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On Generalized Fuzzy Bitopological Spaces*

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ABSTRACT: This paper is devoted to introduce the new classes namely generalized fuzzy bitopological spaces and defined various types of generalized pairwise fuzzy sets in the generalized fuzzy bitopological spaces. Several examples are given to investigate the generalized pairwise fuzzy sets and characterizations of generalized pairwise fuzzy sets are studied in this paper. Especially the intersection (or union) of two generalized pairwise fuzzy closed sets is also a generalized pairwise fuzzy closed set. The intersection of two generalized pairwise fuzzy nowhere dense sets is also a generalized pairwise fuzzy nowhere dense set and a generalized pairwise fuzzy nowhere dense set is a generalized pairwise fuzzy semi-closed set are established. Additionally, a generalized pairwise fuzzy nowhere dense and generalized pairwise fuzzy F_σ-set is a generalized pairwise fuzzy σ-nowhere dense set is investigated in this paper. Furthermore generalized pairwise fuzzy first category spaces, generalized pairwise fuzzy second category spaces and generalized pairwise fuzzy submaximal spaces are defined and some of whose properties are also studied in this paper.

Key Words: Generalized pairwise fuzzy G_δ-set, Generalized pairwise fuzzy nowhere dense set, generalized pairwise fuzzy first category space, generalized pairwise fuzzy submaximal space, generalized fuzzy bitopological space.

Contents

1. Introduction

The notion of fuzzy topological spaces was defined by C.L.Chang [4] in 1968 followed by L.A.Zadeh [10] who first initiated the idea of fuzzy sets in 1965. The concept of generalized closed sets as a generalization of closed sets in topological spaces was first initiated by N.Levine [8]. In 1997, G.Balasubramanian and P.Sundaram [2] have shown that concept of fuzzy generalized closed set in fuzzy setting. The idea of generalized topology was initiated by Á.Császár [5] in 2002. It was J.Chakraborty et al. [3] introduced the idea of generalized fuzzy dense and generalized fuzzy nowhere dense sets in 2016. The meaning of bitopological spaces was first advanced by J.C.Kelly [7] in 1963. The notion of fuzzy bitopological spaces was introduced by A.Kandil [6] in 1989. The purpose of this paper is to generalize the notion of bitopological spaces in fuzzy setting and to study various types of generalized pairwise fuzzy sets and their properties.

2. Preliminaries

In this section, the basic concepts and results such as fuzzy set operations are given.

Definition 2.1 *[4] Let λ and μ be fuzzy sets in X. Then for all $x \in X$,*

1. $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x)$,

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2. $\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$,

3. $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = \max\{\lambda(x), \mu(x)\}$,

4. $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = \min\{\lambda(x), \mu(x)\}$,

5. $\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i \mid i \in I\}$ of fuzzy sets in X, the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined by $\psi(x) = sup_i\{\lambda_i(x) \mid x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x) \mid x \in X\}$.

The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X defined as $1_X(x) = 1$, for all $x \in X$.

Lemma 2.1 *[1] For a fuzzy set λ of a fuzzy space X,*

(a) $1 - cl(\lambda) = int(1 - \lambda)$ *and*

(b) $1 - int(\lambda) = cl(1 - \lambda)$.

Lemma 2.2 *[1] For a family $\mathcal{A} = \{\lambda_a\}$ of fuzzy sets of a fuzzy space X. Then, $\vee\ cl(\lambda_\alpha) \leq cl(\vee \lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\vee\ cl(\lambda_\alpha) = cl(\vee \lambda_\alpha)$. Also $\vee\ int\ (\lambda_\alpha) \leq int(\vee \lambda_\alpha)$.*

Definition 2.2 *[5] Let X be a non-empty set and g_X be a collection of fuzzy subsets of X. Then g_X is called a generalized fuzzy topology on X iff $0_X \in g_X$ and $G_i \in g_X$ for $i \in I \neq \emptyset$ implies $G = \underset{i \in I}{\vee} G_i \in g_X$. The pair (X, g_X) is called a generalized fuzzy topological space. The elements of g_X are called the fuzzy g_X-open sets and the complements are called the fuzzy g_X-closed sets.*

Definition 2.3 *[5] The g_X-closure of a fuzzy subset λ of X is denoted by $c_{g_X}(\lambda)$, defined to be the intersection of all the fuzzy g_X-closed sets including λ and the g_X-interior of λ, denoted by $i_{g_X}(\lambda)$, defined as the union of all the fuzzy g_X-open sets contained in λ.*

Definition 2.4 *[9] A fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy G_δ-set (gpfG_δ-set, for short) if $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are gpfo sets in (X, T_g^1, T_g^2).*

Definition 2.5 *[9] A fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy F_σ-set (gpfF_σ-set, for short) if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are gpfc sets in (X, T_g^1, T_g^2).*

Theorem 2.1 *[9] A fuzzy set λ is a gpfG_δ-set in a gfbt space (X, T_g^1, T_g^2) if and only if $1-\lambda$ is a gpfF_σ-set in (X, T_g^1, T_g^2).*

3. Generalized Fuzzy Bitopological Spaces

Definition 3.1 *Let X be a non-empty set and T_g^i, $(i = 1, 2)$ be a collection of fuzzy subsets of X. Then T_g^i is called a generalized fuzzy topology on X if*

i. $0 \in T_g^i$ *and*

ii. $G_k \in T_g^i$ *for* $k \in I$ *implies* $G = \underset{k \in I}{\vee} G_k \in T_g^i$.

The ordered triple (X, T_g^1, T_g^2) is called a generalized fuzzy bitopological space (henceforth abbreviated as gfbt space), where T_g^1 and T_g^2 are generalized fuzzy topologies on X. The elements of T_g^i are called the fuzzy T_g^i-open sets. The complements of the fuzzy T_g^i-open sets are called the fuzzy T_g^i-closed sets in (X, T_g^1, T_g^2).

Definition 3.2 *[9] A fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy closed set (gpfc set, for short) if $cl_{T_g^i} cl_{T_g^j}(\lambda) \leq \mu$, $(i \neq j \text{ and } i, j = 1, 2)$ whenever $\lambda \leq \mu$ and μ is a generalized pairwise fuzzy open set (gpfo set, for short) in (X, T_g^1, T_g^2).*

The complement of gpfc set in (X, T_g^1, T_g^2) is a gpfo set in (X, T_g^1, T_g^2).

Definition 3.3 *Let (X, T_g^1, T_g^2) be a generalized fuzzy bitopological space and λ be any fuzzy set. The generalized closure and generalized interior of a fuzzy set λ in (X, T_g^1, T_g^2) are respectively denoted as $cl_g(\lambda)$ and $int_g(\lambda)$ are defined as*

(1) $cl_g(\lambda) = \wedge\{\mu \mid \lambda \leq \mu,\ \mu$ is a gpfc set$\}$ and

(2) $int_g(\lambda) = \vee\{\mu \mid \mu \leq \lambda,\ \mu$ is a gpfo set$\}$.

4. Characterizations of Generalized Fuzzy Bitopological Spaces

Definition 4.1 *A fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy semi-closed set (gpfs-c set, for short) $int_{T_g^1} cl_{T^2}(\lambda) \leq \lambda$ and $int_{T_g^2} cl_{T^1}(\lambda) \leq \lambda$ in (X, T_g^1, T_g^2). The complement of a gpfs-c set is a generalized pairwise fuzzy semi-open set (gpfs-o set, for short).*

Proposition 4.1 *If λ and μ are gpfc sets in a gfbt space (X, T_g^1, T_g^2), then $\lambda \vee \mu$ is a gpfc set in (X, T_g^1, T_g^2).*

Proof: Let λ be gpfc set such that $cl_{T_g^i} cl_{T_g^j}(\lambda) \leq \gamma$, $(i \neq j$ and $i, j = 1, 2)$ whenever $\lambda \leq \gamma$ and γ is a gpfo set in (X, T_g^1, T_g^2). Also, let μ be a gpfc set such that $cl_{T_g^i} cl_{T_g^j}(\mu) \leq \gamma$ whenever $\mu \leq \gamma$. Suppose $\lambda \vee \mu \leq \gamma$. Now, $cl_{T_g^i} cl_{T_g^j}(\lambda \vee \mu) = cl_{T_g^i} cl_{T_g^j}(\lambda) \vee cl_{T_g^i} cl_{T_g^j}(\mu) \leq \gamma \vee \gamma = \gamma$ and hence $\lambda \vee \mu$ is a gpfc set in (X, T_g^1, T_g^2). □

Proposition 4.2 *If the fuzzy sets λ and μ are gpfc sets in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $\lambda \wedge \mu$ is a gpfc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfc set such that $cl_{T_g^i} cl_{T_g^j}(\lambda) \leq \gamma$, $(i \neq j$ and $i, j = 1, 2)$ whenever $\lambda \leq \gamma$ and the fuzzy set γ is a gpfo set in (X, T_g^1, T_g^2). Also, let the fuzzy set μ be a gpfc set such that $cl_{T_g^i} cl_{T_g^j}(\mu) \leq \gamma$ whenever $\mu \leq \gamma$. Suppose $\lambda \wedge \mu \leq \gamma$. Now, $cl_{T_g^i} cl_{T_g^j}(\lambda \wedge \mu) \leq cl_{T_g^i} cl_{T_g^j}(\lambda) \wedge cl_{T_g^i} cl_{T_g^j}(\mu) \leq \gamma \wedge \gamma = \gamma$ and hence the fuzzy set $\lambda \wedge \mu$ is a gpfc set in (X, T_g^1, T_g^2). □

Definition 4.2 *The fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy dense set (gpfd set, for short) if $cl_{T_g^1} cl_{T_g^2}(\lambda) = 1 = cl_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2).*

Proposition 4.3 *For the fuzzy set λ in a gfbt space (X, T_g^1, T_g^2),*

(i). if $cl_{T_g^1}(\lambda) = 1 = cl_{T_g^2}(\lambda)$ in (X, T_g^1, T_g^2), then $cl_{T_g^1} cl_{T_g^2}(\lambda) = 1 = cl_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2).

