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Locally Linearly S-closed Spaces

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ABSTRACT: The purpose of the present paper is to study the class of locally linearly S-closed spaces. Characterizations and cardinality bounds for the class of locally linearly S-closed spaces are obtained. It is shown that weakly Lindelöf, Hausdorff, lob and locally linearly S-closed spaces having (i) countable tightness are discrete spaces of countable cardinality; (ii) countable π-character and countable pseudocharacter are of cardinality at most 2^{ω}. In addition, we provide some sufficient conditions for a locally linearly S-closed space to be extremally disconnected. It is shown that Hausdorff (or almost regular), lob, locally linearly S-closed spaces are extremally disconnected. Moreover, it turns out that maximal locally linearly S-closed spaces are also extremally disconnected. Some conditions on functions that preserve (inversely preserve) the property of being locally linearly S-closed are also investigated.

Key Words: cardinality, extremally disconnected, irresolute, lob, maximal, π-character, weakly Lindelöf.

Contents

1. Introduction

The notion of compactness is one of the most significant topological properties with scope going beyond topology to several other fields of mathematics. Several weaker and stronger variants of compactness (such as quasi-compactness, H-closedness and semi-compactness etc.) have been extensively studied in the mathematical literature. Recently, there has been a considerable interest in the study of various compactness like covering properties by using chain covers (i.e. the covers of spaces that are linearly ordered by subset inclusion relation). Some of the examples are linearly H-closedness [2], linearly Lindelöfness [12] and weakly linearly Lindelöfness [12]. A space X is said to be *linearly H-closed* [2] if any open chain cover of X has a member dense in it.

The class of linearly S-closed spaces which generalizes the class of linearly H-closed spaces [2] is introduced recently in [33]. A topological space X is said to be *linearly S-closed* [33] if any semi-open chain cover of X has a member dense in it. In the same paper, the localization of the notion of linearly S-closed spaces is introduced in the following way: a space (X, τ) is *locally linearly S-closed* if each point in X has an open neighbourhood which is linearly S-closed. It was shown that a quasi H-closed space is linearly S-closed if and only if it is locally linearly S-closed [33, Theorem 2.28]. A space X is said to

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be *quasi H-closed (QHC)* if each open cover has a finite subfamily, the closures of whose members covers X.

Several characterizations and fundamental properties for the class of linearly S-closed spaces were investigated and studied in [33]. It was shown that in the class of extremally disconnected spaces, the class of linearly H-closed spaces coincides with the class of linearly S-closed spaces. In the class of lob-spaces, Hausdorff linearly S-closed spaces are extremally disconnected. It was observed that the property of being linearly S-closed space is neither hereditary nor productive. It is interesting to note that unlike compactness, linearly S-closedness is not a closed hereditary property. The class of linearly S-closed spaces is of considerable interest due to its relationship with extremally disconnectedness and being a covering property.

It is easy to see that the class of locally linearly S-closed spaces is larger than that of linearly S-closed spaces. Thus, if a statement P is true for all linearly S-closed spaces, a natural question to ask is whether P also holds for locally linearly S-closed spaces. For example, in [33, Theorem 2.36] it was proved that a first countable, Hausdorff, linearly S-closed space is finite. However, the result fails for locally linearly S-closed spaces (see for instance, Example 4.1 in section 4). It is worth to mention here that there are plenty of first countable, Hausdorff (even perfectly normal) linearly H-closed spaces having infinite cardinality (see [2, Example 2.7]). The purpose of the present paper is to study in greater detail the class of locally linearly S-closed spaces.

The organization of the paper is as follows: In Sections 2 and 3, we provide definitions, notations and preliminary results which are used throughout the paper. In Section 4, we discuss certain properties and characterizations of locally linearly S-closed spaces. In particular, we find that Hausdorff (or almost regular), lob, locally linearly S-closed spaces are extremally disconnected. Moreover, some cardinality bounds for the class of locally linearly S-closed spaces are obtained. It is shown that if X is a weakly Lindelöf, Hausdorff, lob and locally linearly S-closed space having countable tightness, then X is a discrete space of countable cardinality. Furthermore, if X is a weakly Lindelöf, lob, Hausdorff and locally linearly S-closed space having countable π-character and countable pseudocharacter, then $|X| \leq 2^{\omega}$. In Section 5, we study various types of functions that preserve (inversely preserve) locally linearly S-closed spaces. It is shown that locally linearly S-closed spaces are inversely preserved under s-perfect, weakly continuous surjections. Section 6 is concerned with maximality of locally linearly S-closed spaces. It is shown that a maximal locally linearly S-closed space is extremally disconnected.

2. Notations and Terminology

Throughout the paper, (X, τ) (or X) will denote a topological space on which no separation axiom is imposed unless otherwise explicitly stated. For a subset A of (X, τ), the closure, interior and boundary of A in X is denoted by $cl_\tau(A)$ (or $cl_X(A)$), $int_\tau(A)$ (or $int_X(A)$) and $bd_\tau(A)$ (or $bd_X(A)$) respectively. We will use the notations $cl(A)$ or $\overline{A}$, $int(A)$ and $bd(A)$ for closure, interior and boundary of A respectively if there is no confusion about the space and its topology.

Definition 2.1 *A subset A of a space X is said to be:*

(i) semi-open *[14] if there exists an open set O in X such that $O \subset A \subset cl(O)$ or equivalently if $A \subset cl(int(A))$.*

(ii) regular open *if $A = int(cl(A))$.*

(iii) α-open *[17] if $A \subset int(cl(int(A)))$.*

(iv) regular semi-open *if there exists a regular open set O in X such that $O \subset A \subset cl(O)$.*

(v) regular closed *if $A = cl(int(A))$.*

(vi) semi-closed *if $A \supset int(cl(A))$.*

(vii) locally dense or preopen [16] *if $A \subset int(cl(A))$.*

The family of all semi-open, regular open, regular closed and α-open subsets in a space X is denoted by $SO(X)$, $RO(X)$, $RC(X)$ and $\alpha O(X)$ respectively. Let x be a point in a space X. Then, $SO(x)$ denote the collection of semi-open subsets of X which contains x. For any $x \in X$, $RC(x) = \{cl(V) : V \in SO(x)\}$. Each open set in X is α-open in X and each α-open or regular closed set in X is semi-open in X. Semiclosure and semi-interior of a subset A of a space X are defined in a manner analogous to the corresponding definitions of closure and interior. The semiclosure of a subset A (denoted by $sCl(A)$) of a space X is the smallest semiclosed subset of X containing A. Dually, the semi-interior of A (denoted by $sInt(A)$) is the largest semi-open set of X contained in A. For any subset $A \subset X$,

$$int(A) \subset sInt(A) \subset A \subset sCl(A) \subset cl(A).$$

A subset A of a space X is defined to be *linearly S-closed relative to* X [34] if whenever $\mathcal{U}$ is a chain cover of A by semi-open sets in the space X, there exists $U \in \mathcal{U}$ such that $A \subset cl(U)$. Clearly, a space X is linearly S-closed if and only if it is linearly S-closed relative to itself. In [34], it was observed that a subset of a space which is linearly S-closed relative to the space may fail to be linearly S-closed. A subset A of a space X is *linearly S-closed* if it is linearly S-closed in its subspace topology.

