Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.75978

Disjoint Codiskcyclic Operators

El Mostafa Sadouk, Otmane Benchiheb* and Mohamed Amouch

ABSTRACT: This article introduces and examines the concept of disjoint codiskcyclicity for a finite number of operators acting on an infinite-dimensional separable Banach space, providing a corresponding criterion. Furthermore, it characterizes the disjoint codiskcyclicity of finitely many distinct powers of weighted shifts in both unilateral and bilateral cases on a sequence space.

Key Words: Disjoint hypercyclic operator, disjoint supercyclic operator, codiskcyclic operators, weighted shift operators, weighted sequence spaces.

Contents

1. Introduction and preliminaries

Consider X as a separable, complex Banach space with infinite dimension. The set of all operators, i.e., linear and continuous maps acting on X, is represented by $\mathcal{B}(X)$.

An element T of $\mathcal{B}(X)$ is referred to as a *hypercyclic* (respectively, a *supercyclic*) operator if we can find an element x of X such that its orbit under T, i.e., $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$ (respectively, its projective orbit under T, i.e., $\mathbb{C}Orb(T,x) = \{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$) is dense in X. The element x is then referred to as a *hypercyclic* vector (respectively, a *supercyclic* vector) for T.

The appearance of the first example in the theory of hypercyclicity in 1969 by Rolewicz [20] was as follows: if B is the unweighted backward shift on $\ell^2(\mathbb{N})$, then the scaled operator λB is hypercyclic if and only if $|\lambda| > 1$. In a pivotal paper, Salas [23] provided a comprehensive description of the hypercyclicity of the weighted backward shifts acting on $\ell^p(\mathbb{N})$ and $\ell^p(\mathbb{Z})$, where $1 \leq p < \infty$, based on the corresponding weight sequences. From what is presented by Salas, León-Saavedra, and Montes-Rodríguez [15, p. 544], the weighted backward shifts acting on the above spaces are hypercyclic if and only if they satisfy the *hypercyclicity criterion*, where the latter plays a crucial role in the theory of hypercyclicity.

It was established in 1974 by Hilden and Wallen [11] that every unilateral weighted backward shift is supercyclic. After that, Salas [24] presented a study of the supercyclicity of the backward weighted shifts on $\ell^p(\mathbb{Z})$, where $1 \leq p < \infty$, based on their weight sequences, and he discovers the supercyclicity criterion in another article and then proves that bilateral backward weighted shifts are supercyclic if and only if they meet this criterion. For further details on hypercyclicity and supercyclicity, refer to [4,10,11,15,20,23,24].

According to Rolewicz's example above, for every $|\lambda| \leq 1$, λB does not possess the hypercyclic property. This observation led to studying the notions of *disk orbit* and *codisk orbit*. Zeana introduced and examined the notions of *diskcyclic* and *codiskcyclic* operators in her doctoral thesis [12]. They are defined as follows: An element T of $\mathcal{B}(X)$ is referred to as a diskcyclic(respectively, codiskcyclic) operator if we can find an element x of X for which the set

$$\mathbb{D}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, n \in \mathbb{N}\}$$

* Corresponding author

Submitted March 04, 2025. Published August 12, 2025
2010 *Mathematics Subject Classification:* 35B40, 35L70.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
© Soc. Paran. de Mat.

(respectively,

$$\{\beta T^n x : \beta \in \mathbb{C}, |\beta| \geq 1, n \in \mathbb{N}\})$$

is dense in X, the element x is then referred to as a *diskcyclic vector* (respectively, *codiskcyclic vector*) for T.

For a general overview of diskcyclicity and codiskcyclicity, refer to [3,2,12,15,26].

In the following, by the elements $T_1, T_2, \dots, T_N$, we refer to a collection of N elements in $\mathcal{B}(X)$, where $N \geq 2$.

In 2007, Bernal [8], along with Bés and Peris [9], independently studied the characteristics of the following orbit:

$$\{(z, z, \dots, z), (T_1 z, T_2 z, \dots, T_N z), (T_1^2 z, T_2^2 z, \dots, T_N^2 z), \dots\}(z \in X).$$

They focused on the case where one of these orbits is dense in X^N, where X^N is endowed with the product topology. If we can find an element z of X satisfying the above condition, the elements $T_1, T_2, \dots, T_N$ are termed *disjoint hypercyclic*. Numerous studies have investigated the disjoint hypercyclicity of a finite number of shift operators; see [9,8,18].

Definition 1.1 *([9, Definition 2.1]) The sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ of N elements of $\mathcal{B}(X)$, where $N \geq 2$, are disjoint topologically transitive if, for every collection of open, non-empty sets $V_0, V_1, \dots, V_N$ in X, there exists $m \in \mathbb{N}$ such that*

$$V_0 \cap T_{1,m}^{-1}(V_1) \cap \dots \cap T_{N,m}^{-1}(V_N) \neq \emptyset.$$

Additionally, the elements $T_1, T_2, \dots, T_N$ are disjoint topologically transitive if the sequences of iterates $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ are disjoint topologically transitive.

The following criterion, due to Bès and Peris [9].

Definition 1.2 *The sequences $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ of N elements of $\mathcal{B}(X)$, where $N \geq 2$, satisfy the Disjoint Universality Criterion if there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ which satisfies $n_k \overset{k\to\infty}{\longrightarrow} \infty$, dense sets $X_0, X_1, \dots, X_N$ in X, and sequences of mappings $(S_{l,k})_{k\in\mathbb{N}}$, where $S_{l,k} : X_l \longrightarrow X$ for all $1 \leq l \leq N$ and all $k \in \mathbb{N}$ such that for all $x \in X_0$, $y_j \in X_j$ (where $1 \leq j \leq N$), and for all $1 \leq t \leq N$, the following conditions are fulfilled:*

(i) $T_{t,n_k} x \overset{k\to\infty}{\longrightarrow} 0,$

(ii) $S_{j,k} y_j \overset{k\to\infty}{\longrightarrow} 0,$

(iii) $(T_{j,n_k} S_{t,k} - \delta_{t,j} Id_{X_j}) y_j \overset{k\to\infty}{\longrightarrow} 0.$

Additionally, the elements $T_1, T_2, \dots, T_N$ satisfy the Disjoint Hypercyclicity Criterion if their respective sequences of iterates $(T_1^j)_{j=1}^{\infty}, \dots, (T_N^j)_{j=1}^{\infty}$ satisfy the Disjoint Universality Criterion.

The Disjoint Universality Criterion offers enough requirements for establishing disjoint topological transitivity.