(ii). if $int_{T_g^1}(\lambda) = 0 = int_{T_g^2}(\lambda)$ in (X, T_g^1, T_g^2), then $int_{T_g^1} int_{T_g^2}(\lambda) = 0 = int_{T_g^2} int_{T_g^1}(\lambda)$ in (X, T_g^1, T_g^2).

Proof: Let the fuzzy set λ in (X, T_g^1, T_g^2).

(i). Let $cl_{T_g^1}(\lambda) = 1 = cl_{T_g^2}(\lambda)$ in (X, T_g^1, T_g^2). Now $cl_{T_g^1} cl_{T_g^2}(\lambda) = cl_{T_g^1}(1) = 1$. Similarly, $cl_{T_g^2} cl_{T_g^1}(\lambda) = 1$. This implies that $cl_{T_g^1} cl_{T_g^2}(\lambda) = 1 = cl_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2).

(ii). Let $int_{T_g^1}(\lambda) = 0 = int_{T_g^2}(\lambda)$ in (X, T_g^1, T_g^2). Now $int_{T_g^1} int_{T_g^2}(\lambda) = int_{T_g^1}(0) = 0$. Similarly, $int_{T_g^2} int_{T_g^1}(\lambda) = 0$. This implies that $int_{T_g^1} int_{T_g^2}(\lambda) = 0 = int_{T_g^2} int_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2).

□

Remark 4.1 *For the fuzzy set λ in a gfbt space (X, T_g^1, T_g^2),*

(i). if $cl_{T_g^1} cl_{T_g^2}(\lambda) = 1 = cl_{T_g^2} cl_{T_g^1}(\lambda)$ in (X, T_g^1, T_g^2), then it does not imply that $cl_{T_g^1}(\lambda) = 1 = cl_{T_g^2}(\lambda)$ in (X, T_g^1, T_g^2).

(ii). if $int_{T_g^1} int_{T_g^2}(\lambda) = 0 = int_{T_g^2} int_{T_g^1}(\lambda)$ *in* (X, T_g^1, T_g^2)*, then it does not imply that* $int_{T_g^1}(\lambda) = 0 = int_{T_g^2}(\lambda)$ *in* (X, T_g^1, T_g^2)*.*

Proposition 4.4 *If the fuzzy set* λ_k *is a fuzzy* T_g^i $(i = 1, 2)$*-dense set in a gfbt space* (X, T_g^1, T_g^2)*, then the fuzzy set* λ_k *is a gpfd set in* (X, T_g^1, T_g^2)*.*

Proof: Let the fuzzy set λ_k be a fuzzy T_g^i $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then $cl_{T_g^1}(\lambda_k) = 1 = cl_{T_g^2}(\lambda_k)$, in (X, T_g^1, T_g^2). Then by the Proposition 4.3 (i), $cl_{T_1} cl_{T_2}(\lambda_k) = 1 = cl_{T_2} cl_{T_1}(\lambda_k)$, in (X, T_g^1, T_g^2). Hence the fuzzy set λ_k is a gpfd set in (X, T_g^1, T_g^2). □

Definition 4.3 *The fuzzy set* λ *in a gfbt space* (X, T_g^1, T_g^2) *is called a generalized pairwise fuzzy nowhere dense set (gpfnd set, for short) if* $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$*, in* (X, T_g^1, T_g^2)*.*

Example 4.1 *Let* $X = \{a, b, c\}$*. The fuzzy sets* α*,* β *and* γ *are defined on* X *as follows:*
$\alpha : X \to [0, 1]$ *is defined as* $\alpha(a) = 0.5$; $\alpha(b) = 0.4$; $\alpha(c) = 0.6$,
$\beta : X \to [0, 1]$ *is defined as* $\beta(a) = 0.6$; $\beta(b) = 0.2$; $\beta(c) = 0.8$,
$\gamma : X \to [0, 1]$ *is defined as* $\gamma(a) = 0.4$; $\gamma(b) = 0.6$; $\gamma(c) = 0.2$.
Then $T_g^1 = \{0, \alpha, 1\}$ *and* $T_g^2 = \{0, \beta, 1\}$ *are generalized fuzzy topologies on* X*. By computations, one can see that* $cl_{T_g^2}(\gamma) = 1 - \beta$*;* $cl_{T_g^1}(\gamma) = 1 - \alpha$*;* $int_{T_g^1}(1 - \beta) = 0$*;* $int_{T_g^2}(1 - \alpha) = 0$*, in* (X, T_g^1, T_g^2)*. Then,* $int_{T_g^1} cl_{T_g^2}(\gamma) = int_{T_g^2} cl_{T_g^1}(\gamma) = 0$*, in* (X, T_g^1, T_g^2) *and hence the fuzzy set* γ *is a gpfnd set in* (X, T_g^1, T_g^2)*.*

Remark 4.2 *The complement of a gpfnd set in a gfbt space* **need not** *be a gpfnd set. For, consider the example 4.1. By computations,* $int_{T_g^1} cl_{T_g^2}(1 - \gamma) = 1 \neq 0$ *and* $int_{T_g^2} cl_{T_g^1}(1 - \gamma) = 1 \neq 0$*, in* (X, T_g^1, T_g^2)*. Hence the fuzzy set* $1 - \gamma$ *is not a gpfnd set in* (X, T_g^1, T_g^2) *whereas the fuzzy* γ *is a gpfnd set in* (X, T_g^1, T_g^2)*.*

Proposition 4.5 *If the fuzzy set* μ *is a gpfnd set in a gfbt space* (X, T_g^1, T_g^2) *and if* $\lambda \leq \mu$*, for the fuzzy set* λ *in* (X, T_g^1, T_g^2)*, then the fuzzy set* λ *is a gpfnd set in* (X, T_g^1, T_g^2)*.*

Proof: Let the fuzzy set μ be a gpfnd set in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T_g^2}(\mu) = 0 = int_{T_g^1} cl_{T_g^2}(\mu)$, in (X, T_g^1, T_g^2). Since $\lambda \leq \mu$, $int_{T_g^1} cl_{T_g^2}(\lambda) \leq int_{T_g^1} cl_{T_g^2}(\mu)$ and $int_{T_g^2} cl_{T_g^1}(\lambda) \leq int_{T_g^2} cl_{T_g^1}(\mu)$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T_g^2}(\lambda) \leq 0$ and $int_{T_g^2} cl_{T_g^1}(\lambda) \leq 0$. Then $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set λ is a gpfnd set in (X, T_g^1, T_g^2). □

Remark 4.3 *If the fuzzy sets* λ *and* μ *are gpfnd sets in a gfbt space* (X, T_g^1, T_g^2)*, then the fuzzy set* $\lambda \vee \mu$ **need not** *be a gpfnd set in* (X, T_g^1, T_g^2)*. For, consider the following example:*

Example 4.2 *Let* $X = \{a, b, c\}$*. The fuzzy sets* λ*,* μ*,* γ*,* α*,* β *and* ν *are defined as follows:*
$\lambda : X \to [0, 1]$ *is defined as* $\lambda(a) = 0.5$; $\lambda(b) = 0.7$; $\lambda(c) = 0.6$,
$\mu : X \to [0, 1]$ *is defined as* $\mu(a) = 0.4$; $\mu(b) = 0.6$; $\mu(c) = 0.5$,
$\gamma : X \to [0, 1]$ *is defined as* $\gamma(a) = 0.6$; $\gamma(b) = 0.5$; $\gamma(c) = 0.4$,
$\alpha : X \to [0, 1]$ *is defined as* $\alpha(a) = 0.8$; $\alpha(b) = 0.5$; $\alpha(c) = 0.7$,
$\beta : X \to [0, 1]$ *is defined as* $\beta(a) = 0.6$; $\beta(b) = 0.9$; $\beta(c) = 0.4$,
$\nu : X \to [0, 1]$ *is defined as* $\nu(a) = 0.4$; $\nu(b) = 0.7$; $\nu(c) = 0.8$.
Then $T_g^1 = \{0, \lambda, \mu, \gamma, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \gamma, \mu \wedge \gamma, \mu \vee (\lambda \wedge \gamma), \lambda \vee \mu \vee \gamma, 1\}$ *and* $T_g^2 = \{0, \alpha, \beta, \nu, \alpha \vee \beta, \alpha \vee \nu, \beta \vee \nu, \alpha \wedge \beta, \alpha \wedge \nu, \beta \wedge \nu, \alpha \vee (\beta \wedge \nu), \alpha \wedge (\beta \vee \nu), \beta \vee (\alpha \wedge \nu), \beta \wedge (\alpha \vee \nu), \nu \vee (\alpha \wedge \beta), \nu \wedge (\alpha \vee \beta), \alpha \vee \beta \vee \nu, 1\}$ *are generalized fuzzy topologies on* X*.*
Now consider the following fuzzy sets:
$\alpha : X \to [0, 1]$ *and* $\beta : X \to [0, 1]$ *defined on* X*.*
$\eta : X \to [0, 1]$ *is defined as* $\eta(a) = 0.6$; $\eta(b) = 0.3$; $\eta(c) = 0.4$,
$\zeta : X \to [0, 1]$ *is defined as* $\zeta(a) = 0.4$; $\zeta(b) = 0.3$; $\zeta(c) = 0.6$.
By computations, one can see that $int_{T_g^1} cl_{T_g^2}(\eta) = int_{T_g^1}[1 - (\beta \wedge \nu)] = 0$ *and* $int_{T_g^2} cl_{T_g^1}(\eta) = int_{T_g^2}(1 - \mu) = 0$ *in* (X, T_g^1, T_g^2)*. Also,* $int_{T_g^1} cl_{T_g^2}(\zeta) = int_{T_g^1}[1 - (\beta \wedge (\alpha \vee \nu))] = 0$ *and* $int_{T_g^2} cl_{T_g^1}(\zeta) = int_{T_g^2}(1 - \gamma) =$

0, in (X, T_g^1, T_g^2). Hence the fuzzy sets η and ζ are gpfnd sets in (X, T_g^1, T_g^2). But $int_{T_g^2} cl_{T_g^1}(\eta \vee \zeta) = int_{T_g^2}[1 - (\mu \wedge \gamma)] = \alpha \wedge \beta \neq 0$ while $int_{T_g^1} cl_{T_g^2}(\eta \vee \zeta) = 0$, implies that the fuzzy set $\eta \vee \zeta$ is **not** *a gpfnd set in (X, T_g^1, T_g^2).*

