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Investigation of nonlinear Riemann-Liouville fractional differential equations with fractional nonlocal multi-point and integral boundary conditions

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ABSTRACT: We investigate the existence of solutions for a Riemann–Liouville fractional differential equation of order $\alpha \in (2, 3)$ equipped with fractional anti-periodic type nonlocal multi-point and Riemann–Liouville integral boundary conditions in a weighted space. The existence and uniqueness results for the given problem are respectively proved by applying Leray-Schauder's alternative and Banach's contraction mapping principle. The Ulam–Hyers stability for the given problem is also studied. Examples illustrating the main results are offered.

Key Words: Riemann–Liouville fractional derivative and integral operators; nonlocal boundary conditions; existence; fixed Point; Ulam–Hyers stability.

Contents

1. Introduction

Riemann–Liouville fractional differential and integral operators are found to be of great utility in view of their applications in a variety of physical and technical disciplines. Examples include bioengineering [1], fractional dynamics and control [2], self-similar protein dynamics [3], backward diffusion problems [4], viscoelasticity [5], etc. The extensive application of Riemann–Liouville fractional operators motivated many researchers to develop the theoretical aspects of boundary value problems involving these operators, for example, see [6]-[11]. For some interesting results on anti-periodic fractional boundary value problem, we refer the reader to the papers [12]-[16].

In contrast to classical two-point boundary conditions, nonlocal boundary conditions provide a platform to model a phenomenon experiencing changes at arbitrary interior points or sub-segments of its domain [17]. In case of curved boundary structures, integral boundary conditions describe non-uniformities on segments of such structures. Examples include fluid flow problems [18], biomedical applications [19], bacterial self-organization [20], engineering applications [21,22], etc. One can find some useful results on multi-point and integral boundary value problems in the articles [23]-[27].

In 1940, Ulam [28] discussed the stability of a functional equation by presenting the conditions ensuring an approximate solution of this equation to be close to its exact solution. Hyers [29] developed the Ulam's idea of stability more rigorously in the context of Banach spaces in 1941. The concept of stability studied by Ulam and Hyers is known as the Ulam–Hyers stability. Afterward, Rassias [30] applied the idea of Ulam–Hyers stability to a wide class of functional equations, which is now referred to as Ulam–Hyers-Rassias stability [31]. The Ulam-Hyers stability for Black-Scholes equation was studied

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in [32]. For some recent results on Ulam–Hyers stability for fractional differential equations, for instance, see [33]- [36].

In this paper, we investigate the existence, uniqueness and Ulam-Hyers stability for solutions of a nonlinear Riemann–Liouville fractional differential equation of order $\alpha \in (2,3)$ equipped with fractional anti-periodic type nonlocal multi-point and Riemann–Liouville integral boundary conditions in a weighted space. In precise terms, we consider a nonlinear Riemann–Liouville fractional integro-differential equation of the form

$$D^{\alpha}x(t) = g(t, x(t), (\mu_1 x)(t), (\mu_2 x)(t)), \quad 2 < \alpha < 3, \quad t \in \mathcal{J} = [0,T], \quad T > 0, \tag{1.1}$$

subject to fractional anti-periodic type nonlocal multi-point and Riemann–Liouville integral boundary conditions

$$\begin{cases} D^{\alpha-3}x(0^+) + a_1 D^{\alpha-3}x(T^-) = \phi(x), \\ D^{\alpha-2}x(0^+) + a_2 D^{\alpha-2}x(T^-) = v_1 I^{\alpha-2}x(\eta_1) + \displaystyle\sum_{i=1}^{m} \omega_i x(\xi_i), \\ D^{\alpha-1}x(0^+) + a_3 D^{\alpha-1}x(T^-) = v_2 I^{\alpha-1}x(\eta_2) + \displaystyle\sum_{j=1}^{n} \sigma_j x(\zeta_j), \end{cases} \tag{1.2}$$