Theorem 1.1 *([9, Theorem 2.7])Let $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ be N sequences of elements of $\mathcal{B}(X)$, where $N \geq 2$. Then, $(T_{1,j})_{j=1}^{\infty}, \dots, (T_{N,j})_{j=1}^{\infty}$ satisfy the Disjoint Universality Criterion if and only if, for every positive integer r, the sequences $(\oplus_{i=1}^{r} T_{1,j})_{j=1}^{\infty}, \dots, (\oplus_{i=1}^{r} T_{N,j})_{j=1}^{\infty}$ are disjoint topologically transitive on $\bigoplus_{i=1}^{r} X$.*

The elements $T_1, T_2, \dots, T_N$ are *disjoint supercyclic* (respectively, *disjoint diskcyclic*) if there is an element x of X for which $(x, x, \dots, x) \in X^N$ is a supercyclic (respectively, diskcyclic) vector for $\oplus_{i=1}^{N} T_i$ (see [16,17,18,19,25]).

By combining the results mentioned above, we arrive at the following implications:

Disjoint hypercyclicity ⇒ disjoint diskcyclicity ⇒ disjoint supercyclicity.

Furthermore, the inverse implications do not hold in general (see [25]).

Definition 1.3 *([17, Definition 1.2]) The elements $T_1, T_2, \dots, T_N$ are disjoint topologically transitive for supercyclicity if, for every collection of open, non-empty sets $V_0, V_1, \dots, V_N$ in X, there exists $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that*

$$V_0 \cap (\lambda T_1^{-n})(V_1) \cap \cdots \cap (\lambda T_N^{-n})(V_N) \neq \emptyset.$$

When the constant λ in Definition 1.3 satisfies the condition $|\lambda| \geq 1$, in addition to the other conditions of this definition, we say that $T_1, T_2, \dots, T_N$ are disjoint *disk-topologically transitive* (see [25]).

The following criterion is due to Ö. Martin and R. Sanders [19].

Definition 1.4 *The elements $T_1, T_2, \dots, T_N$ satisfy the Disjoint Supercyclicity Criterion if there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ which satisfies $n_k \overset{k\to\infty}{\longrightarrow} \infty$, dense sets $X_0, X_1, \dots, X_N$ in X, a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$, and sequences of mappings $(S_{l,k})_{k\in\mathbb{N}}$, where $S_{l,k} : X_l \longrightarrow X$ for all $1 \leq l \leq N$ and all $k \in \mathbb{N}$, such that for all $x \in X_0$, $y_j \in X_j$ (where $1 \leq j \leq N$), and for all $1 \leq t \leq N$, the following conditions are fulfilled:*

(i) $(T_j^{n_k} S_{t,k} - \delta_{t,j} Id_{X_j}) y_j \overset{k\to\infty}{\longrightarrow} 0$,

(ii) $\lim_{k\to\infty} \|T_t^{n_k} x\| . \left\|\sum_{j=1}^N S_{j,k} y_j\right\| = 0$.

By associating disjoint topological transitivity with codiskcyclicity, we introduce a new notion in linear dynamical systems called *disjoint codiskcyclicity*, which is related to both disjoint topological transitivity and codiskcyclicity.

The article is framed as follows:

Section 2 presents the fundamental definitions of disjoint codiskcyclicity and introduces the disjoint codiskcyclicity criterion. It also derives the relevant properties central to the disjoint codiskcyclicity theory.

Section 3 characterizes the disjoint codiskcyclicity for various powers of weighted bilateral and unilateral shifts.

2. Disjoint codiskcyclicity

Definition 2.1 *The elements $T_1, T_2, \dots, T_N$ are disjoint codiskcyclic if*

$$\{\alpha(T_1^m z, T_2^m z, \dots, T_N^m z); m \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \geq 1\}$$

is dense in X^N for some element x of X, which is then referred to as a disjoint codiskcyclic vector for $T_1, T_2, \dots, T_N$.

In the same way, the following relationships hold:

Disjoint hypercyclicity ⇒ disjoint codiskcyclicity ⇒ disjoint supercyclicity.

Furthermore, the inverse implications do not generally hold (see Section 3).

Definition 2.2 *The elements $T_1, T_2, \dots, T_N$ are disjoint codisk-topologically transitive if, for all collections of open, non-empty sets $V_0, V_1, \dots, V_N$ in X, there are $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ which fulfill $|\alpha| \geq 1$ such that*

$$V_0 \cap T_1^m(\alpha V_1) \cap \cdots \cap T_N^m(\alpha V_N) \neq \emptyset.$$

Equivalently, there are $n \in \mathbb{N}$ and $\beta \in \mathbb{C}$ which fulfill $0 < |\beta| \leq 1$ such that:

$$V_0 \cap T_1^{-n}(\beta V_1) \cap \cdots \cap T_N^{-n}(\beta V_N) \neq \emptyset.$$

In the following result, we explain the relationship between the set of disjoint codiskcyclic vectors and disjoint codisk-topological transitivity for N elements $T_1, T_2, \dots, T_N$ of $\mathcal{B}(X)$, where $N \geq 2$.

Theorem 2.1 *If $T_1, T_2, \ldots, T_N$ are N elements of $\mathcal{B}(X)$, where $N \geq 2$. Then, the following conditions are equivalent:*

(i) $T_1, T_2, \ldots, T_N$ are disjoint codisk-topologically transitive operators.

(ii) The elements $T_1, T_2, \ldots, T_N$ admit a dense G_δ set of disjoint codiskcyclic vectors.

Proof: $(ii) \Rightarrow (i)$. Suppose that $T_1, T_2, \ldots, T_N$ admit a dense G_δ set of disjoint codiskcyclic vectors. Let $V_0, \ldots, V_N$ be a collection of open, nonempty sets in X. Then, we can find an element z of V_0 for which the set

$$\{\alpha(T_1^m z, T_2 z, \ldots, T_N^m z); m \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \geq 1\}$$

is dense in X^N. Thus,

$$(V_1 \oplus \cdots \oplus V_N) \cap \{\alpha(T_1^m z, T_2 z, \ldots, T_N^m z); m \in \mathbb{N}, \alpha \in \mathbb{C}, |\alpha| \geq 1\} \neq \emptyset.$$

Then, we can find $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ which fulfill $|\alpha| \geq 1$ such that for all $1 \leq i \leq N$, we have $\alpha T_i^m z \in V_i$. Since $z \in V_0$, we have

$$V_0 \cap T_1^{-m}(\beta V_1) \cap \cdots \cap T_N^{-m}(\beta V_N) \neq \emptyset,$$

where $\beta = \frac{1}{\alpha}$.