Proposition 4.6 *If the fuzzy sets λ and μ are gpfnd sets in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $\lambda \wedge \mu$ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy sets λ and μ be gpfnd sets in (X, T_g^1, T_g^2). Then, $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$ and $int_{T_g^1} cl_{T_g^2}(\mu) = 0 = int_{T_g^2} cl_{T_g^1}(\mu)$ in (X, T_g^1, T_g^2). Now, $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T_g^2}(\lambda \wedge \mu) \leq int_{T_g^1} cl_{T_g^2}(\lambda)$ and $int_{T_g^2} cl_{T_g^1}(\lambda \wedge \mu) \leq int_{T_g^2} cl_{T_g^1}(\mu)$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T_g^2}(\lambda \wedge \mu) \leq 0$ and $int_{T_g^2} cl_{T_g^1}(\lambda \wedge \mu) \leq 0$, in (X, T_g^1, T_g^2). That is, $int_{T_g^1} cl_{T_g^2}(\lambda \wedge \mu) = 0$ and $int_{T_g^2} cl_{T_g^1}(\lambda \wedge \mu) = 0$, in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T_g^2}(\lambda \wedge \mu) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda \wedge \mu)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set $\lambda \wedge \mu$ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.7 *If the fuzzy set λ is a gpfnd set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $1 - \lambda$ is a gpfd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfnd set in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2). Now $1 - int_{T_g^1} cl_{T_g^2}(\lambda) = 1 - 0 = 1$ and hence $cl_{T_g^1} int_{T_g^2}(1 - \lambda) = 1$, in (X, T_g^1, T_g^2). But $cl_{T_g^1} int_{T_g^2}(1 - \lambda) \leq cl_{T_g^1} cl_{T_g^2}(1 - \lambda)$, in (X, T_g^1, T_g^2). This implies that $1 \leq cl_{T_g^1} cl_{T_g^2}(1 - \lambda)$. Then $cl_{T_g^1} cl_{T_g^2}(1 - \lambda) = 1$, in (X, T_g^1, T_g^2). Similarly, $cl_{T_g^2} cl_{T_g^1}(1 - \lambda) = 1$, in (X, T_g^1, T_g^2). Thus, the fuzzy set $1 - \lambda$ is a gpfd set in (X, T_g^1, T_g^2). □

Remark 4.4 *If the fuzzy set λ is a gpfd set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $1 - \lambda$* **need not** *be a gpfnd set in (X, T_g^1, T_g^2). For, consider the following example:*

Example 4.3 *Let $X = \{a, b, c\}$. The fuzzy sets α, β, γ and λ, μ, ν are defined as follows:*

$\alpha : X \to [0, 1]$ *is defined as* $\alpha(a) = 0.2$; $\alpha(b) = 0.3$; $\alpha(c) = 0.4$,
$\beta : X \to [0, 1]$ *is defined as* $\beta(a) = 0.5$; $\beta(b) = 0.4$; $\beta(c) = 0.7$,
$\gamma : X \to [0, 1]$ *is defined as* $\gamma(a) = 0.4$; $\gamma(b) = 0.5$; $\gamma(c) = 0.6$,
$\lambda : X \to [0, 1]$ *is defined as* $\lambda(a) = 0.4$; $\lambda(b) = 0$; $\lambda(c) = 0$,
$\mu : X \to [0, 1]$ *is defined as* $\mu(a) = 0.7$; $\mu(b) = 0.2$; $\mu(c) = 0$,
$\nu : X \to [0, 1]$ *is defined as* $\nu(a) = 0.5$; $\nu(b) = 0$; $\nu(c) = 0.3$.

Then $T_g^1 = \{0, \alpha, \beta, 1\}$ and $T_g^2 = \{0, \mu_1, \mu_2, 1\}$ are generalized fuzzy topologies on X. By computations, one can see that $cl_{T_g^1} cl_{T_g^2}(1 - \alpha) = cl_{T_g^1}(1) = 1$, in (X, T_g^1, T_g^2). Similarly, $cl_{T_g^2} cl_{T_g^1}(1 - \alpha) = 1$, in (X, T_g^1, T_g^2). Hence $cl_{T_g^1} cl_{T_g^2}(1 - \alpha) = 1 = cl_{T_g^2} cl_{T_g^1}(1 - \alpha)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set $1 - \alpha$ is a gpfd set in (X, T_g^1, T_g^2). Also, $int_{T_g^1} cl_{T_g^2}(\alpha) = int_{T_g^1}(1 - \mu) = \alpha \neq 0$ and $int_{T_g^2} cl_{T_g^1}(\alpha) = int_{T_g^2}(1 - \alpha) = \mu \neq 0$, in (X, T_g^1, T_g^2). Thus, the fuzzy set α is not a gpfnd set in (X, T_g^1, T_g^2).

Proposition 4.8 *If the fuzzy set λ is a gpfnd set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set λ is a gpfsc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfnd set in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T^2}(\lambda) = 0 = int_{T_g^2} cl_{T^1}(\lambda)$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T^2}(\lambda) \leq \lambda$ and $int_{T_g^2} cl_{T^1}(\lambda) \leq \lambda$ in (X, T_g^1, T_g^2). Hence the fuzzy set λ is a gpfsc set in (X, T_g^1, T_g^2). □

Proposition 4.9 *If the fuzzy set λ is a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then $int_{T_g^1}(\lambda) = \lambda$ and $int_{T_g^2}(\lambda) = \lambda$ with $cl_{T_g^1}(\lambda) = 1$ and $cl_{T_g^2}(\lambda) = 1$, in (X, T_g^1, T_g^2). Now $int_{T_g^1} cl_{T_g^2}(1 - \lambda) = 1 - cl_{T_g^1} int_{T_g^2}(\lambda) = 1 - cl_{T_g^1}(\lambda) = 1 - 1 = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^2} cl_{T_g^1}(1 - \lambda) = 0$, in (X, T_g^1, T_g^2). Thus, $int_{T_g^1} cl_{T_g^2}(1 - \lambda) = 0 = int_{T_g^2} cl_{T_g^1}(1 - \lambda)$, in (X, T_g^1, T_g^2). This implies that the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.10 *If the fuzzy set λ is a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $1 - \lambda$ is a gpfsc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then by the Proposition 4.9, the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). Therefore, by the Proposition 4.8, the fuzzy set $1 - \lambda$ is a gpfsc set in (X, T_g^1, T_g^2). □

Proposition 4.11 *If the fuzzy set λ is a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set λ is a gpfso set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then by the Proposition 4.10, the fuzzy set $1 - \lambda$ is a gpfsc set in (X, T_g^1, T_g^2). Therefore, the fuzzy set λ is a gpfso set in (X, T_g^1, T_g^2). □

Proposition 4.12 *If the fuzzy set μ is a gpfnd set in a gfbt space (X, T_g^1, T_g^2) and if $\lambda \leq \mu$, for the fuzzy set λ in (X, T_g^1, T_g^2), then the fuzzy set λ is a gpfsc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set μ be a gpfnd set in (X, T_g^1, T_g^2). Then, $int_{T_g^1} cl_{T^2}(\mu) = 0 = int_{T_g^2} cl_{T^1}(\mu)$, in (X, T_g^1, T_g^2). Since $\lambda \leq \mu$, $int_{T_g^1} cl_{T^2}(\lambda) \leq int_{T_g^1} cl_{T^2}(\mu)$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T^2}(\lambda) \leq 0$, in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T^2}(\lambda) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^2} cl_{T^1}(\lambda) = 0$, in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T^2}(\lambda) = 0 = int_{T_g^2} cl_{T^1}(\lambda)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set λ is a gpfnd set in (X, T_g^1, T_g^2). Then by the Proposition 4.8, the fuzzy set λ is a gpfsc set in (X, T_g^1, T_g^2). □

Proposition 4.13 *If the fuzzy set λ is a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in a gfbt space (X, T_g^1, T_g^2) with $\mu \leq 1 - \lambda$, then the fuzzy set μ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then by the Proposition 4.9, the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T^2}(1 - \lambda) = 0 = int_{T_g^2} cl_{T^1}(1 - \lambda)$, in (X, T_g^1, T_g^2). Since $\mu \leq 1 - \lambda$, $int_{T_g^1} cl_{T^2}(\mu) \leq int_{T_g^1} cl_{T^2}(1 - \lambda)$, in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T^2}(\mu) \leq 0$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1} cl_{T^2}(\mu) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^2} cl_{T^1}(\mu) = 0$, in (X, T_g^1, T_g^2). Thus, $int_{T_g^1} cl_{T^2}(\mu) = 0 = int_{T_g^2} cl_{T^1}(\mu)$, in (X, T_g^1, T_g^2) implies that the fuzzy set μ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.14 *If the fuzzy set λ is a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in a gfbt space (X, T_g^1, T_g^2) with $\mu \leq 1 - \lambda$, then the fuzzy set μ is a gpfsc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo and fuzzy T_g^i, $(i = 1, 2)$-dense set in (X, T_g^1, T_g^2). Then by the Proposition 4.13, the fuzzy set λ is a gpfnd set in (X, T_g^1, T_g^2). Therefore, by the Proposition 4.8, the fuzzy set μ is a gpfsc set in (X, T_g^1, T_g^2). □