where $0 < \eta_1 < \eta_2 < \xi_1 < \xi_2 < \ldots < \xi_m < \zeta_1 < \zeta_2 < \ldots < \zeta_n < T$, D^{α} denotes the Riemann–Liouville fractional derivative operator of order α, $I^{\alpha-1}$ and $I^{\alpha-2}$ respectively, represent the Riemann–Liouville fractional integral operators of order $(\alpha-1)$ and $(\alpha-2)$, $v_1, v_2, a_1, a_2, a_3, \omega_i, \xi_i, \sigma_j$, $\zeta_j \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, $g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : C(\mathcal{J}, \mathbb{R}) \to \mathbb{R}$ are appropriate continuous functions and

$$(\mu_1 x)(t) = \int_0^t \Phi_1(t,s)\, x(s)\, ds, \ \ (\mu_2 x)(t) = \int_0^t \Phi_2(t,s)\, x(s)\, ds, \tag{1.3}$$

with Φ_1 and Φ_2 being continuous functions on $\mathcal{J} \times \mathcal{J}$.

Here, one can notice that the nonlinearity in the equation (1.1) contains integral terms in addition to the unknown function, while boundary conditions (1.2) can be regarded as a combination of Riemann-Liouville fractional derivative and integral operators of different orders and nonlocal multi-point values of the unknown function. The objective of the present work is to enrich the literature on boundary value problems for Riemann-Liouville fractional differential equations.

We arrange the rest of the paper as follows. Section 2 contains background material and a subsidiary result for the linear variant of the problem (1.1)-(1.2). We establish the existence and uniqueness results for the given problem with the aid of Leray-Schauder's alternative and Banach's contraction mapping principle respectively in Section 3. We also discuss the case when the integral terms in the nonlinearity of (1.1) are of the Riemann–Liouville fractional integral type. Illustrative examples are presented in Section 4. We discuss the Ulam–Hyers stability for the problem (1.1)-(1.2) in Section 5. The paper concludes with some interesting remarks.

2. A subsidiary result

Let us first recall some basic concepts of fractional calculus from the text [37].

Definition 2.1 *For $\varphi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of φ, denoted by $I^{\alpha}_{a^+}\varphi$, is defined as*

$$I^{\alpha}_{a^+}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \varphi(s) ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2 *Let $\varphi, \varphi^{(m)} \in L_1[a,b], a,b \in \mathbb{R}$ and $\alpha \in (m-1,m]$, $m \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order α of φ, denoted by $D^{\alpha}_{a+}\varphi$, is given by*

$$D^{\alpha}_{a+}\varphi(t) = \frac{d^m}{dt^m} I^{1-\alpha}_{a^+}\varphi(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-1-\alpha}\varphi(s)ds.$$

In the current work, we write the Riemann–Liouville fractional integral and derivative operators I^q_{a+} and D^q_{a+} as I^q and D^q when $a=0$, respectively.

Lemma 2.1 *Let p and q be positive real numbers. If φ is a continuous function, then*

(i) $I^p I^q \varphi(t) = I^{p+q}\varphi(t)$,

(ii) $D^p I^q \varphi(t) = I^{q-p}\varphi(t)$ *for* $q > p > 0$.

Note that $D^p t^{p-i} = 0, \quad i = 1, 2, \ldots, [p]+1$, where $[p]$ is the largest integer less than p and

$$D^p t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-p+1)} t^{\lambda-p}, \quad \lambda > -1, \quad \lambda \neq p-1, p-2, \ldots, p-n.$$

In the following lemma, we solve the linear version of the equation (1.1) complemented with the boundary data (1.2).