$(i) \Rightarrow (ii)$. We suppose that $\{A_j : j \in \mathbb{N}\}$ forms a basis for the topology of X. By hypothesis, for any $J = (j_1, \ldots, j_N) \in \mathbb{N}^N$, the set

$$\bigcup_{m \geq k} \bigcup_{\alpha_m \in \mathbb{C}, |\alpha_m| \leq 1} (T_1^{-m}(\alpha_m A_{j_1}) \cap \cdots \cap T_N^{-m}(\alpha_m A_{j_N}))$$

is simultaneously a dense and open subset of X, because X is a Baire space. Then, the set of disjoint codiskcyclic vectors for $T_1, T_2, \ldots, T_N$ is given by

$$\bigcap_{J \in \mathbb{N}^N} \bigcap_{k \in \mathbb{N}} \bigcup_{m \geq k} \bigcup_{\alpha_m \in \mathbb{C}, |\alpha_m| \leq 1} (T_1^{-m}(\alpha_m A_{j_1}) \cap \cdots \cap T_N^{-m}(\alpha_m A_{j_N})).$$

It follows that the operators $T_1, T_2, \ldots, T_N$ admit a dense G_δ set of disjoint codiskcyclic vectors. □

Remark 2.1 Suppose that $T_1, T_2, \ldots, T_N$ are N elements of $\mathcal{B}(X)$, where $N \geq 2$, then:

(i) If $T_1, T_2, \ldots, T_N$ are disjoint codisk-topologically transitive, then $(T_i)_{1 \leq i \leq N}$ are codiskcyclic operators, and every disjoint codiskcyclic vector for $T_1, T_2, \ldots, T_N$ is a codiskcyclic vector for any operator T_i such that $1 \leq i \leq N$.

(ii) If $T_1, T_2, \ldots, T_N$ are invertible, then $T_1, T_2, \ldots, T_N$ are disjoint codisk-topologically transitive if and only if $T_1^{-1}, T_2^{-1}, \ldots, T_N^{-1}$ are disjoint disk-topologically transitive.

Proposition 2.1 *(Comparison principle)Assume that X and $\tilde{X}$ are two separable complex Banach spaces. Let $T_1, T_2, \ldots, T_N$ be N elements of $\mathcal{B}(X)$, and $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_N$ be N elements of $\mathcal{B}(\tilde{X})$, where $N \geq 2$. If there exists $\varphi : X \longrightarrow \tilde{X}$, which is a continuous map with dense range such that*

$$\varphi \circ T_i = \tilde{T}_i \circ \varphi \quad for \quad i = 1, 2, \ldots, N,$$

and if x is a disjoint codiskcyclic vector for $T_1, T_2, \ldots, T_N$, then $\varphi(x)$ is a disjoint codiskcyclic vector for $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_N$. Similarly, if $T_1, T_2, \ldots, T_N$ are disjoint codisk-topologically transitive, then $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_N$ are disjoint codisk-topologically transitive.

Proof: Let $x \in X$ be a disjoint codiskcyclic vector for $T_1, T_2, \dots, T_N$. Then we have

$$\begin{aligned}
\overline{\{\alpha(\tilde{T}_1^n(\varphi(x)),\dots,\tilde{T}_N^n(\varphi(x))); n\in\mathbb{N},\alpha\in\mathbb{C},|\alpha|\geq 1\}} &= \overline{\{\alpha(\varphi\circ T_1^n(x),\dots,\varphi\circ T_N^n(x)); n\in\mathbb{N},\alpha\in\mathbb{C},|\alpha|\geq 1\}}\\
&= \overline{\{\alpha\bigoplus_{i=1}^N \varphi(T_1^n\oplus\cdots\oplus T_N^n)(x,\dots,x); n\in\mathbb{N},\alpha\in\mathbb{C},|\alpha|\geq 1\}}\\
&= \overline{\bigoplus_{i=1}^N \varphi\{\alpha(T_1^n\oplus\cdots\oplus T_N^n)(x,\dots,x); n\in\mathbb{N},\alpha\in\mathbb{C},|\alpha|\geq 1\}}\\
&\supset \bigoplus_{i=1}^N \varphi\overline{\{\alpha(T_1^n\oplus\cdots\oplus T_N^n)(x,\dots,x); n\in\mathbb{N},\alpha\in\mathbb{C},|\alpha|\geq 1\}}\\
&= \bigoplus_{i=1}^N \varphi(\bigoplus_{i=1}^N X).
\end{aligned}$$

Then, $\varphi(x)$ is a disjoint codiskcyclic vector for $\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_N$, because φ has a dense range. Finally, it suffices to apply Theorem 2.1 to finish this proof. □

Definition 2.3 *The elements $T_1, T_2, \dots, T_N$ satisfy the disjoint codiskcyclicity criterion if there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ which satisfies $n_k \overset{k\to\infty}{\longrightarrow} \infty$, dense sets $X_0, X_1, \dots, X_N$ in X, a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ which satisfies $|\lambda_k| \geq 1$ for all $k \in \mathbb{N}$, and sequences of mappings $(S_{l,k})_{k\in\mathbb{N}}$, where $S_{l,k} : X_l \longrightarrow X$ for all $1 \leq l \leq N$ and all $k \in \mathbb{N}$, such that for all $x \in X_0$, $y_j \in X_j$ (where $1 \leq j \leq N$), and for all $1 \leq t \leq N$, the following conditions are fulfilled:*

(i) $\lambda_k T_t^{n_k} x \overset{k\to\infty}{\longrightarrow} 0$,

(ii) $\frac{1}{\lambda_k} S_{j,k} y_j \overset{k\to\infty}{\longrightarrow} 0$,

(iii) $(T_j^{n_k} S_{t,k} - \delta_{t,j} Id_{X_j}) y_j \overset{k\to\infty}{\longrightarrow} 0$.

In what follows, we will see that the disjoint codiskcyclicity criterion is enough to establish disjoint codisk-topological transitivity.

Proposition 2.2 *Let $T_1, T_2, \dots, T_N$ be $N \geq 2$ elements of $\mathcal{B}(X)$. Then, the following conditions are equivalent:*

(1) *$T_1, T_2, \dots, T_N$ satisfy the disjoint codiskcyclicity criterion.*

(2) *There exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ that satisfies $n_k \overset{k\to\infty}{\longrightarrow} \infty$, dense sets $X_0, X_1, \dots, X_N$ in X, and sequences of mappings $(S_{l,k})_{k\in\mathbb{N}}$, where $S_{l,k} : X_l \longrightarrow X$ for all $1 \leq l \leq N$ and all $k \in \mathbb{N}$, such that for all $x \in X_0$, $y \in X_j$ (where $1 \leq j \leq N$), and for all $1 \leq t \leq N$, the following conditions are fulfilled:*

  (a) $T_t^{n_k} x \overset{k\to\infty}{\longrightarrow} 0$,

  (b) $(T_j^{n_k} S_{t,k} - \delta_{t,j} Id_{X_j}) y_j \overset{k\to\infty}{\longrightarrow} 0$,

  (c) $\lim_{k\to\infty} \|T_t^{n_k} x\| . \left\| \sum_{j=1}^N S_{j,k} y_j \right\| = 0$.