Proposition 4.15 *If the fuzzy set λ is a gpfo set in a gfbt space (X, T_g^1, T_g^2) such that $cl_{T_g^i}(\lambda) = 1$, $(i = 1, 2)$, then the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfo set in (X, T_g^1, T_g^2) such that $cl_{T_g^i}(\lambda) = 1,\ (i = 1, 2)$. Then $int_{T_g^1}(\lambda) = \lambda$ and $int_{T_g^2}(\lambda) = \lambda$ such that $cl_{T_g^1}(\lambda) = 1$ and $cl_{T_g^2}(\lambda) = 1$, in (X, T_g^1, T_g^2). Now $int_{T_g^1} cl_{T_g^2}(1 - \lambda) = 1 - cl_{T_g^1} int_{T_g^2}(\lambda) = 1 - cl_{T_g^1}(\lambda) = 1 - 1 = 0$. Similarly, $int_{T_g^2} cl_{T_g^1}(1 - \lambda) = 0$. Hence $int_{T_g^1} cl_{T_g^2}(1 - \lambda) = 0 = int_{T_g^2} cl_{T_g^1}(1 - \lambda)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.16 *If the fuzzy set λ is a fuzzy T_g^i-closed set with $int_{T_g^i}(\lambda) = 0,\ (i = 1, 2)$ in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set λ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a fuzzy T_g^i-closed set with $int_{T_g^i}(\lambda) = 0,\ (i = 1, 2)$. Then $cl_{T_g^1}(\lambda) = \lambda$ and $cl_{T_g^2}(\lambda) = \lambda$ with $int_{T_1^1}(\lambda) = 0$ and $int_{T_g^2}(\lambda) = 0$, in (X, T_g^1, T_g^2). Now $int_{T_1} cl_{T_2}(\lambda) = int_{T_1}(\lambda) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_2} cl_{T_1}(\lambda) = 0$, in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set λ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.17 *If the fuzzy set λ is a gpfnd set in a gfbt space (X, T_g^1, T_g^2), then $int_{T_g^1}(\lambda) = 0 = int_{T_g^2}(\lambda)$, in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfnd set in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$, in (X, T_g^1, T_g^2). Now $\lambda \le cl_{T_g^2}(\lambda)$, in (X, T_g^1, T_g^2). Then $int_{T_g^1}(\lambda) \le int_{T_g^1} cl_{T_g^2}(\lambda)$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1}(\lambda) \le 0$. That is, $int_{T_g^1}(\lambda) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^1}(\lambda) = 0$, in (X, T_g^1, T_g^2). Thus, $int_{T_g^1}(\lambda) = 0 = int_{T_g^2}(\lambda)$, in (X, T_g^1, T_g^2). □

Proposition 4.18 *If the fuzzy set λ is a fuzzy T_g^i-open and fuzzy T_g^j, $(i, j = 1, 2$ and $i \ne j)$-dense set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a fuzzy T_g^i-open and fuzzy T_g^j, $(i, j = 1, 2$ and $i \ne j)$-dense set in (X, T_g^1, T_g^2). Then the fizzy set $1 - \lambda$ is a fuzzy T_g^i-closed and $cl_{T_g^j}(\lambda) = 1$, in (X, T_g^1, T_g^2). This implies that $cl_{T_g^i}(1 - \lambda) = 1 - \lambda$ and $1 - cl_{T_g^j}(\lambda) = 0$, in (X, T_g^1, T_g^2). Hence $cl_{T_g^i}(1 - \lambda) = 1 - \lambda$ and $int_{T_g^j}(1 - \lambda) = 0$, in (X, T_g^1, T_g^2). Now, $int_{T_g^j} cl_{T_g^i}(1 - \lambda) = int_{T_g^j}(1 - \lambda) = 0$, $(i, j = 1, 2$ and $i \ne j)$ in (X, T_g^1, T_g^2). That is, $int_{T_g^j} cl_{T_g^i}(1 - \lambda) = 0$ $(i, j = 1, 2$ and $i \ne j)$ in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T_g^2}(1 - \lambda) = 0$ and $int_{T_g^2} cl_{T_g^1}(1 - \lambda) = 0$, in (X, T_g^1, T_g^2). Thus, the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). □

Proposition 4.19 *If the fuzzy set λ is a fuzzy T_g^i-open and fuzzy T_g^j, $(i, j = 1, 2$ and $i \ne j)$-dense set in a gfbt space (X, T_g^1, T_g^2) and if $\mu \le 1 - \lambda$, then the fuzzy set μ is a gpfnd set in (X, T_g^1, T_g^2).*

Proof: Since the fuzzy set λ be a fuzzy T_g^i-open and fuzzy T_g^j, $(i, j = 1, 2$ and $i \ne j)$-dense set in a gfbt space (X, T_g^1, T_g^2) and by the Proposition 4.18, the fuzzy set $1 - \lambda$ is a gpfnd set in (X, T_g^1, T_g^2). Then, $int_{T_g^j} cl_{T_g^i}(1 - \lambda) = 0$, in (X, T_g^1, T_g^2). Since $\mu \le 1 - \lambda$, $int_{T_g^j} cl_{T_g^i}(\mu) \le int_{T_g^j} cl_{T_g^i}(1 - \lambda)$ in (X, T_g^1, T_g^2). Then $int_{T_g^j} cl_{T_g^i}(\mu) \le 0$ in (X, T_g^1, T_g^2). This implies that $int_{T_g^j} cl_{T_g^i}(\mu) = 0$, in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T_g^2}(\mu) = 0 = int_{T_g^2} cl_{T_g^1}(\mu)$, in (X, T_g^1, T_g^2). Thus, the fuzzy set μ is a gpfnd set in (X, T_g^1, T_g^2). □

Definition 4.4 *A fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy first category set (gpffc set, for short) if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Any other fuzzy set in (X, T_g^1, T_g^2) is said to be a generalized pairwise fuzzy second category set (gpfsc set, for short) in (X, T_g^1, T_g^2).*

Example 4.4 *Let $X = \{a, b, c\}$. The fuzzy sets α, β and γ are defined on X as follows:*
$\alpha : X \to [0,1]$ *is defined as* $\alpha(a) = 0.5$; $\alpha(b) = 0.7$; $\alpha(c) = 0.6$,
$\beta : X \to [0,1]$ *is defined as* $\beta(a) = 0.4$; $\beta(b) = 0.6$; $\beta(c) = 0.5$,
$\gamma : X \to [0,1]$ *is defined as* $\gamma(a) = 0.6$; $\gamma(b) = 0.5$; $\gamma(c) = 0.4$.

Then $T_g^1 = \{0, \alpha, \beta, \gamma, \alpha\vee\gamma, \beta\vee\gamma, \alpha\wedge\gamma, \beta\wedge\gamma, \alpha\wedge(\beta\vee\gamma), \alpha\vee\beta\vee\gamma, 1\}$ *and* $T_g^2 = \{0, \alpha, \beta, 1\}$ *are generalized fuzzy topologies on X. By computations, one can see that* $cl_{T_g^2}(1-\alpha) = 1-\alpha$, $cl_{T_g^2}(1-\beta) = 1-\beta$, $cl_{T_g^2}[1-(\alpha\vee\gamma)] = 1-\alpha$, $cl_{T_g^2}[1-(\beta\vee\gamma)] = 1-\beta$, $cl_{T_g^2}[1-(\alpha\wedge(\beta\vee\gamma))] = 1-\gamma$, $int_{T_g^2}(1-\alpha) = 0$, $int_{T_g^2}(1-\beta) = 0$, $int_{T_g^2}[1-(\alpha\vee\gamma)] = 0$, $int_{T_g^2}[1-(\beta\vee\gamma)] = 0$, $int_{T_g^2}[1-(\gamma\wedge(\beta\vee\gamma))] = 0$ *in* (X, T_g^1, T_g^2).

Also, $int_{T_g^1}(1-\gamma) = 0$, $int_{T_g^1}(1-\beta) = 0$, $int_{T_g^1}[1-(\alpha\vee\gamma)] = 0$, $int_{T_g^1}[1-(\beta\vee\gamma)] = 0$, $int_{T_g^1}[1-(\alpha\wedge(\beta\vee\gamma))] = 0$ *and* $cl_{T_g^1}(\nu) = 1-(\beta\wedge\gamma)$, $cl_{T_g^1}(\beta\wedge\gamma) = 1-\gamma$, $cl_{T_g^1}(\alpha\wedge\gamma) = 1-(\alpha\wedge\gamma)$, $cl_{T_g^1}[1-(\alpha\vee\gamma)] = 1-(\alpha\vee\gamma)$, $cl_{T_g^1}[1-(\beta\vee\gamma)] = 1-(\beta\vee\gamma)$, $cl_{T_g^1}(1-\alpha) = 1-\alpha$, $cl_{T_g^1}(1-\beta) = 1-\beta$, $cl_{T_g^1}(1-\gamma) = 1-\gamma$, $cl_{T_g^1}[1-(\alpha\wedge(\beta\vee\gamma))] = 1-(\alpha\wedge(\beta\vee\gamma))$ *in* (X, T_g^1, T_g^2).

Now the fuzzy sets $1-\alpha$, $1-\beta$, $1-(\alpha\vee\gamma)$, $1-(\beta\vee\gamma)$ *and* $1-[\alpha\wedge(\beta\vee\gamma)]$ *are gpfnd sets in* (X, T_g^1, T_g^2). *Therefore,* $\delta = \{(1-\alpha)\vee(1-\beta)\vee[1-(\alpha\vee\gamma)]\vee[1-(\beta\vee\gamma)]\vee[1-(\alpha\wedge(\beta\vee\gamma))]\}$ *is a gpffc set in* (X, T_g^1, T_g^2).