Lemma 2.2 *For $h \in C(\mathcal{J}, \mathbb{R})$, $k \in \mathbb{R}$ and $\Delta \neq 0$, the unique solution of the linear equation*

$$D^{\alpha}x(t) = h(t), \quad 2 < \alpha < 3, \quad t \in \mathcal{J}, \tag{2.1}$$

subject to fractional anti-periodic type nonlocal multi-point and Riemann–Liouville integral boundary conditions

$$\begin{cases} D^{\alpha-3}x(0^+) + a_1 D^{\alpha-3}x(T^-) = k, \\ D^{\alpha-2}x(0^+) + a_2 D^{\alpha-2}x(T^-) = v_1 I^{\alpha-2}x(\eta_1) + \sum\limits_{i=1}^{m}\omega_i x(\xi_i), \\ D^{\alpha-1}x(0^+) + a_3 D^{\alpha-1}x(T^-) = v_2 I^{\alpha-1}x(\eta_2) + \sum\limits_{j=1}^{n}\sigma_j x(\zeta_j), \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) \;=\; & M_1(t)k + \int_0^T \Big[M_2(t)\frac{(T-s)^2}{2} + M_3(t)(T-s) + M_4(t)\Big]h(s)ds \\ & + M_5(t)\sum_{i=1}^{m}\omega_i \int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + M_6(t)\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}h(s)ds \\ & + M_7(t)\sum_{j=1}^{n}\sigma_j \int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + M_8(t)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h(s)ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M_1(t) &= \frac{1}{\Delta}\Big[(B_2E_3 + B_3E_2)\, t^{\alpha-1} - (B_1E_3 - B_3E_1)\, t^{\alpha-2} + (B_1E_2 + B_2E_1)\, t^{\alpha-3}\Big], \\ M_2(t) &= \frac{-a_1}{\Delta}\Big[(B_2E_3 + B_3E_2)\, t^{\alpha-1} - (B_1E_3 - B_3E_1)\, t^{\alpha-2} + (B_1E_2 + B_2E_1)\, t^{\alpha-3}\Big], \\ M_3(t) &= \frac{-a_2}{\Delta}\Big[(A_3E_2 - A_2E_3)\, t^{\alpha-1} + (A_1E_3 + A_3E_1)\, t^{\alpha-2} - (A_1E_2 + A_2E_1)\, t^{\alpha-3}\Big], \end{aligned}$$

$$
\begin{aligned}
M_4(t) &= \frac{-a_3}{\Delta}\Big[(A_2B_3+A_3B_2)\ t^{\alpha-1}-(A_1B_3+A_3B_1)\ t^{\alpha-2}-(A_1B_2-A_2B_1)\ t^{\alpha-3}\Big],\\
M_5(t) &= \frac{1}{\Delta}\Big[(A_3E_2-A_2E_3)\ t^{\alpha-1}+(A_1E_3+A_3E_1)\ t^{\alpha-2}-(A_1E_2+A_2E_1)\ t^{\alpha-3}\Big],\\
M_6(t) &= \frac{v_1}{\Delta}\Big[(A_3E_2-A_2E_3)\ t^{\alpha-1}+(A_1E_3+A_3E_1)\ t^{\alpha-2}-(A_1E_2+A_2E_1)\ t^{\alpha-3}\Big],\\
M_7(t) &= \frac{1}{\Delta}\Big[(A_2B_3+A_3B_2)\ t^{\alpha-1}-(A_1B_3+A_3B_1)\ t^{\alpha-2}-(A_1B_2-A_2B_1)\ t^{\alpha-3}\Big],\\
M_8(t) &= \frac{v_2}{\Delta}\Big[(A_2B_3+A_3B_2)\ t^{\alpha-1}-(A_1B_3+A_3B_1)\ t^{\alpha-2}-(A_1B_2-A_2B_1)\ t^{\alpha-3}\Big],\\
\Delta &= A_1B_2E_3+A_1B_3E_2-A_2B_1E_3+A_2B_3E_1+A_3B_1E_2+A_3B_2E_1,\\
A_1 &= \frac{\Gamma(\alpha)}{2}a_1T^2,\quad A_2=a_1\Gamma(\alpha-1)T,\quad A_3=(a_1+1)\Gamma(\alpha-2),\\
B_1 &= a_2\Gamma(\alpha)T-\frac{\Gamma(\alpha)}{\Gamma(2\alpha-2)}v_1\eta_1^{2\alpha-3}-\sum_{i=1}^{m}\omega_i\xi_i^{\alpha-1},\\
B_2 &= (1+a_2)\Gamma(\alpha-1)-\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-3)}v_1\eta_1^{2\alpha-4}-\sum_{i=1}^{m}\omega_i\xi_i^{\alpha-2},\\
B_3 &= \frac{\Gamma(\alpha-2)}{\Gamma(2\alpha-4)}v_1\eta_1^{2\alpha-5}+\sum_{i=1}^{m}\omega_i\xi_i^{\alpha-3},\\
E_1 &= (1+a_3)\Gamma(\alpha)-\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}v_2\eta_2^{2\alpha-2}-\sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-1},\\
E_2 &= \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)}v_2\eta_2^{2\alpha-3}+\sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-2},\quad E_3=\frac{\Gamma(\alpha-2)}{\Gamma(2\alpha-3)}v_2\eta_2^{2\alpha-4}+\sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-3}.
\end{aligned}
\tag{2.4}
$$

