell 0})}^d \right)^{2/d}$$

$$\| r_{h0}\, v \,-\, v_I \|_{0,\Omega}^2 \,\leq\, C\, h^2 \left( \sum_{J_0} |\, \Omega_{\ell 0}\,| \right)^{1-(2/d)} \left( \left( \sum_{J_0} \| \,v\, \|_{W^{1,d}\ (\Omega_{\ell 0})}^d \right)^{1/d} \right)^2$$

$$\leq\, C\, h^2\, |\,\Omega\,|^{1-(2/d)} \left( \sum_{J_0} \| \,v\, \|_{2,\Omega_{\ell 0}}^2 \right)^2 \quad \leq\, C\, h^2\, \| \,v\, \|_{2,\Omega}^2$$

using Sobolev embedding theorem: $H^2 \hookrightarrow W^{1,d}$ with $(3 < d \leq 6)$.

Triangular inequality and (3.4) give : $\| \,v \,-\, v_I \,\|^2 \,\leq\, C\, h^2\, \| \,v\, \|_{2,\Omega}^2$

Let us now bound the error of the derivatives $\|\partial_i\, v - \partial_{hi}\, v_I\|\,, i \,=\, 1, 2, 3$. We present the proof for the case $i = 1$ since the proofs for the two other cases are similar.
Using (3.9), we can write for $r_{h1}\partial_1 v$ and $\partial_{h1} v_I$

$$\| \,r_{h1}\, \partial_1\, v \,-\, \partial_{h1} v_I \,\|_{0,\Omega}^2 \,=\, \sum_{J_1} \quad \| \,r_{h1}\, \partial_1 v \,-\, \partial_{h1} v_I \,\|_{0,\Omega_{\ell 1}}^2$$

Then

$$\| \,r_{h1}\, \partial_1 v \,-\, \partial_{h1} v_I \,\|_{0,\Omega}^2 \,=\, \sum_{J_1} \quad |\, \Omega_{\ell 1}\,|\ \ (\,\theta_{\ell 1}\,)^2$$

$$\leq\ \ C\ h^2 \left( \sum_{J_1} \quad \| \,v\, \|_{2,\Omega_{\ell 1}}^2 \right) \,\leq\ \ C\ h^2\ \| \,v\, \|_{2,\Omega}^2$$

Triangular inequality and (3.5) give : $\| \,\partial_1\, v \,-\, \partial_{h1} v_I \,\| \,\leq\, C\, h\ \| \,v\, \|_{2,\Omega}$ □

## 4. Concluding Remarks

The main issue of the present work is the presentation of the PCD discretization in 3D order on composite grid. To keep this presentation as simple as possible, this method was only formulated for rectangular parallelepiped bounded open, because the choice of the appropriate meshes is rather simple.

Staying still with rectangular meshes we have investigated the question of feasibility of introducing local mesh refinement without slave nodes. The main question to solve here was to prove the convergence of this discretization and to estimate its convergence rate.

Our conclusion is that, for all function/distribution that is locally $H^2$, the discretization error (interpolation error) has an $O(h)$-convergence rate independently of the presence or not of the local mesh refinement. This convergence rate shows that there is always convergence of the presented discretization in the case where the distributions belong to $H^1$.

In subsequent paper we will discuss the diffusion problem on 3D-composite grid and its convergence with the PCD method. We will present a new error analysis that is not based on the use of the Bramble-Hilbert lemma.

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*Ahmed Tahiri*
*Department of Computer Science,*
*Faculty of Sciences, University Mohammed I,*
*BP 717 , 60000 Oujda, Morocco .*

*tahiriahmed02@yahoo.fr*