For standard definitions and terms used in the paper, one can refer to [8,10].

3. Preliminary Results

To make the paper self contained, we list out here the following results which will be used in the sequel.

Lemma 3.1 *Let X be a space and $A, B \subset X$.*

(i) If $A \in SO(X)$ and $B \in PO(X)$, then $A \cap B$ is semi-open in B and preopen in A.

(ii) [18, Lemma 4.1] If X is an extremally disconnected space and $A \in SO(X)$, then $sCl(A) = cl(A)$.

A filter base $\mathcal{F} = \{F_\alpha\}$ *s-converges* [38] to a point $x \in X$ if for each $V \in SO(x)$ there exists $F_\alpha \in \mathcal{F}$ such that $F_\alpha \subset cl(V)$. Filter base $\mathcal{F} = \{F_\alpha\}$ *s-accumulates* [38] to a point $x \in X$ if for each $V \in SO(x)$ and each $F_\alpha \in \mathcal{F}$, $F_\alpha \cap cl(V) \neq \phi$. By a chain filter-base, we mean a filter-base that is linearly ordered by reverse subset inclusion relation.

Lemma 3.2 *[33] A space X is linearly S-closed if and only if each open chain filter-base in X s-accumulates to a point in X.*

Lemma 3.3 *[34] Let X be a space and A be a subset of X. Then the following statements are equivalent:*

(i) A is linearly S-closed relative to X.

(ii) Any cover of A by regular closed (semi-open) sets in X has a subfamily of strictly smaller cardinality, the closure of the union of whose members contains A.

(iii) Any open chain filter base on X which meets A, s-accumulates to some point in A.

Lemma 3.4 *[34] Suppose A and B are linearly S-closed relative to a space X. Then the following statements are true:*

(i) $int(cl(A))$ and $cl(A)$ are linearly S-closed relative to X.

(ii) If $B \in RO(X)$ (B need not to be necessarily linearly S-closed relative to X), then $A \cap B$ is linearly S-closed relative to X.

Lemma 3.5 *[34] An open (regular open) subset A of a space X is a linearly S-closed subspace of X if and only if it is linearly S-closed relative to X.*

Lemma 3.6 *[34] Let X be a space and $A \subset B \subset X$ where A, B both are open subsets in X. Then A is a linearly S-closed subspace of B if and only if A is a linearly S-closed subspace of X.*

4. Locally Linearly S-closed Spaces

We will use Greek letters κ, λ, ... to denote the infinite cardinal numbers and α, β, γ,... to denote the infinite ordinal numbers. ω is used to denote the smallest infinite cardinal, ω_1 for the smallest uncountable cardinal. Since any linear order having no maximum element has a cofinal subset indexed by a regular cardinal, therefore we may assume that any chain cover of a space X is of the form $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, where κ is a regular infinite cardinal and $U_\alpha \subset U_\beta$ whenever $\alpha < \beta$.

A space X is said to be *S-closed* [38] (resp. *countably S-closed* [6]) if any semi-open cover (resp. countable semi-open cover) of X has a finite subfamily, the closures of whose members cover X. In [33], it was proved that the class of linearly S-closed spaces lies between the class of S-closed spaces and the class of countably S-closed spaces. A space X is said to be *locally S-closed* [20] (resp. *locally countably S-closed* [7]) if every point has an open neighborhood which is an S-closed (resp. countably S-closed) subspace of X. Thus, it is easy to see that the class of locally linearly S-closed spaces lies between the classes of locally S-closed spaces and locally countably S-closed spaces. Locally linearly S-closedness is linked to other generalized compactness like covering properties as shown in figure 1 below. Plain straight arrows denote the implications that hold for any space while additional properties are needed for the implications denoted by dotted arrows.

$$
\begin{tikzcd}
\text{Locally S-closed} \arrow[r] & \text{Locally linearly S-closed} \arrow[d, dashed, bend right, "QHC"'] \arrow[r] & \text{Locally countably S-closed} \\
\text{S-closed} \arrow[u] \arrow[r] \arrow[d] & \text{Linearly S-closed} \arrow[u] \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "Semi-Lindel\ddot{o}f"] & \text{Countably S-closed} \arrow[u] \arrow[d] \arrow[l, dashed, bend right, "CCC"'] \\
\text{QHC} \arrow[r] & \text{Linearly H-closed} \arrow[r] \arrow[u, dashed, bend right, "e.d."'] \arrow[dl, dashed, "regular\ and\ Lindel\ddot{o}f"] & \text{Feebly Compact} \arrow[d] \arrow[u, dashed, bend right, "e.d."'] \\
\text{Compact} \arrow[u] \arrow[r] & \text{Countably compact} \arrow[r] & \text{Pseudocompact}
\end{tikzcd}
$$

Figure 1

The converses are not true in general. Following are the examples:

Example 4.1 *The set of positive integers ω with the discrete topology is an example of a locally linearly S-closed space which fails to be linearly S-closed because a discrete space is linearly S-closed if and only if it is finite [33, Example 2.2].*

Remark 4.2 *In [33, Theorem 2.36], it was shown that a first countable, Hausdorff, linearly S-closed space is finite always. However, the above example (Example 4.1) shows that the result fails if we replace linearly S-closedness by locally linearly S-closedness.*

Example 4.3 *[6, Example 4.2] Let $Y = \beta\omega \setminus \{p\}$, where $p \in \beta\omega \setminus \omega$. Let Y_1 and Y_2 be two disjoint copies of Y and let $X = Y_1 \cup Y_2 \cup \{p\}$. For any subset $A \subseteq \beta\omega \setminus \{p\}$, denote the corresponding subsets of Y_1 and Y_2 by A_1 and A_2 respectively. Define a topology τ on X in the following way: for any $x \in X$, if $x \in Y_1$ (resp. $x \in Y_2$), then the basic open neighborhoods of x in (X, τ) are the sets of the form V_1 (resp. V_2), where V is open subset of $\beta\omega \setminus \{p\}$. For every open neighborhood W of p in $\beta\omega$, a basic open neighborhood of p in (X, τ) is $\{p\} \cup (W \setminus \{p\})_1 \cup (W \setminus \{p\})_2$. Then X is a compact Hausdorff, linearly S-closed space [33, Example 4.2] and hence locally linearly S-closed space. Furthermore, the space X is not extremally disconnected. Since compact Hausdorff spaces are almost regular and almost regular, locally S-closed spaces are extremally disconnected [21, Theorem 3.5], X is not locally S-closed.*

We obtain the following characterization for locally linearly S-closed spaces.