(3) *For each $r \in \mathbb{N}$, the elements $\oplus_{i=1}^r T_1, \dots, \oplus_{i=1}^r T_N$ are disjoint codisk-topologically transitive on $\bigoplus_{i=1}^r X$.*

Proof: (1) ⇒ (2) Let $x \in X_0$ and $y_j \in X_j$, where $1 \leq j \leq N$, and let $1 \leq t \leq N$. Since $|\lambda_k| \geq 1$ for all $k \in \mathbb{N}$, and by hypothesis (i) of Definition 2.3, we have

$$\|T_t^{n_k} x\| = \frac{1}{|\lambda_k|} |\lambda_k| \|T_t^{n_k} x\| \overset{k\to\infty}{\longrightarrow} 0 .$$

According to hypothesis (ii) in Definition 2.3, we also have

$$\|T_t^{n_k}x\| \cdot \left\|\sum_{j=1}^{N} S_{j,k}x_j\right\| \leq |\lambda_k|\|T_t^{n_k}x\|(\sum_{j=1}^{N}\frac{1}{|\lambda_k|}\|S_{j,k}x_j\|) \overset{k\to\infty}{\longrightarrow} 0 \ .$$

Thus, we obtain (a) and (c), while (b) follows similarly from hypothesis (iii) in Definition 2.3.

(2) $\Rightarrow$ (3) Let $\varepsilon > 0$. Consider the vectors $e = (e_1, \dots, e_r), f_1 = (f_{1,1}, \dots, f_{1,r}), \dots, f_N = (f_{N,1}, \dots, f_{N,r})$ in $\bigoplus_{i=1}^r X$. We select $x_j \in X_0$ and $y_{i,j} \in X_i$ such that $\|e_j - x_j\| < \frac{\varepsilon}{2}$ and $\|f_{i,j} - y_{i,j}\| < \frac{\varepsilon}{4}$ for all integers $1 \leq i \leq N$ and $1 \leq j \leq r$. Through (i), (ii), and (iii), we can choose $k \in \mathbb{N}$ such that, for all $1 \leq l \leq N$ and $1 \leq j, m \leq r$, we have

$$\|T_l^{n_k}x_j\| < \frac{\varepsilon}{2}, \tag{2.1}$$

$$\|f_{l,j} - \sum_{i=1}^{N} T_l^{n_k}S_{i,k}y_{i,j}\| < \frac{\varepsilon}{2}, \tag{2.2}$$

and

$$\|T_l^{n_k}x_j\|\|\sum_{i=1}^{N} S_{i,k}y_{i,m}\| < \frac{\varepsilon^2}{4}. \tag{2.3}$$

If $\max\{\|T_t^{n_k}x_i\|: 1 \leq t \leq N, 1 \leq i \leq r\} \neq 0$, put $\beta = \frac{2}{\varepsilon}\max\{\|T_t^{n_k}x_i\|: 1 \leq t \leq N, 1 \leq i \leq r\}$. Thus, $0 < \beta < 1$. Define $\hat{x}_j = x_j + \beta\sum_{i=1}^{N} S_{i,k}y_{i,j}$ and $\alpha = \frac{1}{\beta}$. Using (2.3), we have

$$\|e_j - \hat{x}_j\| \leq \|e_j - x_j\| + \beta\|\sum_{i=1}^{N} S_{i,k}y_{i,j}\| < \varepsilon.$$

Now, since $T_l^{n_k}\hat{x}_j = T_l^{n_k}x_j + \beta y_{l,j}$, we get

$$\begin{aligned}
\|f_{l,j} - \alpha T_l^{n_k}\hat{x}_j\| &= \|f_{l,j} - \sum_{i=1}^{N} T_l^{n_k}S_{i,k}y_{i,j} - \alpha T_l^{n_k}x_j\| \\
&\leq \|f_{l,j} - \sum_{i=1}^{N} T_l^{n_k}S_{i,k}y_{i,j}\| + \alpha\|T_l^{n_k}x_j\| \\
&\leq \|f_{l,j} - \sum_{i=1}^{N} T_l^{n_k}S_{i,k}y_{i,j}\| + \frac{\varepsilon}{2\max\{\|T_t^{n_k}x_i\|: 1 \leq t \leq N, 1 \leq i \leq r\}}\|T_l^{n_k}x_j\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore,

$$(\hat{x}_1, \dots, \hat{x}_r) \in \oplus_{i=1}^r B(e_i, \varepsilon) \cap \frac{1}{\alpha}(\oplus_{i=1}^r T_1)^{-n_k}(\oplus_{i=1}^r B(f_{1,i}, \varepsilon)) \cap \dots \cap \frac{1}{\alpha}(\oplus_{i=1}^r T_N)^{-n_k}(\oplus_{i=1}^r B(f_{N,i}, \varepsilon)).$$

This proves (3).

(2) $\Rightarrow$ (3) Assume that for all positive integers r, the elements $\oplus_{i=1}^r T_1, \dots, \oplus_{i=1}^r T_N$ are disjoint codisk-topologically transitive on $\bigoplus_{i=1}^r X$. Then, there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ that satisfies $n_k \overset{k\to\infty}{\longrightarrow} \infty$, and a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ that satisfies $|\lambda_k| \geq 1$ for all $k \in \mathbb{N}$, such that the sequences $(\oplus_{i=1}^r \lambda_k T_1^{n_k})_{k=1}^{\infty}, \dots, (\oplus_{i=1}^r \lambda_k T_N^{n_k})_{k=1}^{\infty}$ are disjoint topologically transitive on $\bigoplus_{i=1}^r X$. According to Theorem 1.1, this means that the sequences $(\lambda_k T_1^{n_k})_{k=1}^{\infty}, \dots, (\lambda_k T_N^{n_k})_{k=1}^{\infty}$ satisfy the Disjoint Universality Criterion. Since $|\lambda_k| \geq 1$ for all $k \in \mathbb{N}$, we deduce that $T_1, T_2, \dots, T_N$ satisfy the disjoint codiskcyclicity criterion. □

Remark 2.2 Disjoint codisk-topological transitivity implies disjoint topological transitivity for supercyclicity. However, the converse is not generally true, as seen in the next section.

3. Powers of disjoint codiskcyclic weighted shifts

We begin here by defining the spaces $l^2(\mathbb{N},w)$, $l^2(\mathbb{Z},w)$, $c_0(\mathbb{N},w)$, and $c_0(\mathbb{Z},w)$, such that the sequence $w=(w_i)_{i\in I}\subset\mathbb{R}$, satisfying $w_i\geq 1$ for all $i\in I$, where $I=\mathbb{Z}$ or $\mathbb{N}$.