Proof: Operating the integral operator I^α to (2.1), we obtain

$$x(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+I^\alpha h(t), \tag{2.5}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are unknown arbitrary constants.
From (2.5), we have

$$
\begin{cases}
D^{\alpha-1}x(t)=c_1\Gamma(\alpha)+I^1h(t),\\
D^{\alpha-2}x(t)=c_1\Gamma(\alpha)t+c_2\Gamma(\alpha-1)+I^2h(t),\\
D^{\alpha-3}x(t)=c_1\dfrac{\Gamma(\alpha)}{2}t^2+c_2\Gamma(\alpha-1)t+c_3\Gamma(\alpha-2)+I^3h(t).
\end{cases}
\tag{2.6}
$$

Using (2.6) in the boundary conditions (2.2) together with notation (2.4), we obtain

$$
\begin{cases}
A_1c_1+A_2c_2+A_3c_3=J_1,\\
B_1c_1+B_2c_2-B_3c_3=J_2,\\
E_1c_1-E_2c_2-E_3c_3=J_3,
\end{cases}
\tag{2.7}
$$

where

$$
\begin{aligned}
&J_1=k-a_1I^3h(T),\ J_2=v_1I^{2\alpha-2}h(\eta_1)+\sum_{i=1}^{m}\omega_iI^\alpha h(\xi_i)-a_2I^2h(T),\\
&J_3=v_2I^{2\alpha-1}h(\eta_2)+\sum_{j=1}^{n}\sigma_jI^\alpha h(\zeta_j)-a_3I^1h(T).
\end{aligned}
$$

Solving the system (2.7) for c_1, c_2 and c_3, we find that

$$c_1=\frac{1}{\Delta}\big[(B_2E_3+B_3E_2)J_1-(A_2E_3-A_3E_2)J_2+(A_2B_3+A_3B_2)J_3\big],$$

$$c_2 = \frac{1}{\Delta}\big[(B_3E_1 - B_1E_3)J_1 + (A_1E_3 + A_3E_1)J_2 - (A_1B_3 + A_3B_1)J_3\big],$$
$$c_3 = \frac{1}{\Delta}\big[(B_1E_2 + B_2E_1)J_1 - (A_1E_2 + A_2E_1)J_2 - (A_1B_2 - A_2B_1)J_3\big].$$

Inserting the above values of c_1, c_2 and c_3 in (2.5) and using (2.4), we get the solution (2.3). The converse of the lemma follows by direct computation. □

3. Main Results

Let $C(\mathcal{J},\mathbb{R})$ be the Banach space of all continuous real-valued functions from $\mathcal{J} \to \mathbb{R}$ endowed with the supremum norm $\|x\| = \sup_{t\in\mathcal{J}} |x(t)|$. For $t \in \mathcal{J}$, we define $x_{\tilde{\rho}}(t) = t^{\tilde{\rho}}x(t)$, $\tilde{\rho} > 0$, and let $C_{\tilde{\rho}}(\mathcal{J},\mathbb{R})$ be the space of all functions $x_{\tilde{\rho}}$ such that $x \in C(\mathcal{J},\mathbb{R})$ which turns out to be a Banach space when endowed with the norm $\|x\|_{\tilde{\rho}} = \sup_{t\in\mathcal{J}}\{t^{\tilde{\rho}}|x(t)|\}$.