Proposition 4.20 *If the fuzzy set μ is a gpffc set in a gfbt space (X, T_g^1, T_g^2) and if $\lambda \le \mu$, for a fuzzy set λ in (X, T_g^1, T_g^2), then the fuzzy set λ is a gpffc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set μ be a gpffc set in (X, T_g^1, T_g^2). Then $\mu = \vee_{k=1}^{\infty}(\mu_k)$, where the fuzzy sets (μ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Now $\lambda\wedge\mu = \lambda\wedge(\vee_{k=1}^{\infty}(\mu_k) = \vee_{k=1}^{\infty}(\lambda\wedge\mu_k)$. Since $\lambda \le \mu$, $\lambda\wedge\mu = \lambda$ in (X, T_g^1, T_g^2). Therefore, $\lambda = \vee_{k=1}^{\infty}(\lambda\wedge\mu_k)$. Since $(\lambda\wedge\mu_k) \le \mu_k$ and the fuzzy sets (μ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) and by the Proposition 4.5, the fuzzy sets $(\lambda\wedge\mu_k)$'s are gpfnd sets in (X, T_g^1, T_g^2). Hence $\lambda = \vee_{k=1}^{\infty}(\lambda\wedge\mu_k)$, where the fuzzy sets $(\lambda\wedge\mu_k)$'s are gpfnd sets in (X, T_g^1, T_g^2) implies that the fuzzy set λ is a gpffc set in (X, T_g^1, T_g^2). □

Proposition 4.21 *If the gpfG_δ-set λ in a gfbt space (X, T_g^1, T_g^2) such that $cl_{T_g^i}(\lambda) = 1$, $(i = 1, 2)$, then the fuzzy set $1-\lambda$ is a gpffc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfG_δ-set such that $cl_{T_g^i}(\lambda) = 1$, $(i = 1, 2)$, in (X, T_g^1, T_g^2). Then $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfo sets in (X, T_g^1, T_g^2). Now $cl_{T_g^i}(\lambda) = cl_{T_g^i}(\wedge_{k=1}^{\infty}(\lambda_k)) \le \wedge_{k=1}^{\infty}\left(cl_{T_g^i}(\lambda_k)\right)$, $1 \le \wedge_{k=1}^{\infty} cl_{T_g^i}(\lambda_k)$. That is, $\wedge_{k=1}^{\infty} cl_{T_g^i}(\lambda_k) = 1$. This implies that $cl_{T_g^i}(\lambda_k) = 1$ $(i = 1, 2)$, in (X, T_g^1, T_g^2). Also, $1-\lambda = 1-\wedge_{k=1}^{\infty}(\lambda_k) = \vee_{k=1}^{\infty}(1-\lambda_k) \longrightarrow (A)$. Since the fuzzy sets (λ_k)'s are gpfo sets, $(1-\lambda_k)$'s are gpfc sets in (X, T_g^1, T_g^2). This implies that $cl_{T_g^i}(1-\lambda_k) = 1-\lambda_k$, $(i = 1, 2)$, in (X, T_g^1, T_g^2). Since $cl_{T_g^i}(\lambda_k) = 1$, $1 - cl_{T_g^i}(\lambda_k) = 0$. Then $int_{T_g^i}(1-\lambda_k) = 0$, in (X, T_g^1, T_g^2). Now $int_{T_g^2} cl_{T_g^1}(1-\lambda_k) = int_{T_g^2}(1-\lambda_k) = 0$ and $int_{T_g^1} cl_{T_g^2}(1-\lambda_k) = int_{T_g^1}(1-\lambda_k) = 0$. Hence $int_{T_g^2} cl_{T_g^1}(1-\lambda_k) = 0$ and $int_{T_g^1} cl_{T_g^2}(1-\lambda_k) = 0$, in (X, T_g^1, T_g^2). This implies that the fuzzy sets $(1-\lambda_k)$'s are gpfnd sets in (X, T_g^1, T_g^2). Therefore, from (A), the fuzzy set $1-\lambda$ is a gpffc set in (X, T_g^1, T_g^2). □

Proposition 4.22 *If the gpfnd set λ in a gfbt space (X, T_g^1, T_g^2) is a gpfF_σ-set, then the fuzzy set λ is a gpffc set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfnd set in (X, T_g^1, T_g^2) such that $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfc sets in (X, T_g^1, T_g^2). Then $cl_{T_g^i}(\lambda) = cl_{T_g^i}(\vee_{k=1}^{\infty}(\lambda_k)) \ge \vee_{k=1}^{\infty} cl_{T_g^i}(\lambda_k)$, $(i = 1, 2)$, by the Lemma 2.2. Since the fuzzy sets (λ_k)'s are gpfc sets in (X, T_g^1, T_g^2), $cl_{T_g^i}(\lambda_k) = \lambda_k$. Hence $\vee_{k=1}^{\infty}(\lambda_k) \le cl_{T_g^i}(\lambda)$. That is, $\vee_{k=1}^{\infty}(\lambda_k) \le cl_{T_g^1}(\lambda)$ and $\vee_{k=1}^{\infty}(\lambda_k) \le cl_{T_g^2}(\lambda)$. Then $int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k)) \le int_{T_g^2} cl_{T_g^1}(\lambda)$ and $int_{T_g^1}(\vee_{k=1}^{\infty}(\lambda_k)) \le int_{T_g^1} cl_{T_g^2}(\lambda)$, in (X, T_g^1, T_g^2). Since the fuzzy set λ is a gpfnd set, $int_{T_g^1} cl_{T_g^2}(\lambda) = int_{T_g^2} cl_{T_g^1}(\lambda) = 0$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$ and $int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2).

Since $\vee_{k=1}^{\infty}(int_{T_g^2}(\lambda_k)) \leq int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k))$ and $\vee_{k=1}^{\infty}(int_{T_g^1}(\lambda_k)) \leq int_{T_g^1}(\vee_{k=1}^{\infty}(\lambda_k))$, $\vee_{k=1}^{\infty}(int_{T_g^2}(\lambda_k)) = 0$ and $\vee_{k=1}^{\infty}(int_{T_g^1}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^2}(\lambda_k) = 0$ and $int_{T_g^1}(\lambda_k) = 0$, in (X, T_g^1, T_g^2). Hence $int_{T_g^1} cl_{T_g^2}(\lambda_k) = int_{T_g^1}(\lambda_k) = 0$ and $int_{T_g^2} cl_{T_g^1}(\lambda_k) = int_{T_g^2}(\lambda_k) = 0$, in (X, T_g^1, T_g^2) and hence the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Therefore $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) implies that the fuzzy set λ is a gpffc set in (X, T_g^1, T_g^2). □

Proposition 4.23 *If the fuzzy set λ is a gpffc set in a gfbt space (X, T_g^1, T_g^2), then $1 - \lambda = \wedge_{k=1}^{\infty}(\mu_k)$, where the fuzzy sets (μ_k)'s are gpfd sets in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpffc set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Now $1 - \lambda = 1 - \vee_{k=1}^{\infty}(\lambda_k) = \wedge_{k=1}^{\infty}(1 - \lambda_k)$. Since the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) and by the Proposition 4.7, the fuzzy sets $(1 - \lambda_k)$'s are gpfd sets in (X, T_g^1, T_g^2). Put $\mu_k = 1 - \lambda_k$. $1 - \lambda = \wedge_{k=1}^{\infty}(\mu_k)$, the fuzzy sets (μ_k)'s are gpfd sets in (X, T_g^1, T_g^2). □

Proposition 4.24 *If the fuzzy set λ is a gpffc set in a gfbt space (X, T_g^1, T_g^2), then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpffc set in (X, T_g^1, T_g^2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda k)$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Since the fuzzy sets (λ_k)'s are gpfnd sets and by the Proposition 4.8, the fuzzy sets (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2). This implies that $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2). □

Proposition 4.25 *If the fuzzy set μ is a gpffc set in a gfbt space (X, T_g^1, T_g^2) and if $\lambda \leq \mu$ for a fuzzy set λ in (X, T_g^1, T_g^2), then $\lambda = \vee_{k=1}^{\infty}(\nu_k)$, where the fuzzy sets (ν_k)'s are gpfsc sets in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set μ be a gpffc set in (X, T_g^1, T_g^2). Then $\mu = \vee_{k=1}^{\infty}(\mu_k)$, where the fuzzy sets (μ_i)'s are gpfnd sets in (X, T_g^1, T_g^2). Now $\lambda \wedge \mu = \lambda \wedge (\vee_{k=1}^{\infty}(\mu_k)) = \vee_{k=1}^{\infty}(\lambda \wedge \mu_k)$. Since $\lambda \leq \mu$, $\lambda \wedge \mu = \lambda$ in (X, T_g^1, T_g^2). Therefore $\lambda = \vee_{k=1}^{\infty}(\lambda \wedge \mu_k)$. Let $\lambda \wedge \mu_k = \nu_k$. Also, since $\lambda \wedge \mu_k = \nu_k \leq \mu_k$ and (μ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) and by the Proposition 4.12, (ν_k)'s are gpfsc sets in (X, T_g^1, T_g^2). Hence $\lambda = \vee_{k=1}^{\infty}(\nu_k)$, where the fuzzy sets (ν_k)'s are gpfsc sets in (X, T_g^1, T_g^2). □

Definition 4.5 *If the fuzzy set λ is a gpffc set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy $1 - \lambda$ is called a generalized pairwise fuzzy residual set (gpfr set, for short) in (X, T_g^1, T_g^2).*

Example 4.5 *From the above example 4.4, the fuzzy set* $1 - \delta = 1 - [\{(1 - \alpha) \vee (1 - \beta) \vee [1 - (\alpha \vee \gamma)] \vee [1 - (\beta \vee \gamma)] \vee [1 - (\alpha \wedge (\beta \vee \gamma))]\}] = (\alpha) \wedge (\beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma) \wedge [\alpha \wedge (\beta \vee \gamma)]$ *is a gpfr set in* (X, T_g^1, T_g^2).

Proposition 4.26 *If the fuzzy set λ is a gpfr set in a gfbt space (X, T_g^1, T_g^2) and if $\lambda \leq \mu$ for a fuzzy set μ in (X, T_g^1, T_g^2), then the fuzzy set μ is a gpfr set in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfr set in (X, T_g^1, T_g^2). Then the fuzzy set $1 - \lambda$ is a gpffc set in (X, T_g^1, T_g^2). Since $\lambda \leq \mu$ for a fuzzy set μ in (X, T_g^1, T_g^2), $1 - \lambda \geq 1 - \mu$ in (X, T_g^1, T_g^2). Also, since the fuzzy set $1 - \lambda$ is a gpffc set in (X, T_g^1, T_g^2) and by the Proposition 4.20, the fuzzy set $1 - \mu$ is a gpffc set in (X, T_g^1, T_g^2). This implies that the fuzzy set μ is a gpfr set in (X, T_g^1, T_g^2). □

Proposition 4.27 *If the fuzzy set λ is a gpfr set in a gfbt space (X, T_g^1, T_g^2), then the fuzzy set $\lambda = \wedge_{k=1}^{\infty}(\mu_k)$, where (μ_k)'s are gpfso sets in (X, T_g^1, T_g^2).*

Proof: Let the fuzzy set λ be a gpfr set in (X,T_g^1,T_g^2). Then the fuzzy set $1-\lambda$ is a gpffc set in (X,T_g^1,T_g^2) and $1-\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X,T_g^1,T_g^2). Then the fuzzy set $\lambda = 1-\vee_{k=1}^{\infty}(\lambda_k) = \wedge_{k=1}^{\infty}(1-\lambda_k)$, in (X,T_g^1,T_g^2). Since the fuzzy sets (λ_k)'s are the gpfnd sets and by the Proposition 4.8, (λ_k)'s are gpfsc sets in (X,T_g^1,T_g^2). This implies that $(1-\lambda_k)$'s are gpfso sets in (X,T_g^1,T_g^2). Let $1-\lambda_k = \mu_k$. Therefore, the fuzzy set $\lambda = \wedge_{k=1}^{\infty}(\mu_k)$, where (μ_k)'s are gpfso sets in (X,T_g^1,T_g^2). □