The space $l^2(I,w)$, defined over the index set I, where $I=\mathbb{N}$ or $\mathbb{Z}$, is characterized as follows:

$$l^2(I,w):=\{x=(x_n)_n\in\mathbb{C}^I:\sum_{n\in I}w_n^2|x_n|^2<\infty\}. \tag{3.1}$$

As a result, the space $l^2(I,w)$, where $I=\mathbb{N}$ or $\mathbb{Z}$, is a separable Banach space when provided with the norm

$$\|x\|=(\sum_{n\in I}w_n^2|x_n|^2)^{1/2}.$$

The space $c_0(\mathbb{N},w)$, defined over the index set $\mathbb{N}$, is characterized as follows:

$$c_0(\mathbb{N},w):=\{x=(x_n)_n\in\mathbb{C}^{\mathbb{N}}:\lim_{n\to\infty}w_n|x_n|=0\}. \tag{3.2}$$

Similarly, the space $c_0(\mathbb{Z},w)$, defined over the index set $\mathbb{Z}$, is described by

$$c_0(\mathbb{Z},w):=\{x=(x_i)_i\in\mathbb{C}^{\mathbb{Z}}:\lim_{|i|\to\infty}w_i|x_i|=0\}. \tag{3.3}$$

As a result, the space $c_0(I,w)$, where $I=\mathbb{N}$ or $\mathbb{Z}$, is a separable Banach space when provided with the sup-norm

$$\|x\|=\sup_{n\in I}w_n|x_n|,$$

see [16].

3.1. Bilateral shifts

In the following, let X be either $c_0(\mathbb{Z},w)$ or $l^2(\mathbb{Z},w)$. Then, if $(e_k)_{k\in\mathbb{Z}}$ is the canonical basis of X and if $a=(a_k)_{k\in\mathbb{Z}}\subset\mathbb{C}$ is a sequence of nonzero scalars that is bounded, the bilateral backward weighted shift $B_a:X\longrightarrow X$ is the bounded operator defined by

$$B_a e_n=a_n e_{n-1}\quad(n\in\mathbb{Z}).$$

Theorem 3.1 *Let $B_{a_1},B_{a_2},\ldots,B_{a_N}$ be N bilateral backward weighted shifts acting on X, where $N\geq 2$. Then, if $r_1,r_2,\ldots,r_N$ are integers such that $1\leq r_1<r_2<\cdots<r_N$, the following assertions are equivalent:*

a) $B_{a_1}^{r_1},B_{a_2}^{r_2},\ldots,B_{a_N}^{r_N}$ are disjoint codisk-topologically transitive.

b) Given any $\varepsilon>0$ and $q\in\mathbb{N}$, one can find an $m\in\mathbb{N}$ with $m>2q$, such that for $|j|,|k|\leq q$, we obtain:

If $1\leq l,s\leq N$,

$$w_{j-r_lm}\left|\prod_{i=j-r_lm+1}^{j}a_{l,i}\right|\quad<\quad\varepsilon, \tag{3.4}$$

and

$$\frac{w_{j-r_lm}|\prod_{i=j-r_lm+1}^{j}a_{l,i}|w_{k+r_sm}}{|\prod_{i=k+1}^{k+r_sm}a_{s,i}|}<\varepsilon. \tag{3.5}$$

If $1\leq s<l\leq N$,

$$w_{j+(r_l-r_s)m}\left|\frac{\prod_{i=j+(r_l-r_s)m+1}^{j+r_lm}a_{s,i}}{\prod_{i=j+1}^{j+r_lm}a_{l,i}}\right|<\varepsilon, \tag{3.6}$$

and

$$w_{j+(r_s-r_l)m}\left|\frac{\prod_{i=j+(r_s-r_l)m+1}^{j+r_sm}a_{l,i}}{\prod_{i=j+1}^{j+r_sm}a_{s,i}}\right|<\varepsilon. \tag{3.7}$$

c) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \dots, B_{a_N}^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Proof: *a)* $\Rightarrow$ *c)*. Given $\varepsilon>0$ and $q\in\mathbb{N}$, select δ which satisfies $0<\delta<1/2$ and $\frac{\delta}{(1-\delta)}<\varepsilon$. Suppose that $x=\sum_{k\in\mathbb{Z}}x_ke_k\in X$ and $\alpha\in\mathbb{C}$ satisfies $|\alpha|\geq 1$, and let $m\in\mathbb{N}$ with $m>2q$ such that

$$\|x-\sum_{|j|\leq q}e_j\|<\delta \quad and \quad \|\alpha B_{a_l}^{r_lm}x-\sum_{|j|\leq q}e_j\|<\delta, \quad where \quad 1\leq l\leq N.$$

Therefore,

$$|x_j-1|w_j<\delta, \quad where \quad |j|\leq q, \tag{3.8}$$

and

$$|x_j|w_j<\delta, \quad where \quad |j|>q. \tag{3.9}$$

And therefore,

$$1-\frac{\delta}{w_j}<\left|\alpha\left(\prod_{i=j+1}^{j+r_lm}a_{l,i}\right)x_{j+r_lm}\right|<1+\frac{\delta}{w_j}, \quad where \quad |j|\leq q, \tag{3.10}$$

and

$$\left|\alpha\left(\prod_{i=j+1}^{j+r_lm}a_{l,i}\right)x_{j+r_lm}\right|w_j<\delta, \quad where \quad |j|>q. \tag{3.11}$$

Now suppose that $1\leq l\leq N$ and $|j|\leq q$ are fixed. Since $j-r_lm<-q$ and since $w_j\geq 1$ ($j\in\mathbb{Z}$), combining (3.8) and (3.11) we obtain

$$\begin{aligned} w_{j-r_lm}\left|\prod_{i=j-r_lm+1}^{j}a_{l,i}\right| &< \frac{\delta}{|\alpha|(1-\frac{\delta}{w_j})} \\ &< \frac{\delta}{1-\delta} \\ &< \varepsilon. \end{aligned}$$

Thus, we get (3.4).

Finally, since the properties (3.8), (3.9), (3.10), and (3.11) are satisfied, we can use the proof $(a)\Rightarrow(b)$ of [16, Theorem 4.1] to see that

$$\frac{w_{j-r_lm}|\prod_{i=j-r_lm+1}^{j}a_{l,i}|w_{k+r_sm}}{|\prod_{i=k+1}^{k+r_sm}a_{s,i}|}<\varepsilon, \quad where \quad 1\leq l,s\leq N,$$

and that:

$$w_{j+(r_l-r_s)m}\left|\frac{\prod_{i=j+(r_l-r_s)m+1}^{j+r_lm}a_{s,i}}{\prod_{i=j+1}^{j+r_lm}a_{l,i}}\right|<\varepsilon,$$

$$w_{j+(r_s-r_l)m}\left|\frac{\prod_{i=j+(r_s-r_l)m+1}^{j+r_sm}a_{l,i}}{\prod_{i=j+1}^{j+r_sm}a_{s,i}}\right|<\varepsilon, \quad where \quad 1\leq s<l\leq N.$$

Thus, we get (3.5), (3.6) and (3.7).