By Lemma 2.2, the problem (1.1)-(1.2) can be transformed into a fixed point problem as

$$x = \mathcal{G}x,$$

where $\mathcal{G} : C_{3-\alpha}(\mathcal{J},\mathbb{R}) \to C_{3-\alpha}(\mathcal{J},\mathbb{R})$ is an operator defined by

$$\begin{aligned}
(\mathcal{G}x)(t) \quad = \quad & M_1(t)\phi(x) + \int_0^T \Big[M_2(t)\frac{(T-s)^2}{2} + M_3(t)(T-s) + M_4(t)\Big] \\
& \times g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_5(t)\sum_{i=1}^m \omega_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_6(t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{2\alpha-3}}{\Gamma(2\alpha-2)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_7(t)\sum_{j=1}^n \sigma_j \int_0^{\zeta_j} \frac{(\zeta_j - s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_8(t)\int_0^{\eta_2} \frac{(\eta_2 - s)^{2\alpha-2}}{\Gamma(2\alpha-1)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds. \qquad (3.1)
\end{aligned}$$

Observe that the fixed points of the operator $\mathcal{G}$ are solution to the problem (1.1)-(1.2).

Lemma 3.1 *Let $g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $\phi \in C(\mathcal{J},\mathbb{R})$. Then, the operator $\mathcal{G} : C_{3-\alpha}(\mathcal{J},\mathbb{R}) \to C_{3-\alpha}(\mathcal{J},\mathbb{R})$ is completely continuous.*

Proof: The operator $\mathcal{G}$ is continuous, since g is continuous. Let $\mathcal{E}$ be a bounded set in $C_{3-\alpha}(\mathcal{J},\mathbb{R})$. Then, we can find positive constants N_g and N_ϕ such that $|g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))| \le N_g$, $|\phi(x)| \le N_\phi$, $\forall x \in \mathcal{E}$, $t \in \mathcal{J}$. In consequence, we have

$$\begin{aligned}
\|\mathcal{G}x\|_{3-\alpha} \quad = \quad & \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|M_1(t)\phi(x) + \int_0^T \Big[M_2(t)\frac{(T-s)^2}{2} + M_3(t)(T-s) + M_4(t)\Big] \\
& \times g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_5(t)\sum_{i=1}^m \omega_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds \\
& + M_6(t)\int_0^{\eta_1} \frac{(\eta_1 - s)^{2\alpha-3}}{\Gamma(2\alpha-2)} g(s, x(s), (\mu_1x)(s), (\mu_2x)(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+M_7(t)\sum_{j=1}^{n}\sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_8(t)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\Bigg|\Bigg\}\\
\leq\ &\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[|M_1(t)|N_\phi+N_g\int_0^T\Big[|M_2(t)|\frac{(T-s)^2}{2}+|M_3(t)|(T-s)+|M_4(t)|\Big]ds\\
&+N_g|M_5(t)|\sum_{i=1}^{m}|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds+N_g|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}ds\\
&+N_g|M_7(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}ds+N_g|M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}ds+\\
&+N_g\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg]\Bigg\}\\
\leq\ &N_\phi\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_1(t)|+N_g\Big[\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_2(t)|\frac{T^3}{6}+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_3(t)|\frac{T^2}{2}\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_4(t)|T\Big]+N_g\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_5(t)|)\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\\
&+N_g\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_6(t)|\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}+N_g\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_7(t)|\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}\\
&+N_g\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_8(t)|\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}+N_g\sup_{t\in\mathcal{J}}\left\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}\\
&=N_\phi\delta_1+\Big[\delta_2\frac{T^3}{6}+\delta_3\frac{T^2}{2}+\delta_4T\ +\delta_5\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}+\delta_6\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}\\
&+\delta_7\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}+\delta_8\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{T^3}{\Gamma(\alpha+1)}\Big]N_g,
\end{aligned}
$$

where

$$\delta_j=\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_j(t)|,\quad j=1,2,\ldots,8. \tag{3.2}$$