Proposition 4.28 *If the fuzzy set λ is a gpfr set in a gfbt space (X,T_g^1,T_g^2) and if $\lambda \leq \mu$ for a fuzzy set μ in (X,T_g^1,T_g^2), then the fuzzy set $\mu = \wedge_{k=1}^{\infty}(\mu_k)$, where the fuzzy sets (μ_k)'s are gpfso sets in (X,T_g^1,T_g^2).*

Proof: Let the fuzzy set λ be a gpfr set in (X,T_g^1,T_g^2). Then, the fuzzy set $1-\lambda$ is a gpffc set in (X,T_g^1,T_g^2). Since $\lambda \leq \mu$ for a fuzzy set μ in (X,T_g^1,T_g^2), $1-\lambda \geq 1-\mu$, in (X,T_g^1,T_g^2). Also, since the fuzzy set $1-\lambda$ is a gpffc set and $1-\lambda \geq 1-\mu$ in (X,T_g^1,T_g^2) and by the Proposition 4.20, the fuzzy set $1-\mu$ is a gpffc set in (X,T_g^1,T_g^2). Then, by the Proposition 4.25, $1-\mu = \vee_{k=1}^{\infty}(\nu_k)$, where the fuzzy sets (ν_k)'s are gpfsc sets in (X,T_g^1,T_g^2). Now $\mu = 1-\vee_{k=1}^{\infty}(\nu_k) = \wedge_{k=1}^{\infty}(1-\nu_k)$. Let $1-\nu_k = \mu_k$. Since (ν_k)'s are gpfsc sets, the fuzzy sets $(1-\nu_k)$'s are gpfso sets in (X,T_g^1,T_g^2). This implies that the fuzzy set $\mu = \wedge_{k=1}^{\infty}(\mu_k)$, where the fuzzy sets (μ_k)'s are gpfso sets in (X,T_g^1,T_g^2). □

Definition 4.6 *A fuzzy set λ in a gfbt space (X,T_g^1,T_g^2) is called a generalized pairwise fuzzy σ-nowhere dense set (gpfσ-nd set, for short) if λ is a gpfF_σ-set in (X,T_g^1,T_g^2) such that $int_{T_g^1}int_{T_g^2}(\lambda) = int_{T_g^2}int_{T_g^1}(\lambda) = 0$.*

Example 4.6 *Let $X = \{a,b,c\}$. The fuzzy sets λ, μ, δ and η are defined on X as follows:*

$\lambda : X \to [0,1]$ *is defined as* $\lambda(a) = 0.2;\quad \lambda(b) = 0.7;\quad \lambda(c) = 0.5,$

$\mu : X \to [0,1]$ *is defined as* $\mu(a) = 0.7;\quad \mu(b) = 0.5;\quad \mu(c) = 0.3,$

$\delta : X \to [0,1]$ *is defined as* $\delta(a) = 0.5;\quad \delta(b) = 0.7;\quad \delta(c) = 0.5,$

$\eta : X \to [0,1]$ *is defined as* $\eta(a) = 0.5;\quad \eta(b) = 0.2;\quad \eta(c) = 0.8.$

Then $T_g^1 = \{0,\mu,\eta,\delta,\mu\vee\eta,\mu\vee\delta,\eta\vee\delta,\mu\wedge\eta,\mu\wedge\delta,\eta\wedge\delta,\mu\vee[\eta\wedge\delta],\eta\vee[\mu\wedge\delta],\delta\wedge[\mu\vee\eta],\mu\vee\eta\vee\delta,1\}$ *and* $T_g^2 = \{0,\lambda,\mu,\eta,\lambda\vee\mu,\lambda\vee\eta,\mu\vee\eta,\lambda\wedge\mu,\lambda\wedge\eta,\mu\wedge\eta,\lambda\vee[\mu\wedge\eta],\mu\vee[\lambda\wedge\eta],\eta\vee[\lambda\wedge\mu],\lambda\wedge[\mu\vee\eta],\mu\wedge[\lambda\vee\eta],\eta\wedge[\lambda\vee\mu],\mu\vee\eta\vee\delta,1\}$ *are generalized fuzzy topologies on X. The fuzzy sets* $\mu,\eta,\delta,\lambda\vee\eta,\mu\vee\eta,\mu\vee\delta,\mu\wedge\eta,\mu\wedge\delta,\eta\wedge\delta,\mu\vee[\eta\wedge\delta],\eta\vee[\lambda\wedge\mu],\mu\vee\eta\vee\delta,1$ *are gpfo sets in (X,T_g^1,T_g^2). The fuzzy sets* $1-[\mu\wedge\eta] = (1-\mu)\vee(1-\eta)\vee(1-[\mu\wedge\delta])\vee(1-[\eta\wedge\delta])\vee(1-[\mu\vee(\eta\wedge\delta)])$ *and* $\delta\wedge(\mu\vee\eta) = (1-\delta)\vee(1-[\lambda\vee\eta])\vee(1-[\mu\vee\eta])\vee(1-[\mu\vee\delta])\vee(1-[\eta\vee(\lambda\wedge\mu)])\vee(1-[\mu\vee\eta\vee\delta])$ *are gpfF_σ-sets in (X,T_g^1,T_g^2). Also* $int_{T_g^1}int_{T_g^2}(1-[\mu\wedge\eta]) = \delta \neq 0$ *and* $int_{T_g^2}int_{T_g^1}(1-[\mu\wedge\eta]) = \delta\vee[\mu\wedge\eta] \neq 0$ *and* $int_{T_g^1}int_{T_g^2}(\delta\wedge[\mu\vee\eta]) = \mu\wedge\delta \neq 0$ *and* $int_{T_g^2}int_{T_g^1}(\delta\wedge[\mu\vee\eta]) = \mu\wedge[\lambda\vee\eta] \neq 0$. *Hence* $1-[\mu\wedge\eta]$ *and* $\delta\wedge[\mu\vee\eta]$ *are not gpfσ-nd sets in (X,T_g^1,T_g^2). Now* $1-(\eta\vee[\lambda\wedge\mu]) = (1-[\lambda\vee\eta])\vee(1-[\mu\vee\eta])\vee(1-[\mu\vee\eta\vee\delta])$ *is a gpfF_σ-set in (X,T_g^1,T_g^2) such that* $int_{T_g^1}int_{T_g^2}(1-(\eta\vee[\lambda\wedge\mu])) = 0$ *and* $int_{T_g^2}int_{T_g^1}(1-(\eta\vee[\lambda\wedge\mu])) = 0$ *and hence* $1-(\eta\vee[\lambda\wedge\mu])$ *is a gpfσ-nd set in (X,T_g^1,T_g^2).*

Proposition 4.29 *In a gfbt space (X,T_g^1,T_g^2), λ is a gpfσ-nd set in (X,T_g^1,T_g^2) if and only if $1-\lambda$ is a gpfd and gpfG_δ-set in (X,T_g^1,T_g^2).*

Proof: Let λ be a gpfσ-nd set in (X,T_g^1,T_g^2). Then λ is a gpfF_σ-set such that $int_{T_g^1}int_{T_g^2}(\lambda) = 0$ and $int_{T_g^2}int_{T_g^1}(\lambda) = 0$, in (X,T_g^1,T_g^2). This implies that $1-\lambda$ is a gpfG_δ-set in (X,T_g^1,T_g^2) and $1-int_{T_g^1}int_{T_g^2}(\lambda) = 1$ and also $1-int_{T_g^2}int_{T_g^1}(\lambda) = 1$. That is, $1-\lambda$ is a gpfG_δ-set and $cl_{T_g^1}cl_{T_g^2}(1-\lambda) = 1$ and also $cl_{T_g^2}cl_{T_g^1}(1-\lambda) = 1$. Hence $1-\lambda$ is a gpfd and gpfG_δ-set in (X,T_g^1,T_g^2).

Conversely, let λ be a gpfd and gpfG_δ-set in (X,T_g^1,T_g^2). Then $cl_{T_g^1}cl_{T_g^2}(\lambda) = 1$ and $cl_{T_g^2}cl_{T_g^1}(\lambda) = 1$ and also $1-\lambda$ is a gpfF_σ-set in (X,T_g^1,T_g^2). Now $int_{T_g^1}int_{T_g^2}(1-\lambda) = 1-cl_{T_g^1}cl_{T_g^2}(\lambda) = 1-1 = 0.$ and also $int_{T_g^2}int_{T_g^1}(1-\lambda) = 1-cl_{T_g^2}cl_{T_g^1}(\lambda) = 1-1 = 0$. Hence $1-\lambda$ is a gpfF_σ-set such that $int_{T_g^1}int_{T_g^2}(1-\lambda) = 0$ and $int_{T_g^2}int_{T_g^1}(1-\lambda) = 0$. Therefore $1-\lambda$ is a gpfσ-nd set in (X,T_g^1,T_g^2). □

Proposition 4.30 *If a gpfnd set λ is a gpfF_σ-set in a gfbt space in (X, T_g^1, T_g^2), then λ is a gpfσ-nd set in (X, T_g^1, T_g^2).*

Proof: Let the gpfnd set λ be a gpfF_σ-set in (X, T_g^1, T_g^2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where the fuzzy sets (λ_k)'s are gpfc sets in (X, T_g^1, T_g^2). Since λ is a gpfnd set, $int_{T_g^1} cl_{T_g^2}(\lambda) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda)$ in (X, T_g^1, T_g^2). Now $int_{T_g^1}(\lambda) \leq int_{T_g^1} cl_{T_g^2}(\lambda)$, implies that $int_{T_g^1}(\lambda) \leq 0$. That is, $int_{T_g^1}(\lambda) = 0$ in (X, T_g^1, T_g^2) and hence $int_{T_g^2} int_{T_g^1}(\lambda) = int_{T_g^2}(0) = 0$ and also $int_{T_g^1} int_{T_g^2}(\lambda) = 0$. Since λ is a gpfF_σ-set such that $int_{T_g^1} int_{T_g^2}(\lambda) = 0 = int_{T_g^2} int_{T_g^1}(\lambda)$, λ is a gpfσ-nd set in (X, T_g^1, T_g^2). □