Theorem 4.4 *For a space X, the following statements are equivalent:*

(i) X is locally linearly S-closed.

(ii) Each point in X has an open neighbourhood O which is linearly S-closed relative to X.

(iii) Each point in X has an open neighbourhood O such that $cl(O)$ and $int(cl(O))$ are linearly S-closed relative to X.

(iv) Each point in X has an open neighbourhood O such that $int(cl(O))$ is a linearly S-closed subspace of X.

(v) Each point in X has an open neighbourhood O which is locally linearly S-closed.

Proof: $(i) \Leftrightarrow (ii)$ is followed by using Lemma 3.5.
$(ii) \rightarrow (iii)$ is followed by using Lemma 3.4(i).
$(iii) \rightarrow (iv)$ is followed by using Lemma 3.5.
$(iv) \rightarrow (i)$ is followed by the definition of locally linearly S-closed spaces.
$(i) \rightarrow (v)$: Let x be any point in X. Since X is locally linearly S-closed, there exists an open neighbourhood $O \ni x$ such that O is a linearly S-closed subspace of X. Since every linearly S-closed space is locally linearly S-closed, O is locally linearly S-closed.
$(v) \rightarrow (i)$: From the hypothesis, for any point $x \in X$ there exists an open neighbourhood $O \ni x$ such that O is a locally linearly S-closed subspace of X. Therefore there exists an open neighbourhood U_x of x in the subspace O such that U_x is a linearly S-closed subspace of O. Since U_x is open in O and O is open in X, in view of Lemma 3.6, $U_x \ni x$ is an open linearly S-closed subspace of the space X. This shows that X is locally linearly S-closed. □

Proposition 4.5 *If X is a locally linearly S-closed space and $A \in RO(X)$, then A is locally linearly S-closed.*

Proof: Suppose X is locally linearly S-closed. In view of Theorem 4.4(ii), for any $x \in A \subset X$ there exists an open neighbourhood $O \ni x$ in X which is linearly S-closed relative to X. Since $A \in RO(X)$ and O is linearly S-closed relative to X, by using Lemma 3.4(ii), $O \cap A$ is linearly S-closed relative to X. Therefore, in view of Lemma 3.5 and Lemma 3.6, $O \cap A \ni x$ is an open linearly S-closed subspace of A. This shows that A is a locally linearly S-closed subspace of X. □

A space X is called *perfect* (resp., *RC-perfect* [24]) if each open set in X is an F_σ-set (resp., countable union of regular closed sets) in X. A space X is called *km-perfect* [6] if for each regular open set U and each $x \notin U$, there exists a sequence $\{G_n : n \in \omega\}$ of open sets such that $\cup\{G_n : n \in \omega\} \subseteq U \subseteq \cup\{\overline{G_n} : n \in \omega\}$ and $x \notin \cup\{\overline{G_n} : n \in \omega\}$. Dlaska et. al [6, Theorem 3.1] obtained that if X is extremally disconnected, or hereditarily Lindelöf and Hausdorff, or second countable and Hausdorff, or RC-perfect, or regular and perfect, then X is km-perfect. In our next results, we obtain some sufficient conditions for a locally linearly S-closed space to be extremally disconnected. We start here with the following result.

Proposition 4.6 *Locally linearly S-closed, km-perfect spaces are extremally disconnected.*

Proof: In [7, Theorem 2.6], it was shown that locally countably S-closed, km-perfect spaces are extremally disconnected. Therefore, the result follows by the fact that each locally linearly S-closed space is locally countably S-closed. □

Corollary 4.7 *A locally linearly S-closed space is extremally disconnected if and only if it is km-perfect.*

Proof: Since extremally disconnected spaces are km-perfect, therefore the result follows directly by using Proposition 4.6. □

A space X is called *lob-space* [5] if each point in X has an open neighbourhood base which is linearly ordered by reverse subset inclusion relation. The class of lob-spaces is quite large including the first countable spaces, the non-Archimedean spaces, the protometrizable spaces and the spaces having orthobases (for definitions one can refer to [23]).

Theorem 4.8 *Locally linearly S-closed, Hausdorff, lob-spaces are extremally disconnected.*

Proof: Suppose that X is a Hausdorff, lob and locally linearly S-closed space. Assume that X is not extremally disconnected. Then there exists a proper regular open subset G of X such that $X - cl(G) \neq \phi$ and $cl(G) - G \neq \phi$. Let $x_0 \in cl(G) - G$. Then for each open neighbourhood U of x_0, $U \cap G \neq \phi$. Since X is a lob-space, there exists an open neighbourhood chain filter base $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ at x_0 in X. Furthermore, since X is locally linearly S-closed, by using Theorem 4.4(ii), there exists an open set $O \ni x_0$ which is linearly S-closed relative to X. In view of Lemma 3.4(ii), the subset $A = G \cap O$ is linearly S-closed relative to X. Clearly, for each $\alpha < \kappa$, $B_\alpha \cap A = B_\alpha \cap O \cap G \neq \phi$. Since A is linearly S-closed relative to X and $\mathcal{B}$ is an open chain filter base in X which meets A, by using Lemma 3.3 (iii), $\mathcal{B}$ s-accumulates to some point $y_0 \in A$. Note that $x_0 \neq y_0$ and X is Hausdorff. Therefore there exist disjoint open sets V and W in X such that $x_0 \in V$ and $y_0 \in W$. Since filter base $\mathcal{B}$ converges to x_0, there exists a $\gamma < \kappa$ such that $B_\beta \subset V$ for each β, where $\gamma < \beta < \kappa$. This shows that $B_\beta \cap cl(W) = \phi$ for each β, where $\gamma < \beta < \kappa$, a contradiction to the fact that $\mathcal{B}$ s-accumulates to y_0. Thus, X is an extremally disconnected space. □