$b) \Rightarrow c)$. From $b)$, we can construct integers $1 \leq n_1 < n_2 < \dots$ so that for each $q \in \mathbb{N}$, we have

$$w_{j-r_l n_q} \left| \prod_{i=j-r_l n_q+1}^{j} a_{l,i} \right| < \frac{1}{q} \quad (1 \leq l \leq N), \tag{3.12}$$

$$\frac{w_{j-r_l n_q} |\prod_{i=j-r_l n_q+1}^{j} a_{l,i}| w_{k+r_s n_q}}{|\prod_{i=k+1}^{k+r_s n_q} a_{s,i}|} < \frac{1}{q} \quad (1 \leq l, s \leq N), \tag{3.13}$$

and for $1 \leq s < l \leq N$,

$$w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| < \frac{1}{q}, \tag{3.14}$$

$$w_{j+(r_s-r_l)n_q} \left| \frac{\prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i}}{\prod_{i=j+1}^{j+r_s n_q} a_{s,i}} \right| < \frac{1}{q}. \tag{3.15}$$

Let $X_l = span\{e_j : j \in \mathbb{Z}\}$ for all $1 \leq l \leq N$. Then, $X_1, X_2, \dots, X_N$ are dense sets in X. By (3.12), $B_{a_l}^{r_l n_q} x \overset{k\to\infty}{\longrightarrow} 0$ for all $x \in X_0$.

Now, let the maps $S_l : X_l \longrightarrow X$, where $1 \leq l \leq N$, be defined as

$$S_l e_j = \frac{e_{j+1}}{a_{l,j+1}} (j \in \mathbb{Z}).$$

Based on these properties, we can use the proof $(b) \Rightarrow (c)$ of [16, Theorem 4.1] to see that

$$(B_{a_l}^{r_l n_k} S_i^{r_i n_k} - \delta_{i,l} Id_{X_l}) x_l \overset{k\to\infty}{\longrightarrow} 0 \text{ for all } x_l \in X_l \ (1 \leq i, l \leq N),$$

and

$$\lim_{k\to\infty} \left\| B_{a_l}^{r_l n_k} x \right\| . \left\| \sum_{j=1}^{N} S_j^{r_j n_k} x_j \right\| = 0 \text{ for any } x \in X_0 \text{ and any } x_j \in X_j \text{ where } 1 \leq j, l \leq N.$$

Therefore, $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \dots, B_{a_N}^{r_N}$ satisfy the disjoint codiskcyclicity criterion. □

If the backward weighted shifts in Theorem 3.1 are invertible, it follows that

Corollary 3.1 *Let $B_{a_1}, B_{a_2}, \dots, B_{a_N}$ be N invertible bilateral backward weighted shifts acting on X, where $N \geq 2$. Then, if $r_1, r_2, \dots, r_N$ are integers such that $1 \leq r_1 < r_2 < \dots < r_N$, the following assertions are equivalent:*

a) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \dots, B_{a_N}^{r_N}$ are disjoint codisk-topologically transitive.

b) There are integers $n_1, n_2, n_3, \dots$ satisfying $1 \leq n_1 < n_2 < \dots$, such that for each $j \in \mathbb{N}$, the following is true:

If $1 \leq s < l \leq N$,

$$\lim_{q\to\infty} w_{j+(r_l-r_s)n_q} \frac{\left| \prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i} \right|}{\left| \prod_{i=j+1}^{j+r_l n_q} a_{l,i} \right|} = 0,$$

and

$$\lim_{q\to\infty} w_{j+(r_s-r_l)n_q} \frac{\left| \prod_{i=j+(r_s-r_l)n_q+1}^{j+r_s n_q} a_{l,i} \right|}{\left| \prod_{i=j+1}^{j+r_s n_q} a_{s,i} \right|} = 0.$$

If $1 \le s, l \le N$,

$$\lim_{q\to\infty} w_{-r_l n_q} \left| \prod_{i=-r_l n_q}^{1} a_{l,i} \right| = 0,$$

and

$$\lim_{q\to\infty} \max \left\{ \frac{w_{-r_l n_q} |\prod_{i=-r_l n_q}^{1} a_{l,i}| w_{r_s n_q}}{|\prod_{i=1}^{r_s n_q} a_{s,i}|} \right\} = 0.$$

c) $B_{a_1}^{r_1}, B_{a_2}^{r_2}, \ldots, B_{a_N}^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Next, we investigate two specific cases of Theorem 3.1.

Corollary 3.2 *Let* B_a *be a bilateral backward weighted shifts acting on* X. *Then, for all integers* $1 \le r_1 < r_2 < \cdots < r_N$, *where* $N \ge 2$, *the following conditions are equivalent:*

a) $B_a^{r_1}, B_a^{r_2}, \ldots, B_a^{r_N}$ *are disjoint codisk-topologically transitive.*

b) *Given any* $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *one can find an* $m \in \mathbb{N}$ *with* $m > 2q$, *such that for* $|j|, |k| \le q$, *we obtain:*

If $1 \le l, s \le N$,

$$w_{j-r_l m} \left| \prod_{i=j-r_l m+1}^{j} a_i \right| < \varepsilon,$$

and

$$\frac{w_{j-r_l m} |\prod_{i=j-r_l m+1}^{j} a_i| w_{k+r_s m}}{|\prod_{i=k+1}^{k+r_s m} a_i|} < \varepsilon.$$

If $1 \le s < l \le N$,

$$\frac{w_{j+(r_l-r_s)m}}{\left|\prod_{i=j+1}^{j+(r_l-r_s)m} a_i\right|} < \varepsilon \quad \text{and} \quad w_{j+(r_s-r_l)m} \left| \prod_{i=j+(r_s-r_l)m+1}^{j} a_i \right| < \varepsilon.$$

c) $B_a^{r_1}, B_a^{r_2}, \ldots, B_a^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Corollary 3.3 *Let* B *be the unweighted backward shift defined on* X *as follows:*

$$Be_k = e_{k-1} \text{ for all } k \in \mathbb{Z}.$$

Then, if $r_1, r_2, \ldots, r_N$ *are* N *integers such that* $1 \le r_1 < r_2 < \cdots < r_N$, *and if* $\lambda_1, \lambda_2, \ldots, \lambda_N$ *are* N *complex numbers, where* $N \ge 2$, *the following conditions are equivalent:*

a) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \ldots, \lambda_N B^{r_N}$ *are disjoint codisk-topologically transitive.*

b) *Given any* $\varepsilon > 0$ *and* $q \in \mathbb{N}$, *one can find an* $m \in \mathbb{N}$ *with* $m > 2q$, *such that for* $|j|, |k| \le q$, *we obtain:*

If $1 \le l, s \le N$,

$$w_{j-r_l m} |\lambda_l|^m \quad < \quad \varepsilon,$$

and

$$w_{j-r_l m} w_{k+r_s m} \left| \frac{\lambda_l}{\lambda_s} \right|^m \quad < \quad \varepsilon.$$