Thus, it follows that $\mathcal{G}(\mathcal{E})<\infty$. Hence $\mathcal{G}(\mathcal{E})$ is uniformly bounded. To show that $\mathcal{G}(\mathcal{E})$ is equicontinuous, we take $\nu_1,\nu_2\in\mathcal{J}$ with $\nu_1<\nu_2$ and $x\in\mathcal{E}$. Then, we obtain

$$
\begin{aligned}
&|\nu_2^{3-\alpha}(\mathcal{G}x)(\nu_2)-\nu_1^{3-\alpha}(\mathcal{G}x)(\nu_1)|\\
=\ &\Bigg|\nu_2^{3-\alpha}\Bigg[M_1(\nu_2)\phi(x)+\int_0^T\Big[M_2(\nu_2)\frac{(T-s)^2}{2}+M_3(\nu_2)(T-s)+M_4(\nu_2)\Big]\\
&\times g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_5(\nu_2)\sum_{i=1}^{m}\omega_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_6(\nu_2)\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+M_7(\nu_2)\sum_{j=1}^{n}\sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_8(\nu_2)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+\int_0^{\nu_2}\frac{(\nu_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\Bigg]\\
&-\nu_1^{3-\alpha}\Bigg[M_1(\nu_1)\phi(x)+\int_0^T\left[M_2(\nu_1)\frac{(T-s)^2}{2}+M_3(\nu_1)(T-s)+M_4(\nu_1)\right]\\
&\times g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_5(\nu_1)\sum_{i=1}^{m}\omega_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_6(\nu_1)\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_7(\nu_1)\sum_{j=1}^{n}\sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+M_8(\nu_1)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\\
&+\int_0^{\nu_1}\frac{(\nu_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\Bigg]\Bigg|\\
\le\ & N_\phi|\nu_2^{3-\alpha}M_1(\nu_2)-\nu_1^{3-\alpha}M_1(\nu_1)|+\Bigg[|\nu_2^{3-\alpha}M_2(\nu_2)-\nu_1^{3-\alpha}M_2(\nu_1)|\frac{T^3}{6}\\
&+|\nu_2^{3-\alpha}M_3(\nu_2)-\nu_1^{3-\alpha}M_3(\nu_1)|\frac{T^2}{2}+|\nu_2^{3-\alpha}M_4(\nu_2)-\nu_1^{3-\alpha}M_4(\nu_1)|T\\
&+|\nu_2^{3-\alpha}M_5(\nu_2)-\nu_1^{3-\alpha}M_5(\nu_1)|\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}+|\nu_2^{3-\alpha}M_6(\nu_2)-\nu_1^{3-\alpha}M_6(\nu_1)|\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}\\
&+|\nu_2^{3-\alpha}M_7(\nu_2)-\nu_1^{3-\alpha}M_7(\nu_1)|\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}+|\nu_2^{3-\alpha}M_8(\nu_2)-\nu_1^{3-\alpha}M_8(\nu_1)|\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}\\
&+\frac{1}{\Gamma(\alpha+1)}|\nu_2^3-\nu_1^3|+\frac{2}{\Gamma(\alpha+1)}\nu_2^{3-\alpha}(\nu_2-\nu_1)^\alpha\Bigg]N_g\\
=\ & \frac{N_\phi}{|\Delta|}\Big[|B_2E_3+B_3E_2||\nu_2^2-\nu_1^2|+|B_1E_3-B_3E_1||\nu_2-\nu_1|\Big]\\
&+\frac{N_g|\nu_2-\nu_1|}{|\Delta|}\Bigg[\frac{|a_1|T^3}{6}|B_1E_3-B_3E_1|+\frac{|a_2|T^2}{2}|A_1E_3+A_3E_1|\\
&+|a_3|T|A_1B_3+A_3B_1|+|A_1E_3+A_3E_1|\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\\
&+|A_1E_3+A_3E_1||v_1|\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}+|A_1B_3+A_3B_1|\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}\\
&+|A_1B_3+A_3B_1||v_2|\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}\Bigg]
\end{aligned}
$$