Corollary 4.9 *First countable, locally linearly S-closed, Hausdorff spaces are extremally disconnected.*

Proof: In view of Theorem 4.8, the result follows by the fact that first countable spaces are lob. □

Corollary 4.10 *First countable, Hausdorff, locally linearly S-closed spaces are discrete always.*

Proof: In view of Corollary 4.9, the result directly follows by the fact that first countable, Hausdorff, extremally disconnected spaces are discrete always. □

A topological space X is *almost regular* [32] if for any regular closed subset A and a point $x \notin A$ of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $\{x\} \subset V$. It can be easily seen that a regular space is almost regular. Soundararajan [36, Theorem 3.1] proved that almost regular, semi-regular space is regular and hence Hausdorff too. A space may be almost regular without being regular (see [32, Example 3.1]). However, an almost regular space may fail to be Hausdorff even if it is T_1. Moreover, none of the property among almost regularity and Hausdorffness implies each other. The space X in [36, Proposition 2.1] is an example of a space which is neither Hausdorff nor almost regular. On the other hand, an example of a space which is almost regular but not Hausdorff is $\mathbb{R}$ together with the countable complement topology on it. An example of a Hausdorff space that is not almost regular is given below.

Example 4.11 *[37] Let $X = \mathbb{Z}^+$ be the set of positive integers. Define a topology τ on X generated by basis $\mathcal{B} = \{U_a(b) : a,\ b \in X$ and $gcd(a, b) = 1\}$, where $U_a(b) = \{b + na \in X : n \in \mathbb{Z}\}$. Then the space X is Hausdorff but neither regular nor Urysohn. Since almost regular Hausdorff spaces are Urysohn [32, Theorem 3.2], therefore X is not almost regular.*

Theorem 4.12 *Locally linearly S-closed, almost regular, lob-spaces are extremally disconnected.*

Proof: Let X be a locally linearly S-closed, almost regular, lob-space. Assume that X is not extremally disconnected. Then, proceeding in the similar manner as in the proof of Theorem 4.8, there exists a proper regular open subset G of X and a point $x_0 \in cl(G) - G$ such that for each open neighbourhood U of x_0, $U \cap G \neq \phi$. Further, suppose that $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ is an open neighbourhood chain filter base at x_0 in X. Since X is locally linearly S-closed, in view of Theorem 4.4(iv), there exists an open set $O \ni x_0$ such that $int(cl(O))$ is linearly S-closed relative to X. Put $A = G \cap U$ where $U = int(cl(O))$. Then the family $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ where $F_\alpha = B_\alpha \cap A$ form an open chain filter base in the subspace A. Since A is regular open in X, by using Lemma 3.4 (ii) and Lemma 3.5, the subspace A is linearly S-closed and hence, from Lemma 3.2, $\mathcal{F}$ s-accumulates to some point $y_0 \in A$. Note that $X - A \in RC(X)$ and $y_0 \notin X - A$. Since X is an almost regular space, there exist disjoint open sets V and W in X such that $X - A \subset V$ and $y_0 \in W$. By construction, the filter base $\mathcal{B}$ converges to x_0. Since $x_0 \in X - A \subset V$, there exists some $\gamma < \kappa$ such that $B_\beta \subset V$ for each β, where $\gamma < \beta < \kappa$. This shows that $F_\beta \subset V$ for each β, where $\gamma < \beta < \kappa$ and therefore $F_\beta \cap cl(W) = \phi$. Since $y_0 \in W$, this gives a contradiction to the fact that $\mathcal{F}$ s-accumulates to $y_0 \in A$. Thus, X is an extremally disconnected space. □

Corollary 4.13 *First countable, almost regular, locally linearly S-closed space is extremally disconnected.*

Proof: In view of Theorem 4.12, it follows from the fact that first countable spaces are lob. □

A space X is called *weakly regular* [32] if and only if for each point $x \in X$ and each regular open set U containing x, there exists an open set V containing x such that $V \subset cl(V) \subset U$. Almost regular spaces are weakly regular. However, in general a weakly regular space may fail to be almost regular (see [32, Example 2.1]). In the class of T_1-spaces, almost regularity coincides with weak regularity. Therefore

Corollary 4.14 *In the class of T_1-spaces; weakly regular, lob, locally linearly S-closed spaces are extremally disconnected.*

Definition 4.1 *Let X be a topological space. Then*

(i) X *is* weakly first countable *[1] if and only if at each point $x \in X$ there is a decreasing sequence $\{B(n, x) : n \in \omega\}$ of (not necessarily open) subsets of X such that $U \subset X$ is open if and only if for each $x \in U$ there is $n(x) \in \omega$ with $B(n(x), x) \subset U$.*

(ii) X *is* sequential *[8] if each of its sequentially closed subsets is closed. A subset $A \subset X$ is* sequentially closed *if A contains all limits of convergent sequences of points in A.*

(iii) the weak Lindelöf degree *of X (denoted as $wL(X)$) is the smallest infinite cardinal κ such that each open cover of X has a subcollection of cardinality $\leq \kappa$ whose union is dense in X. If $wL(X) = \omega$, then X is called* weakly Lindelöf *[10].*

Theorem 4.15 *[3, Theorem 2.1 and Corollary 2.2, 2.3] Let X be an extremally disconnected, Hausdorff space. Then*

(i) $c(X) \leq wL(X)t(X)$

(ii) $c(X) \leq wL(X)\pi_\chi(X)$

(iii) $|X| \leq \pi_\chi(X)^{wL(X)t(X)\psi_c(X)}$ *and*

(iv) $|X| \leq 2^{wL(X)\pi_\chi(X)\psi(X)}$

where $c(X)$, $t(X)$, $\pi_\chi(X)$, $\psi(X)$, and $\psi_c(X)$ denote respectively, the cellularity, the tightness, the π-character, the pseudocharacter, and the closed pseudocharacter of space X.

Combining Theorems 4.8 and 4.15, we obtain that

Theorem 4.16 *Suppose X is a lob, Hausdorff and locally linearly S-closed space. Then*

(i) $c(X) \leq wL(X)t(X)$

(ii) $c(X) \leq wL(X)\pi_\chi(X)$

(iii) $|X| \leq \pi_\chi(X)^{wL(X)t(X)\psi_c(X)}$

(iv) $|X| \leq 2^{wL(X)\pi_\chi(X)\psi(X)}$

Theorem 4.17 *[5, Theorem 2.2] Let X be a T_1, lob-space. Then the following statements are equivalent:*

(i) X is first countable.