If $1 \leq s < l \leq N$,

$$w_{j+(r_l-r_s)m}\left|\frac{\lambda_s}{\lambda_l}\right|^m < \varepsilon \quad \textit{and} \quad w_{j+(r_s-r_l)m}\left|\frac{\lambda_l}{\lambda_s}\right|^m < \varepsilon.$$

c) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \ldots, \lambda_N B^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Example 3.1 *Let* $w_0 = (1)_{j\in\mathbb{Z}}$ *represent a positive constant sequence, and let* $a = (a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ *be a sequence defined by:*

$$a_k = \begin{cases} \frac{1}{2}, & \textit{if } k \in \{-2^n+1, \ldots, -2^n+n\} \quad \textit{for} \quad \textit{some} \quad \textit{odd} \quad n \in \mathbb{N} \\ 2, & \textit{if } k \in \{-2^n-n+1, \ldots, -2^n\} \quad \textit{or} \quad k = 2^n \quad \textit{for} \quad \textit{some} \quad \textit{odd} \quad n \in \mathbb{N} \\ 1, & \textit{otherwise.} \end{cases}$$

Therefore:

$$a = (\ldots, 1, 2, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \ldots, 1, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 2, \frac{1}{2}, [1], 1, 2, 1, \ldots),$$

where [.] *represent the* 0^{th} *column of* $(a_k)_{k\in\mathbb{Z}}$.

If B_a *is the bilateral weighted backward shift associated with the sequence* $(a_k)_{k\in\mathbb{Z}}$*, then the operators* B_a, B_a^2 *are disjoint topologically transitive for supercyclicity (see [16]). By Corollary 3.2 and since*

$$\left|\prod_{i=1-m}^{0} a_i\right| < 1 \quad \textit{when} \quad \left|\prod_{i=1-2m}^{0} a_i\right| = 1 \quad \textit{for} \quad \textit{all} \quad m \in \mathbb{N},$$

it follows that B_a, B_a^2 *are not disjoint codisk-topologically transitive.*

3.2. Unilateral shifts

In this subsection, we will demonstrate that disjoint topologically codisk-transitive can coincide with disjoint topologically transitive for supercyclicity in certain cases.

In the following, let X be either $c_0(\mathbb{N}, w)$ or $l^2(\mathbb{N}, w)$. Then, if $a = (a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a sequence of nonzero scalars that is bounded, the unilateral backward weighted shift $T_a : X \longrightarrow X$ is the bounded operator defined by

$$T_a(x_0, x_1, \ldots) = (a_1 x_1, a_2 x_2, \ldots), \text{ where } (x_0, x_1, \ldots) \in X.$$

Theorem 3.2 *Let* $T_{a_1}, T_{a_2}, \ldots, T_{a_N}$ *be* N *unilateral backward weighted shifts acting on* X*, where* $N \geq 2$*. Then, if* $r_1, r_2, \ldots, r_N$ *are integers such that* $1 \leq r_1 < r_2 < \cdots < r_N$*, the following assertions are equivalent:*

a) $T_{a_1}^{r_1}, T_{a_2}^{r_2}, \ldots, T_{a_N}^{r_N}$ *are disjoint codisk-topologically transitive.*

b) $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ *are disjoint topologically transitive for supercyclicity.*

c) Given any $\varepsilon > 0$ *and* $q \in \mathbb{N}$*, one can find an* $m \in \mathbb{N}$ *with* $m > q$*, such that for* $0 \leq j \leq q$*, we obtain: If* $1 \leq s < l \leq N$,

$$w_{j+(r_l-r_s)m}\left|\frac{\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}}{\prod_{i=j+1}^{j+r_l m} a_{l,i}}\right| \quad < \quad \varepsilon.$$

d) $T_{a_1}^{r_1}, T_{a_2}^{r_2}, \ldots, T_{a_N}^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Proof: First, we prove that $c)$ implies $d)$. Conditions $b)$ and $c)$ are known to be equivalent from [16, Theorem 3.2]. Finally, the implications $d) \Rightarrow a)$ and $a) \Rightarrow b)$ hold trivially, which concludes the demonstration of the equivalence of conditions $a) - d)$ of the theorem.

$c) \Rightarrow d)$: From $c)$, if $0 \leq j \leq q$ and $1 \leq s < l \leq N$, one can find integers $1 \leq n_1 < n_2 < \dots$ such that

$$w_{j+(r_l-r_s)n_q} \left| \frac{\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_l n_q} a_{s,i}}{\prod_{i=j+1}^{j+r_l n_q} a_{l,i}} \right| < \frac{1}{q}. \tag{3.16}$$

Let $X_l = span\{e_j : j \in \mathbb{N}\}$ for all $1 \leq l \leq N$. Then, $X_1, X_2, \dots, X_N$ are dense sets in X, and $T_{a_l}^{r_l n_q} x \overset{k\to\infty}{\longrightarrow} 0$ for all $x \in X_0$. Define the maps $S_l : X_l \longrightarrow X$ $(1 \leq l \leq N)$ as follows:

$$S_l e_j = \frac{1}{a_{l,j+1}} e_{j+1} \quad (j \in \mathbb{N}).$$

Based on these properties, we can use the proof $(b) \Rightarrow (c)$ of [16, Theorem 3.2] to see that

$$(T_{a_l}^{r_l n_k} S_i^{r_i n_k} - \delta_{i,l} Id_{X_i}) y_i \overset{k\to\infty}{\longrightarrow} 0 \text{ for all } y_i \in X_i, \text{ where } 1 \leq i, l \leq N,$$

and

$$\lim_{k\to\infty} \left\| T_{a_l}^{r_l n_k} x \right\| . \left\| \sum_{j=1}^N S_j^{n_k} x_j \right\| = 0 \text{ for any } x \in X_0 \text{ and any } x_j \in X_j \text{ where } 1 \leq j, l \leq N.$$

Therefore, $T_{a_1}^{r_1}, T_{a_2}^{r_2}, \dots, T_{a_N}^{r_N}$ satisfy the disjoint codiskcyclicity criterion. □

Below, we derive two particular cases of Theorem 3.1.