Proposition 4.31 *If λ is a gpfd set in a gfbt space (X, T_g^1, T_g^2) such that $\mu \leq 1 - \lambda$, where μ is a gpfF_σ-set, then μ is a gpfσ-nd set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfd set in (X, T_g^1, T_g^2) such that $\mu \leq 1-\lambda$. Then $cl_{T_g^1} cl_{T_g^2}(\lambda) = 1$ and $cl_{T_g^2} cl_{T_g^1}(\lambda) = 1$, in (X, T_g^1, T_g^2) such that $int_{T_g^1} int_{T_g^2}(\mu) \leq int_{T_g^1} int_{T_g^2}(1 - \lambda) = 1 - cl_{T_g^1} cl_{T_g^2}(\lambda) = 1 - 1 = 0$. This implies that $int_{T_g^1} int_{T_g^2}(\mu) \leq 0$ and hence $int_{T_g^1} int_{T_g^2}(\mu) = 0$. Similarly, $int_{T_g^2} int_{T_g^1}(\mu) = 0$. Since μ is a gpfF_σ-set in (X, T_g^1, T_g^2) such that $int_{T_g^1} int_{T_g^2}(\mu) = 0$ and $int_{T_g^2} int_{T_g^1}(\mu) = 0$, μ is a gpfσ-nd set in (X, T_g^1, T_g^2). □

Proposition 4.32 *If λ is a gpfnd and gpfF_σ-set in a gfbt space (X, T_g^1, T_g^2), then λ is a gpfσ-nd set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfnd set in (X, T_g^1, T_g^2). Then $int_{T_g^1} cl_{T_g^2}(\lambda) = 0$ and $int_{T_g^2} cl_{T_g^1}(\lambda) = 0$. Now $\lambda \leq cl_{T_g^1}(\lambda)$ implies that $int_{T_g^2}(\lambda) \leq int_{T_g^2} cl_{T_g^1}(\lambda) = 0$. It follows that $int_{T_g^1} int_{T_g^2}(\lambda) \leq int_{T_g^1}(0) = 0$. That is, $int_{T_g^1} int_{T_g^2}(\lambda) = 0$. Similarly, $int_{T_g^2} int_{T_g^1}(\lambda) = 0$. Since λ is a gpfF_σ-set such that $int_{T_g^1} int_{T_g^2}(\lambda) = 0$ and $int_{T_g^2} int_{T_g^1}(\lambda) = 0$, λ is a gpfσ-nd set in (X, T_g^1, T_g^2). □

Proposition 4.33 *If λ is a gpfσ-nd set in a gfbt space (X, T_g^1, T_g^2), then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0 = int_{T_g^2} int_{T_g^1}(\lambda_k)$ and $cl_{T_g^1} cl_{T_g^2}(\lambda_k) = \lambda_k = cl_{T_g^2} cl_{T_g^1}(\lambda_k)$, in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfσ-nd set in (X, T_g^1, T_g^2). Then λ is a gpfF_σ-set in (X, T_g^1, T_g^2) such that $int_{T_g^1} int_{T_g^2}(\lambda) = int_{T_g^2} int_{T_g^1}(\lambda) = 0$. Since λ is a gpfF_σ-set in (X, T_g^1, T_g^2), $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are gpfc sets in (X, T_g^1, T_g^2). That is, $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $cl_{T_g^1}(\lambda_k) = \lambda_k$ and $cl_{T_g^2}(\lambda_k) = \lambda_k$ implies that $cl_{T_g^1} cl_{T_g^2}(\lambda_k) = cl_{T_g^1}(\lambda_k) = \lambda_k$ and $cl_{T_g^2} cl_{T_g^1}(\lambda_k) = cl_{T_g^2}(\lambda_k) = \lambda_k$, in (X, T_g^1, T_g^2). Also, since $int_{T_g^1} int_{T_g^2}(\lambda) = 0$, $int_{T_g^1} int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Now $\vee_{k=1}^{\infty}[int_{T_g^1} int_{T_g^2}(\lambda_k)] \leq int_{T_g^1}\{\vee_{k=1}^{\infty}[int_{T_g^2}(\lambda_k)]\} \leq int_{T_g^1} int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k))$ implies that $\vee_{k=1}^{\infty}[int_{T_g^1} int_{T_g^2}(\lambda_k)] \leq int_{T_g^1} int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k))$ and $int_{T_g^1} int_{T_g^2}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Then $\vee_{k=1}^{\infty}[int_{T_g^1} int_{T_g^2}(\lambda_k)] \leq 0$. That is, $\vee_{k=1}^{\infty}[int_{T_g^1} int_{T_g^2}(\lambda_k)] = 0$ and $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^2} int_{T_g^1}(\lambda_k) = 0$, in (X, T_g^1, T_g^2). Therefore $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0 = int_{T_g^2} int_{T_g^1}(\lambda_k)$ and $cl_{T_g^1} cl_{T_g^2}(\lambda_k) = \lambda_k = cl_{T_g^2} cl_{T_g^1}(\lambda_k)$, in (X, T_g^1, T_g^2). □

Proposition 4.34 *If λ is a gpfσ-nd set in a gfbt space (X, T_g^1, T_g^2), then $1 - \lambda = \wedge_{k=1}^{\infty}(\mu_k)$, where $int_{T_g^1} int_{T_g^2}(\mu_k) = \mu_k = int_{T_g^2} int_{T_g^1}(\mu_k)$ and $cl_{T_g^1} cl_{T_g^2}(\mu_k) = 1 = cl_{T_g^2} cl_{T_g^1}(\mu_k)$, in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfσ-nd set in (X, T_g^1, T_g^2). Then by the Proposition 4.33, $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0 = int_{T_g^2} int_{T_g^1}(\lambda_k)$ and $cl_{T_g^1} cl_{T_g^2}(\lambda_k) = \lambda_k = cl_{T_g^2} cl_{T_g^1}(\lambda_k)$, in (X, T_g^1, T_g^2). This implies that $1 - \lambda = 1 - \vee_{k=1}^{\infty}(\lambda_k)$, where $1 - int_{T_g^1} int_{T_g^2}(\lambda_k) = 1 = 1 - int_{T_g^2} int_{T_g^1}(\lambda_k)$ and $1 - cl_{T_g^1} cl_{T_g^2}(\lambda_k) = 1 - \lambda_k = 1 - cl_{T_g^2} cl_{T_g^1}(\lambda_k)$. Then $1 - \lambda = \wedge_{k=1}^{\infty}(1 - \lambda_k)$, where $cl_{T_g^1} cl_{T_g^2}(1 - \lambda_k) = 1 = cl_{T_g^2} cl_{T_g^1}(1 - \lambda_k)$ and $int_{T_g^1} int_{T_g^2}(1 - \lambda_k) = 1 - \lambda_k = int_{T_g^2} int_{T_g^1}(1 - \lambda_k)$, in (X, T_g^1, T_g^2). Let $1 - \lambda_k = \mu_k$. Therefore, $1 - \lambda = \wedge_{k=1}^{\infty}(\mu_k)$, where $cl_{T_g^1} cl_{T_g^2}(\mu_k) = 1 = cl_{T_g^2} cl_{T_g^1}(\mu_k)$ and $int_{T_g^1} int_{T_g^2}(\mu_k) = \mu_k = int_{T_g^2} int_{T_g^1}(\mu_k)$, in (X, T_g^1, T_g^2). □

Proposition 4.35 *If λ is a gpfnd set in a gfbt space (X, T_g^1, T_g^2) such that $\mu \leq \lambda$, where μ is a gpfF_σ-set in (X, T_g^1, T_g^2), then μ is a gpfσ-nd set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfnd set such that $\mu \leq \lambda$, where μ is a gpfF_σ-set in (X, T_g^1, T_g^2). Since λ is a gpfnd set in (X, T_1, T_2) and by the Proposition 4.7, $1-\lambda$ is a gpfd set in (X, T_g^1, T_g^2). Now $\mu \leq \lambda$, in (X, T_g^1, T_g^2). Then $\mu \leq 1-(1-\lambda)$, in (X, T_g^1, T_g^2). Also, since μ is a gpfF_σ-set and $1-\lambda$ is a gpfd set in (X, T_g^1, T_g^2). By the Proposition 4.31, μ is a gpfσ-nd set in (X, T_g^1, T_g^2). □

Definition 4.7 *A non-zero fuzzy set λ in a gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy somewhere dense set (gpfsd set, for short) if $int_{T_g^1} cl_{T_g^2}(\lambda) \neq 0$ and $int_{T_g^2} cl_{T_g^1} \neq 0$.*

Proposition 4.36 *If λ is a gpfσ-nd set in a gfbt space (X, T_g^1, T_g^2), then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) = 0 = int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k))$ and (λ_k)'s are gpfnd sets or gpfsd sets in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfσ-nd set in (X, T_g^1, T_g^2). Then by the Proposition 4.33, $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0 = int_{T_g^2} int_{T_g^1}(\lambda_k)$ and $cl_{T_g^1} cl_{T_g^2}(\lambda_k) = \lambda_k = cl_{T_g^2} cl_{T_g^1}(\lambda_k)$, in (X, T_g^1, T_g^2). Since $int_{T_g^1} int_{T_g^2}(\lambda_k) = 0$, $int_{T_g^1} int_{T_g^2}(cl_{T_g^1} cl_{T_g^2}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Now $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) \leq int_{T_g^1} int_{T_g^2}(cl_{T_g^1} cl_{T_g^2}(\lambda_k))$ and $int_{T_g^1} int_{T_g^2}(cl_{T_g^1} cl_{T_g^2}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Then $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) \leq 0$, in (X, T_g^1, T_g^2). This implies that $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Similarly, $int_{T_g^2}(int_{T_g^1} cl_{T_g^2}(\lambda_k)) = 0$, in (X, T_g^1, T_g^2). Therefore, $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) = 0 = int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k))$, in (X, T_g^1, T_g^2). Now the following two cases arises:
Case (i)**:** Suppose that $int_{T_g^2} cl_{T_g^1}(\lambda_k) = 0 = int_{T_g^2} cl_{T_g^1}(\lambda_k)$ implies that (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2).
Case (ii)**:** Suppose that $\mu_k \neq 0$. Then $int_{T_g^2} cl_{T_g^1}(\lambda_k) \neq 0 \neq int_{T_g^2} cl_{T_g^1}(\lambda_k)$, implies that (λ_k)'s are gpfsd sets in (X, T_g^1, T_g^2).