(ii) *X is weakly first countable.*

(iii) *X is sequential.*

(iv) *X has countable tightness.*

(v) *Each point of X is a G_δ set.*

Theorem 4.18 *Let X be a weakly Lindelöf, Hausdorff, lob and locally linearly S-closed space. If in addition, X has either countable tightness (or is weakly first countable or is sequential or each point of X is a G_δ set), then the space X is a discrete space of countable cardinality.*

Proof: Suppose X is a Hausdorff, lob-space having countable tightness (or is weakly first countable or is sequential or each point of X is a G_δ set). By using Theorem 4.17, X is first countable and therefore in view of Corollary 4.10, X is a discrete space. Furthermore, since X is weakly Lindelöf, in view of Theorems 4.17 and 4.16(i), $c(X) \leq \omega$. This shows that $|X| \leq \omega$. □

Theorem 4.19 *Let X be a weakly Lindelöf, first countable, Hausdorff and locally linearly S-closed space. If in addition,*

(i) *X has countable π-character, then $|X| \leq \omega$.*

(ii) *either $t(X) \leq \mathfrak{c}$ or $\pi_\chi(X) \leq \mathfrak{c}$, then $|X| \leq \mathfrak{c}$.*

Proof: (i) Since X is a weakly Lindelöf, Hausdorff, locally linearly S-closed space having countable π-character, by using Theorem 4.16(ii), $c(X) \leq \omega$. Therefore, the result directly follows by Corollary 4.10.

(ii) Proceeding in the same manner as in proof of (i), $c(X) \leq \mathfrak{c}$ and therefore the result directly follows by Corollary 4.10. □

Theorem 4.20 *If X is a weakly Lindelöf, lob, Hausdorff and locally linearly S-closed space having countable π-character and countable pseudocharacter, then $|X| \leq 2^\omega$.*

Proof: The result directly follows by Theorem 4.16(iv). □

5. Invariance Properties

Our main focus in this section is to investigate some conditions on functions for the images (inverse images) of locally linearly S-closed spaces to be locally linearly S-closed. We recall here the following definitions of functions on topological spaces which will be used in this section.

Definition 5.1 *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be*

(i) semi-continuous *(resp.,* irresolute *[26]) if inverse image of each open (resp., semi-open) subset of Y is semi-open in X.*

(ii) weakly continuous *[13] if for each point $x \in X$ and each open set $V \ni f(x)$, there exists an open set $U \ni x$ such that $f(U) \subset cl(V)$.*

(iii) almost continuous (*a.c.S.*) *[31] (in sense of Singhal and Singhal) if for each point $x \in X$ and each open set $V \ni f(x)$ there exists an open set $U \ni x$ such that $f(U) \subset int(cl(V))$.*

(iv) almost continuous (*a.c.H.*) *[11] (in sense of Husain) if for each point $x \in X$ and each neighbourhood V of $f(x)$ in Y, $cl(f^{-1}(V))$ is a neighbourhood of x in X.*

(v) almost open *[39] if $f^{-1}(cl(V)) \subset cl(f^{-1}(V))$ for each open set $V \subset Y$.*

(vi) weakly θ-irresolute *[9] if inverse image of each regular closed subset of Y is semi-open in X.*

(vii) slightly continuous *[28] if preimage of each clopen set in Y is open in X or equivalently, if $f(cl(U)) \subset cl(f(U))$ for each open set $U \subset X$.*

(viii) semi-open *(resp., pre-semi-open[26]) if image of each open (resp., semi-open) subset of X is semi-open in Y.*

Following are some well known results:

Lemma 5.1 *Suppose X and Y are topological spaces.*

(i) [26, Theorem 1.6] A function $f : X \longrightarrow Y$ is irresolute if and only if for each $B \subset Y$, $sCl(f^{-1}(B)) \subset f^{-1}(sCl(B))$.

(ii) [26, Theorem 1.2] An open, continuous map is irresolute.

(iii) [19, Theorem 7] An open, semi-continuous map is irresolute.

Lemma 5.2 *A function $f : X \longrightarrow Y$ is almost open if and only if*

(i) [35, Lemma 3.9] for each semi-open set $V \subset Y$, $f^{-1}(cl(V)) \subset cl(f^{-1}(V))$.

(ii) [25] for each open set $U \subset X$, $f(U) \subset int(cl(f(U)))$.

Lemma 5.3 *[29, Proposition 3.7] If $f : X \longrightarrow Y$ is a semi-continuous function from an extremally disconnected space X to a space Y, then f is slightly continuous.*

5.1. Images of Locally Linearly S-closed Spaces

The following example shows that the continuous image of a locally linearly S-closed space may fail to be locally linearly S-closed.

Example 5.1.1 *Let X be a countably infinite set and p be a particular point of X. Define a topology τ on X such that a subset $A \subset X$ is τ-open if complement of A is either finite or includes p. Then the space (X, τ) is a first countable, Hausdorff space. The identity map $id_X : (X, \tau_d) \to (X, \tau)$, where τ_d is a discrete topology on X, is a continuous surjection. Note that the discrete space (X, τ_d) is locally linearly S-closed. Since (X, τ) is not extremally disconnected, in view of Corollary 4.9, (X, τ) fails to be locally linearly S-closed.*

We now provide some restrictions on functions under which the property of being locally linearly S-closed is preserved.

Lemma 5.1.2 *Let $f : X \longrightarrow Y$ be a continuous irresolute map and A be a subset of X. If A is linearly S-closed relative to X, then $f(A)$ is linearly S-closed relative to Y.*

Proof: Suppose that $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is any chain cover of $f(A)$ by semi-open sets in Y. Then $\{f^{-1}(V_\alpha) : \alpha < \kappa\}$ forms a chain cover of A by semi-open sets in X. Since A is linearly S-closed relative to X, there exists $\beta < \kappa$ such that $A \subset cl(f^{-1}(V_\beta))$. Since f is continuous, $cl(f^{-1}(V_\beta)) \subset f^{-1}(cl(V_\beta))$. This shows that $A \subset f^{-1}(cl(V_\beta))$ and hence $f(A) \subset cl(V_\beta)$. Thus, $f(A)$ is linearly S-closed relative to Y. □

Corollary 5.1.3 *Sets linearly S-closed relative to a space are preserved under continuous open maps.*

Proof: Since open continuous maps are irresolute, the result directly follows by Lemma 5.1.2. □

Theorem 5.1.4 *An open, continuous image of a locally linearly S-closed space is locally linearly S-closed.*