Corollary 3.4 *Let T_a be a unilateral backward weighted shift acting on X. Then, if $r_1, r_2, \dots, r_N$ are integers such that $1 \leq r_1 < r_2 < \dots < r_N$, where $N \geq 2$, the following assertions are equivalent:*

a) *$T_a^{r_1}, T_a^{r_2}, \dots, T_a^{r_N}$ are disjoint codisk-topologically transitive.*

b) *$T_a^{r_1}, T_a^{r_2}, \dots, T_a^{r_N}$ are disjoint topologically transitive for supercyclicity.*

c) *Given any $\varepsilon > 0$ and $q \in \mathbb{N}$, one can find an $m \in \mathbb{N}$ with $m > q$, such that for $0 \leq j \leq q$, we obtain: If $1 \leq s < l \leq N$,*

$$\frac{w_{j+(r_l-r_s)m}}{\left| \prod_{i=j+1}^{j+(r_l-r_s)m} a_i \right|} < \varepsilon.$$

d) *$T_a^{r_1}, T_a^{r_2}, \dots, T_a^{r_N}$ satisfy the disjoint codiskcyclicity criterion.*

Corollary 3.5 *Let T be the unweighted backward shift defined on X as follows:*

$$T(x_0, x_1, \dots) = (x_1, x_2, \dots), \text{ where } (x_0, x_1, \dots) \in X.$$

Then, if $r_1, r_2, \dots, r_N$ are N integers such that $1 \leq r_1 < r_2 < \dots < r_N$, and if $\lambda_1, \lambda_2, \dots, \lambda_N$ are N complex numbers, where $N \geq 2$, the following conditions are equivalent:

a) *$\lambda_1 T^{r_1}, \lambda_2 T^{r_2}, \dots, \lambda_N T^{r_N}$ are disjoint codisk-topologically transitive.*

b) *$\lambda_1 T^{r_1}, \lambda_2 T^{r_2}, \dots, \lambda_N T^{r_N}$ are disjoint topologically transitive for supercyclicity.*

c) *Given any $\varepsilon > 0$ and $q \in \mathbb{N}$, one can find an $m \in \mathbb{N}$ with $m > q$, such that for $0 \leq j \leq q$, we obtain: If $1 \leq l < s \leq N$,*

$$w_{j+(r_l-r_s)m} \left| \frac{\lambda_s}{\lambda_l} \right|^m < \varepsilon.$$

d) $\lambda_1 T^{r_1}, \lambda_2 T^{r_2}, \ldots, \lambda_N T^{r_N}$ *satisfy the disjoint codiskcyclicity criterion.*

Remark 3.1 *We can use [19, Proposition 3.2] to show that there exist two unilateral weighted backward shifts* T_1,T_2 *on* $\ell^2(\mathbb{N}, w)$ *that are disjoint topologically transitive for supercyclicity. Therefore, by Theorem 3.1,* T_1, T_2 *are disjoint codisk-topologically transitive, but the shifts* T_1, T_2 *fail to be disjoint hypercyclic.*

Acknowledgment.
The authors are sincerely grateful to the handling editor and the anonymous referee for their thorough reading, constructive criticism, and insightful suggestions, which have greatly contributed to improving the quality of the manuscript during the revision process.

References

1. Amouch, M., Bachir, A., Benchiheb, O., Mecheri, S. (2023). Weakly recurrent operators. *Mediterranean Journal of Mathematics*, **20**(3), 169.
2. Amouch, M., Benchiheb, O. (2020). Diskcyclicity of sets of operators and applications. *Acta Mathematica Sinica, English Series*, **36**, 1203–1220.
3. Amouch, M., Benchiheb, O. (2022). Codiskcyclic sets of operators on complex topological vector spaces. *Proyecciones Journal of Mathematics*, **41**(6), 1439–1456.
4. Bayart, F., Matheron, E. (2009). *Dynamics of linear operators*. Cambridge University Press.
5. Benchiheb, O., Amouch, M. (2024). On recurrent sets of operators. *Boletim da Sociedade Paranaense de Matemática*, **42**, 1–9.
6. Benchiheb, O., Amouch, M. (2024). Subspace-super recurrence of operators. *Filomat*, **38**(9), 3093–3103.
7. Benchiheb, O., Sadek, F., Amouch, M. (2023). On super-rigid and uniformly super-rigid operators. *Afrika Matematika*, **34**(1), 6.
8. Bernal-González, L. (2007). Disjoint hypercyclic operators. *Studia Mathematica*, **2**, 113–131.
9. Bès, J., Peris, A. (2007). Disjointness in hypercyclicity. *Journal of Mathematical Analysis and Applications*, **336**, 297–315.
10. Godefroy, G., Shapiro, J.H. (1991). Operators with dense, invariant, cyclic vector manifolds. *Journal of Functional Analysis*, **98**, 229–269.
11. Hilden, H.M., Wallen, L.J. (1994). Some cyclic and non-cyclic vectors of certain operators. *Indiana University Mathematics Journal*, **23**, 557–565.
12. Jamil, Z.Z. (2002). *Cyclic phenomena of operators on Hilbert space*. Thesis, University of Baghdad.
13. Karim, N., Benchiheb, O., Amouch, M. (2024). Faber-hypercyclic semigroups of linear operators. *Filomat*, **38**(25), 8869–8876.
14. Karim, N., Benchiheb, O., Amouch, M. (2024). Disjoint strong transitivity of composition operators. *Collectanea Mathematica*, **75**(1), 171–187.
15. León-Saavedra, F., Montes-Rodríguez, A. (1997). Linear structure of hypercyclic vectors. *Journal of Functional Analysis*, **148**, 524–545.
16. Liang, Y.X., Zhou, Z.H. (2014). Disjoint supercyclic powers of weighted shifts on weighted sequence spaces. *Turkish Journal of Mathematics*, **38**, 1007–1022.
17. Liang, Y.X., Zhou, Z.H. (2018). Disjoint supercyclic weighted composition operators. *Bulletin of the Korean Mathematical Society*, **55**(4), 1137–1147.
18. Martin, Ö. (2011). *Disjoint hypercyclic and supercyclic composition operators*. PhD thesis, Bowling Green State University.
19. Martin, Ö., Sanders, R. (2016). Disjoint supercyclic weighted shifts. *Integral Equations and Operator Theory*, **85**, 191–220.
20. Rolewicz, S. (1969). On orbits of elements. *Studia Mathematica*, **32**, 17–22.
21. Sadouk, E.M., Benchiheb, O., Amouch, M. (2024). Disjoint topologically super-recurrent operators. *Filomat*, **38**(30), 10495–10504.
22. Sadouk, E.M., Benchiheb, O., Amouch, M. (2024). C-Recurrent operators. *Palestine Journal of Mathematics*, **13**(4).
23. Salas, H. (1995). Hypercyclic weighted shifts. *Transactions of the American Mathematical Society*, **347**(3), 993–1004.
24. Salas, H. (1999). Supercyclicity and weighted shifts. *Studia Mathematica*, **135**(1), 55–74.
25. Wang, C., Zhou, Z.H., Zhang, L. (2022). Disjoint diskcyclicity of weighted shifts. *Open Mathematics*, **20**, 820–828.

26. Wang, Y., Zhou, H.G. (2018). Disk-cyclic and codisk-cyclic weighted pseudo-shifts. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, **25**(2), 209–224.

Otmane Benchiheb, El Mostafa Sadouk,, and Mohamed Amouch,
Department of Mathematics,
Chouaib Doukkali University,
Faculty of science El Jadida,
Morocco.
E-mail address: otmane.benchiheb@gmail.com // benchiheb.o@ucd.ac.ma
E-mail address: elmostaphasadouk@gmail.com
E-mail address: amouch.m@ucd.ac.ma