Hence $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where $int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k)) = 0 = int_{T_g^1}(int_{T_g^2} cl_{T_g^1}(\lambda_k))$ and (λ_k)'s are gpfnd sets or gpfsd sets in (X, T_g^1, T_g^2). □

Definition 4.8 *A gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy first category space (gpffc space, for short) if the fuzzy set 1_X is a gpffc set in (X, T_g^1, T_g^2). That is, $1_X = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Otherwise (X, T_g^1, T_g^2) will be called a generalized pairwise fuzzy second category space (gpfsc space, for short).*

Proposition 4.37 *If $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$, for gpfd sets (λ_k)'s in a gfbt space (X, T_g^1, T_g^2), then the gfbt space (X, T_g^1, T_g^2) is a gpfsc space.*

Proof: Let the fuzzy sets (λ_k)'s ($k = 1$ to ∞) be the gpfd sets in (X, T_g^1, T_g^2) such that $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$. It has to be proved that (X, T_g^1, T_g^2) is a gpfsc space. Assume the contrary. Suppose that (X, T_g^1, T_g^2) is a gpffc space. Then, $\vee_{k=1}^{\infty}(\mu_k) = 1_X$, where the fuzzy sets (μ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). This implies that $1 - \vee_{k=1}^{\infty}(\mu_k) = 0$, in (X, T_g^1, T_g^2). Then $\wedge_{k=1}^{\infty}(1-\mu_k) = 0$, in (X, T_g^1, T_g^2). Since the fuzzy sets (μ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) and by the Proposition 4.7, the fuzzy sets $(1-\mu_k)$'s are gpfd sets in (X, T_g^1, T_g^2). Let $1-\mu_k = \lambda_k$. Thus, $\wedge_{k=1}^{\infty}(\lambda_k) = 0$, where the fuzzy sets (λ_k)'s are gpfd sets in (X, T_g^1, T_g^2) which is a contradiction to the hypothesis that $\wedge_{k=1}^{\infty}(\lambda_k) \neq 0$, for gpfd sets (λ_k)'s in (X, T_g^1, T_g^2). Therefore, our assumption is wrong. Thus, the gfbt space (X, T_g^1, T_g^2) is a gpfsc space. □

Proposition 4.38 *If the gfbt space (X, T_g^1, T_g^2) is a gpffc space, then $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where the fuzzy set (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2).*

Proof: Let the gfbt space (X, T_g^1, T_g^2) be a gpffc space. Then $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where the fuzzy sets (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Since the fuzzy sets (λ_k)'s are the gpfnd sets and by the Proposition 4.8, the fuzzy sets (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2). Therefore $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where the fuzzy set (λ_k)'s are gpfsc sets in (X, T_g^1, T_g^2). □

Definition 4.9 *A gfbt space (X, T_g^1, T_g^2) is called a generalized pairwise fuzzy submaximal space (gpfs space, for short) if each gpfd set in (X, T_g^1, T_g^2) is a gpfo set in (X, T_g^1, T_g^2). That is, if for a fuzzy set λ in (X, T_g^1, T_g^2) such that $cl_{T_g^i}(\lambda) = 1$, $(i = 1, 2)$, then λ is a gpfo set in (X, T_g^1, T_g^2).*

Example 4.7 *Let $X = \{a, b, c\}$. The fuzzy sets α, β and γ are defined on X as follows:*
$\alpha : X \to [0, 1]$ *is defined as* $\alpha(a) = 0.2$; $\alpha(b) = 0.3$; $\alpha(c) = 0.4$,
$\beta : X \to [0, 1]$ *is defined as* $\beta(a) = 0.8$; $\beta(b) = 0.7$; $\beta(c) = 0.6$,
$\gamma : X \to [0, 1]$ *is defined as* $\gamma(a) = 0.4$; $\gamma(b) = 0$; $\gamma(c) = 0$.
Then $T_g^1 = \{0, \alpha, \beta, 1\}$ and $T_g^2 = \{0, \beta, \gamma, 1\}$ are generalized fuzzy topologies on X. By computations, one can see that $cl_{T_g^1} cl_{T_g^2}(\beta) = cl_{T_g^1}(1) = 1$ and $cl_{T_g^2} cl_{T_g^1}(\beta) = cl_{T_g^2}(1 - \alpha) = 1$, in (X, T_g^1, T_g^2) and thus β is a gpfd set in (X, T_g^1, T_g^2). Clearly, β is a gpfo set in (X, T_g^1, T_g^2) and hence (X, T_g^1, T_g^2) is a gpfs space.

Example 4.8 *Let $X = \{a, b, c\}$. The fuzzy sets λ_1, λ_2 and λ_3 are defined on X as follows:*
$\lambda_1 : X \to [0, 1]$ *is defined as* $\lambda_1(a) = 0.25$; $\lambda_1(b) = 0$; $\lambda_1(c) = 0$,
$\lambda_2 : X \to [0, 1]$ *is defined as* $\lambda_2(a) = 0.75$; $\lambda_2(b) = 0.5$; $\lambda_2(c) = 0$,
$\lambda_3 : X \to [0, 1]$ *is defined as* $\lambda_3(a) = 0.8$; $\lambda_3(b) = 0$; $\lambda_3(c) = 1$.
Then $T_g^1 = \{0, \lambda_1, \lambda_2, 1\}$ and $T_g^2 = \{0, \lambda_1, \lambda_3, 1\}$ are generalized fuzzy topologies on X. By computations, one can see that the only gpfd set in (X, T_g^1, T_g^2) is λ_3. But λ_3 is not a gpfo set in (X, T_g^1, T_g^2). Therefore, (X, T_g^1, T_g^2) is not a gpfs space.

Proposition 4.39 *If a gfbt space (X, T_g^1, T_g^2) is a gpfs space and λ is a gpffc set, then $1 - \lambda$ is a gpfG_δ-set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpffc set in (X, T_g^1, T_g^2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2). Since (λ_k)'s are gpfnd sets in (X, T_g^1, T_g^2) and by the Proposition 4.7, $(1 - \lambda_k)$'s are gpfd sets in (X, T_g^1, T_g^2). Since (X, T_g^1, T_g^2) is a gpfs space, $(1 - \lambda_k)$'s are gpfo sets in (X, T_g^1, T_g^2). Now $1 - \lambda = 1 - \vee_{k=1}^{\infty}(\lambda_k) = \wedge_{k=1}^{\infty}(1 - \lambda_k)$. Thus, $1 - \lambda = \wedge_{k=1}^{\infty}(1 - \lambda_k)$, where $(1 - \lambda_k)$'s are gpfo sets in (X, T_g^1, T_g^2), implies that $1 - \lambda$ is a gpfG_δ-set in (X, T_g^1, T_g^2). □

Proposition 4.40 *If a gfbt space (X, T_g^1, T_g^2) is a gpfs space, then every gpffc set is a gpfF_σ-set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpffc set in (X, T_g^1, T_g^2). Since (X, T_g^1, T_g^2) is a gpfs space and by the Proposition 4.39, $1 - \lambda$ is a gpfG_δ-set in (X, T_g^1, T_g^2) and hence λ is a gpfF_σ-set in (X, T_g^1, T_g^2). □

Proposition 4.41 *If a gfbt space (X, T_g^1, T_g^2) is a gpfs space, then every gpfr set is a gpfG_δ-set in (X, T_g^1, T_g^2).*

Proof: Let λ be a gpfr set in (X, T_g^1, T_g^2). Then $1 - \lambda$ is a gpffc set (X, T_g^1, T_g^2). Since (X, T_g^1, T_g^2) is a gpfs space and by the Proposition 4.40, $1 - \lambda$ is a gpfF_σ-set in (X, T_g^1, T_g^2). Therefore λ is a gpfG_δ-set in (X, T_g^1, T_g^2). □

5. Conclusions

The new classes namely the generalized fuzzy bitopological spaces have been introduced and defined various types of generalized pairwise fuzzy sets in the generalized fuzzy bitopological spaces in this paper. Several examples given to investigate the generalized pairwise fuzzy sets and characterizations of generalized pairwise fuzzy sets have studied in this paper. Additionally generalized pairwise fuzzy first category spaces, generalized pairwise fuzzy second category spaces and generalized pairwise fuzzy submaximal spaces were defined and some of whose properties were also studied in this paper.

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References

1. Azad, K. K., *On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly contiunity*, J. Math. Anal. Appl. 52, 14-32, (1981).
2. Balasubramanian, G., Sundaram, P., *On some generalizations of fuzzy continuous functions*, Fuzzy Sets and Systems, 89(1), 93-100, (1997).
3. Chakraborty, J., Bhattacharya, B., Paul, A., *Some properties of generalized fuzzy hyperconnected spaces*, Ann. Fuzzy Math. Inform. 12(5), 659-668, (2016).
4. Chang, C. L., *Fuzzy topological spaces*, J. Math. Anal. Appl. 24, 182-190, (1968).
5. Császár, Á., *Generalized topology, generalized continuity*, Acta Math. Hungar. 96(4), 351-357, (2002).
6. Kandil, A., El-Shafee, M. E., *Biproximities and fuzzy bitopological spaces*, Simen Stevin, 63, 45-66, (1989).
7. Kelly, J. C., *Bitopological spaces*, Proc. London Math. Soc. 13, 71-89, (1963).
8. Levine, N., *Generalized closed sets in topological spaces*, Rend. Circ. Mat. Palemo. 19, 89-96, (1970).
9. Thangaraj, G., Chandiran, V., Ashokkumar, P., *On pairwise fuzzy generalized Volterra spaces*, Adv. Math. Sci. J. 9(8), 6035-6047, (2020).
10. Zadeh, L. A., *Fuzzy sets*, Inform. and Control, 8, 338-353, (1965).

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