Proof: Let $f : X \longrightarrow Y$ be an open, continuous surjection from a locally linearly S-closed space X to a space Y. Suppose $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Since X is a locally linearly S-closed space, by using Theorem 4.4(ii), there exists an open neighbourhood $O \ni x$ which is linearly S-closed relative to X. Since f is open and continuous, in view of Corollary 5.1.3, $f(O) \subset Y$ is an open neighbourhood of y which is linearly S-closed relative to Y. Thus, again by using Theorem 4.4(ii), Y is locally linearly S-closed. □

Corollary 5.1.5 *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces. If $\Pi_{\alpha\in\Lambda} X_\alpha$ is locally linearly S-closed, then X_α is locally linearly S-closed for each $\alpha \in \Lambda$.*

Proof: For each $\alpha \in \Lambda$, the projection map $\pi_\alpha : \Pi_{\alpha\in\Lambda} X_\alpha \longrightarrow X_\alpha$ is an open, continuous surjection. Therefore, the result directly follows by Theorem 5.1.4. □

Lemma 5.1.6 *Let $f : X \longrightarrow Y$ be a slightly continuous, weakly θ-irresolute function from a space X to a space Y. If $A \subset X$ is linearly S-closed relative to X, then $f(A)$ is linearly S-closed relative to Y.*

Proof: Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f(A)$ by regular closed subsets of Y. Since f is weakly θ-irresolute, $\hat{\mathcal{V}} = \{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ forms a cover of A by semi-open sets in X. By using Lemma 3.3(ii), there exists a subfamily $\{f^{-1}(V_{\alpha_i}) : \alpha_i \in I \subset \Lambda\}$, where $|I| < |\Lambda|$, of $\hat{\mathcal{V}}$ such that $A \subset cl(\underset{\alpha_i\in I}{\cup} f^{-1}(V_{\alpha_i}))$. Since for each $\alpha_i \in I$, $f^{-1}(V_{\alpha_i})$ is semi-open in X, there exist open subsets U_{α_i} of X such that $U_{\alpha_i} \subset f^{-1}(V_{\alpha_i}) \subset cl(U_{\alpha_i})$. Thus, $A \subset cl(\underset{\alpha_i\in I}{\cup} f^{-1}(V_{\alpha_i})) \subset cl(\underset{\alpha_i\in I}{\cup} cl(U_{\alpha_i})) \subset cl(\underset{\alpha_i\in I}{\cup} U_{\alpha_i})$. Since f is slightly continuous and $(\underset{\alpha_i\in I}{\cup} U_{\alpha_i})$ is open in X, therefore $f(cl(\underset{\alpha_i\in I}{\cup} U_{\alpha_i})) \subset cl(f(\underset{\alpha_i\in I}{\cup} U_{\alpha_i})) \subset cl(\underset{\alpha_i\in I}{\cup} f(U_{\alpha_i}))$. Consequently, $f(A) \subset f(cl(\underset{\alpha_i\in I}{\cup} U_{\alpha_i})) \subset cl(\underset{\alpha_i\in I}{\cup} f(U_{\alpha_i})) \subset cl(\underset{\alpha_i\in I}{\cup} V_{\alpha_i})$. By using Lemma 3.3(ii), $f(A)$ is linearly S-closed relative to Y. □

Corollary 5.1.7 *Let $f : X \longrightarrow Y$ be a function from a space X to a space Y and A be linearly S-closed relative to X. Then $f(A)$ is linearly S-closed relative to Y if any one of the following holds:*

(i) f is slightly continuous, semi-continuous and almost open map.

(ii) f is semi-continuous, almost open and X is an extremally disconnected space.

(iii) f is a continuous and almost open map.

Proof: (i) It is easy to verify that a semi-continuous, almost open map is weakly θ-irresolute. Therefore, the result directly follows from Lemma 5.1.6.

(ii) In view of Lemma 5.3, proof directly follows from (i).

(iii) Since each continuous function is both semi-continuous and slightly continuous, therefore (iii) follows from (i). □

Remark 5.1.8 *We want to point out here that a continuous, irresolute map may fail to be open (see [26, Example 1.1]). Continuous maps as well as irresolute maps are semi-continuous. On the other hand, a semi-continuous map need not to be necessarily continuous (irresolute). Furthermore, in general a slightly continuous map may fail to be continuous (see [29, Example 3.6]). However, the following examples show that semi-continuous and slightly continuous functions are independent of each other.*

Example 5.1.9 *[28, Example 2.11] Let $\mathbb{R}$ be the usual topological space and $\mathbb{S}$ be the Sorgenfrey line. Then the identity function $id : \mathbb{R} \to \mathbb{S}$ is a semi-continuous function which is not slightly continuous.*

Example 5.1.10 *[28, Example 2.12] Let $X = \{a, b\}$. Suppose (X, τ_1) is an indiscrete space and (X, τ_2) is the Sierpiński's space. Then the identity map $id : (X, \tau_1) \to (X, \tau_2)$ is a slightly continuous and weakly θ-irresolute map that fails to be semi-continuous.*

Theorem 5.1.11 *If $f : X \longrightarrow Y$ be a continuous and almost open surjection from a locally linearly S-closed space X to a space Y, then Y is locally linearly S-closed.*

Proof: Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Since X is locally linearly S-closed, there exists an open set $U \ni x$ such that U is linearly S-closed relative to X. From corollary 5.1.7(iii), $f(U) \ni y$ is linearly S-closed relative to the space Y. In view of Lemma 3.4(i), $int(cl(f(U)))$ is linearly S-closed relative to Y. Since f is almost open, it follows from Lemma 5.2(ii) that $y \in f(U) \subset int(cl(f(U)))$. Thus, $int(cl(f(U)))$ is an open neighbourhood of y in Y which is linearly S-closed relative to Y. This shows that Y is locally linearly S-closed. □

Theorem 5.1.12 *If $f : X \longrightarrow Y$ is a semi-continuous, almost open, surjection from an extremally disconnected space X to a space Y and X is locally linearly S-closed, then Y is locally linearly S-closed.*

Proof: In view of Corollary 5.1.7(ii), proof of the theorem is identical to that of Theorem 5.1.11. Hence we omit here. □

5.2. Inverse Images of Locally Linearly S-closed Spaces

We now provide conditions on functions under which the inverse image of locally linearly S-closed space is locally linearly S-closed.

Lemma 5.2.1 *Let $f : X \longrightarrow Y$ be an almost open, pre-semi-open map from a space X to a space Y. If $A \subset Y$ is linearly S-closed relative to Y, then $f^{-1}(A)$ is linearly S-closed relative to X.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a chain cover of $f^{-1}(A)$ by semi-open sets in X. Since f is a pre-semi-open map, $\mathcal{U}^* = \{f(U_\alpha) : \alpha < \kappa\}$ forms a chain cover of A by semi-open sets in Y. Thus, there exists a $\beta < \kappa$ such that $A \subset cl_Y(f(U_\beta))$. This implies that $f^{-1}(A) \subset f^{-1}(cl_Y(f(U_\beta)))$. Since $f(U_\beta)$ is semi-open in Y, by using Lemma 5.2(i), $f^{-1}(cl_Y(f(U_\beta))) \subset cl_X(f^{-1}(f(U_\beta))) \subset cl_X(U_\beta)$. This implies that $f^{-1}(A) \subset cl_X(U_\beta)$. Thus $f^{-1}(A)$ is linearly S-closed relative to X. □

Lemma 5.2.2 *[25, Theorem 4] A function $f : X \longrightarrow Y$ is almost continuous (a.c.H) if and only if $f^{-1}(V) \subset int(cl(f^{-1}(V)))$ for each open set $V \subset Y$.*

Theorem 5.2.3 *Let $f : X \longrightarrow Y$ be an almost open, pre-semi-open, almost continuous (a.c.H) map from a space X to a space Y. If Y is locally linearly S-closed, then X is locally linearly S-closed.*

Proof: Since Y is locally linearly S-closed space, for any $x \in X$ there exists an open neighbourhood V of $f(x)$ which is linearly S-closed relative to Y. Since f is an almost open, pre-semi-open map, in view of Lemma 5.2.1, $f^{-1}(V)$ is linearly S-closed relative to X. Furthermore, since f is almost continuous (a.c.H), therefore by using Lemma 5.2.2, $x \in f^{-1}(V) \subset int(cl(f^{-1}(V)))$. In view of Lemma 3.4(i), $int(cl(f^{-1}(V)))$ is an open neighbourhood of x which is linearly S-closed relative to X. This shows that X is locally linearly S-closed. □

Every open, continuous (or semi-open, continuous) function is pre-semi-open ([14, Theorem 9], [4, Theorem 11]). Moreover, an almost continuous (a.c.H), semi-open (or weakly continuous, semi-open) map is pre-semi-open [22, Theorem 2.5].

Corollary 5.2.4 *Let $f : X \longrightarrow Y$ be an almost open, semi-open, almost continuous (a.c.H) map from a space X to a space Y. If Y is locally linearly S-closed, then X is locally linearly S-closed.*

Proof: In view of Theorem 5.2.3, it follows from the fact that an almost continuous (a.c.H), semi-open map is pre-semi-open. □

A mapping $f : X \to Y$ is called *s-perfect* [27] if image of each semi-closed subset A of the space X is semi-closed in Y and for each $y \in Y$, $f^{-1}(y)$ is semi-compact relative to X. A subset A of a space X is *semi-compact relative to X* [30] if any cover of A by semi-open sets in X has a finite subcover.

Lemma 5.2.5 *Let $f : X \longrightarrow Y$ be an s-perfect surjection. If $A \subset Y$ is linearly S-closed relative to Y, then $f^{-1}(A)$ is linearly S-closed relative to X.*

Proof: Suppose $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is any chain cover of $f^{-1}(A)$ by semi-open sets in X. Since f is a s-perfect surjection, for each $\alpha < \kappa$, $f(X - U_\alpha)$ is semi-closed in Y. Let $V_\alpha = (Y - f(X - U_\alpha))$. Then each V_α is a semi-open subset of Y. Now, we will show that the collection $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is a chain cover of A by semi-open sets in Y. Note that for any $y \in A$, the family $\mathcal{U}$ form a chain cover of $f^{-1}(y)$ by semi-open sets in X. Since f is s-perfect map, $f^{-1}(y)$ is semi-compact relative to X and therefore $\mathcal{U}$ has a finite subfamily whose union contains $f^{-1}(y)$. This implies that there exists $\gamma < \kappa$ such that $\{f^{-1}(y)\} \subset U_\alpha$ for each α, where $\gamma < \alpha < \kappa$. Hence, $y \in (Y - f(X - U_\alpha)) = V_\alpha$ for each α, where $\gamma < \alpha < \kappa$. This shows that the family $\mathcal{V}$ forms a chain cover of A by semi-open sets in Y. Since A is linearly S-closed relative to Y, there exists a $\beta < \kappa$ such that $A \subset cl_Y(V_\beta)$. Now, we claim that $f^{-1}(A) \subset cl_X(U_\beta)$. Let us assume that $f^{-1}(A) \nsubseteq cl_X(U_\beta)$. Then $f^{-1}(A) \cap (X - cl_X(U_\beta)) \neq \phi$ which in turn implies that $f^{-1}(A) \cap (X - U_\beta) \neq \phi$. Thus, we obtain that $A \cap f(X - U_\beta) \neq \phi$. Consequently, $A \nsubseteq V_\beta \subset cl_Y(V_\beta)$ which is a contradiction. Thus, $f^{-1}(A) \subset cl_X(U_\beta)$. This shows that $f^{-1}(A)$ is linearly S-closed relative to X. □

Corollary 5.2.6 *If $f : X \longrightarrow Y$ be an s-perfect surjection from a space X to a linearly S-closed space Y, then X is linearly S-closed.*

Proof: The result follows by using Lemma 5.2.5 and the fact that a space is linearly S-closed if and only if it is linearly S-closed relative to itself. □

Theorem 5.2.7 *Let $f : X \longrightarrow Y$ be an s-perfect, almost continuous (a.c.H) (or weakly continuous) surjection. If Y is locally linearly S-closed space, then X is locally linearly S-closed.*

Proof: In view of Lemma 5.2.5, proof of the theorem is very similar to that of Theorem 5.2.3 and hence omitted. □

Lemma 5.2.8 *Let Y be an extremally disconnected space and $f : X \longrightarrow Y$ be a semi-open surjection with $f^{-1}(f(U)) \subset cl_X(U)$ for each semi-open set $U \subset X$. If f is almost continuous (a.c.H) (or weakly continuous) and $A \subset Y$ is linearly S-closed relative to Y, then $f^{-1}(A)$ is linearly S-closed relative to X.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be any chain cover of $f^{-1}(A)$ by semi-open sets in X. Since f is almost continuous (a.c.H) (or weakly continuous) semi-open map, we note that f is pre-semi-open ([22, Theorem 2.5]) and therefore the family $\{f(U_\alpha) : \alpha < \kappa\}$ forms a semi-open chain cover of A by semi-open sets in Y. Furthermore, since A is linearly S-closed relative to Y, there exists $\beta < \kappa$ such that $A \subset cl_Y(f(U_\beta))$. Since f is semi-open, $f^{-1}(sCl_Y(f(U_\beta))) \subset cl_X(f^{-1}(f(U_\beta)))$. Since Y is extremally disconnected, by using Lemma 3.1(ii), $f^{-1}(A) \subset f^{-1}(cl_Y(f(U_\beta))) \subset f^{-1}(sCl_Y(f(U_\beta))) \subset cl_X(f^{-1}(f(U_\beta))) \subset cl_X(U_\beta)$. This shows that $f^{-1}(A)$ is linearly S-closed relative to X. □

Theorem 5.2.9 *Let $f : X \longrightarrow Y$ be a semi-open surjection with $f^{-1}(f(U)) \subset cl_X(U)$ for each semi-open set U in X. If f is almost continuous (a.c.H) (or weakly continuous) and Y is an extremally disconnected locally linearly S-closed space, then X is locally linearly S-closed.*

Proof:

(a.) Suppose f is almost continuous (a.c.H). By using Lemma 5.2.8, proof of the theorem is very similar to that of Theorem 5.2.3 and hence omitted.

(b.) Suppose f is weakly continuous. Since Y is locally linearly S-closed, by using Theorem 4.4(iii), for any point $x \in X$ there exists an open neighbourhood V of $f(x)$ such that $cl_Y(V)$ is linearly S-closed relative to Y. This implies that $f^{-1}(cl_Y(V))$ is linearly S-closed relative to X (from Lemma 5.2.8). Since f is weakly continuous, $x \in f^{-1}(V) \subset int(f^{-1}(cl_Y(V))) \subset int(cl(f^{-1}(cl_Y(V))))$. It follows from Lemma 3.4(i) that $int(cl(f^{-1}(cl_Y(V))))$ is an open neighbourhood of x in X which is linearly S-closed relative to X. Therefore in view of Theorem 4.4(ii), X is locally linearly S-closed.

□

Corollary 5.2.10 *Let $f : X \longrightarrow Y$ be a semi-open surjection with $f^{-1}(f(U)) \subset cl_X(U)$ for each semi-open set U in X. If f is almost continuous (a.c.H) (or weakly continuous) and Y is km-perfect locally linearly S-closed space, then X is locally linearly S-closed.*

Proof: Suppose that Y is a km-perfect locally linearly S-closed space. Therefore, by using Proposition 4.6, Y is extremally disconnected. Hence, in view of Theorem 5.2.9, X is locally linearly S-closed. □

Corollary 5.2.11 *Let $f : X \longrightarrow Y$ be a semi-open surjection with $f^{-1}(f(U)) \subset cl_X(U)$ for each semi-open set U in X. If f is almost continuous (a.c.H) (or weakly continuous) and Y is Hausdorff (or almost regular), lob and locally linearly S-closed space, then X is locally linearly S-closed.*

Proof: Since Y is Hausdorff (or almost regular), lob and locally linearly S-closed space, by using Theorem 4.8 (resp., Theorem 4.12), Y is extremally disconnected. Thus, the result follows from Theorem 5.2.9. □

6. Maximality

Let X be a space together with two topologies τ_1 and τ_2 defined on it. τ_2 is called an *expansion* (resp., *proper expansion*) of τ_1 if $\tau_1 \subset \tau_2$ (resp., $\tau_1 \subsetneq \tau_2$). A space (X, τ) together with a property R is said to be maximal R if no proper expansion of τ satisfies the property R. For any space (X, τ), a topology τ' finer than τ is termed as *simple extension* [15] of τ if there exists a subset $A \notin \tau$ of X such that $\tau' = \{U \cup (V \cap A) : U, V \in \tau\}$. In that case, it is written as $\tau' = \tau(A)$. It is easy to see that the simple extensions are proper expansions of the original topology.

Lemma 6.1 *Let $f : X \longrightarrow Y$ be an irresolute map from a space X to a space Y. If G is a linearly S-closed subspace of X and G is preopen in X, then $f(G)$ is a linearly S-closed subspace of Y.*

Proof: In [33, Proposition 2.8], it was shown that the irresolute image of a linearly S-closed space is linearly S-closed. Therefore, to prove that $f(G)$ is a linearly S-closed subspace of Y, it is enough to show that the restriction map $f|_G : G \longrightarrow f(G)$ is an irresolute map. Suppose that $U \in SO(f(G))$. Then there exists an $U_0 \in SO(Y)$ such that $U = U_0 \cap f(G)$. Since f is an irresolute map, $f^{-1}(U_0) \in SO(X)$. Furthermore, since $G \subset X$ is preopen in X, by using Lemma 3.1(i), $f^{-1}(U_0) \cap G \in SO(G)$. Thus, $f|_G^{-1}(U) = f^{-1}(U_0) \cap G$ is a semi-open subset of the subspace G. This shows that the mapping $f|_G : G \longrightarrow f(G)$ is an irresolute map. □

Theorem 6.2 *A maximal locally linearly S-closed space is extremally disconnected.*

Proof: Suppose that (X, τ) is a maximal locally linearly S-closed space. Assume that (X, τ) is not an extremally disconnected space. Then there exists a proper τ-regular closed subset F of X which is not τ-open. Therefore, the simple extension $\tau(F) = \{U \cup (V \cap F) : U, V \in \tau\}$ is a proper expansion of τ. Let $id_X : (X, \tau) \longrightarrow (X, \tau(F))$ be the identity map on X. It is easy to see that id_X is an open map. Furthermore, for each $U, V \in \tau$, the subsets $U \cup (V \cap F)$ are τ-semi-open in X and hence the mapping id_X is a semi-continuous map. Now, we will show that $(X, \tau(F))$ is a locally linearly S-closed space. Since (X, τ) is a locally linearly S-closed space, for any $x \in X$ there exists an open neighbourhood $O \ni x$ such that O is a linearly S-closed subspace of (X, τ). Since id_X is an open, semi-continuous map and open sets are preopen, by using Lemma 5.1(iii) and Lemma 6.1, $id_X(O) \ni x$ is an open linearly S-closed subspace of $(X, \tau(F))$. This shows that $(X, \tau(F))$ is a locally linearly S-closed space, a contradiction to the fact that (X, τ) is maximal locally linearly S-closed. Thus, (X, τ) is extremally disconnected. □

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