$$+\frac{N_g|\nu_2^2-\nu_1^2|}{|\Delta|}\Bigg[\frac{|a_1|T^3}{6}|B_2E_3+B_3E_2|+\frac{|a_2|T^2}{2}|A_3E_2-A_2E_3|$$
$$+|a_3|T|A_2B_3+A_3B_2|+|A_2E_3-A_3E_2|\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}$$
$$+|A_2E_3-A_3E_2||v_1|\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}+|A_2B_3+A_3B_2|\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}$$
$$+|A_2B_3+A_3B_2||v_2|\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}\Bigg]+\frac{N_g}{\Gamma(\alpha+1)}|\nu_2^3-\nu_1^3|+\frac{2N_g}{\Gamma(\alpha+1)}\nu_2^{3-\alpha}(\nu_2-\nu_1)^\alpha, \tag{3.3}$$

which tends to zero as $\nu_2\to\nu_1$ independent of $x\in\mathcal{E}$. Thus, $\mathcal{G}(\mathcal{E})$ is equicontinuous. In view of the forgoing arguments, it follows that $\mathcal{G}$ is completely continuous. This completes the proof. □

Now, we present our first existence result, which is based on the Leray-Schauder's alternative [38, Theorem 2.4, p.4].

Theorem 3.1 *Let $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, $\phi\in C(\mathcal{J},\mathbb{R})$ and there exist positive constants N_g, N_ϕ such that $|g(t,x(t),(\mu_1x)(t),(\mu_2x)(t))|\le N_g$, $|\phi(x)|\le N_\phi$, $\forall x\in\mathbb{R}$, $t\in\mathcal{J}$. Then, the problem (1.1) − (1.2) has at least one solution on $\mathcal{J}$.*

By Lemma 3.1, we know that $\mathcal{G}$ is completely continuous. So, the conclusion of the Leray-Schauder alternative will be applicable once it is shown that the set $\mathcal{W}=\{t^{3-\alpha}x\in\mathbb{R}:t^{3-\alpha}x=\tau t^{3-\alpha}\mathcal{G}x, 0<\tau<1\}$ is bounded. For $x\in\mathcal{W}$, we have $|t^{3-\alpha}x(t)|=|\tau t^{3-\alpha}\mathcal{G}x(t)|<t^{3-\alpha}|\mathcal{G}x(t)|$. Using the arguments employed in the proof of Lemma 3.1, we obtain

$$\|x\|_{3-\alpha} < N_\phi\delta_1+\Bigg[\delta_2\frac{T^3}{6}+\delta_3\frac{T^2}{2}+\delta_4T+\delta_5\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}+\delta_6\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}$$
$$+\delta_7\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)}+\delta_8\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{T^3}{\Gamma(\alpha+1)}\Bigg]N_g<\infty, \tag{3.4}$$

which implies that the set $\mathcal{W}$ is bounded. Thus, by Leray-Schauder's alternative, we deduce that the operator $\mathcal{G}$ has at least one fixed point, which is indeed a solution to the problem (1.1)-(1.2).

Theorem 3.2 *Let $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous functions and $\phi\in C(\mathcal{J},\mathbb{R})$. In addition, the following conditions hold:*

(H_1) *There exist positive functions $L_1(t),L_2(t),L_3(t)$ such that*

$$|g(t,x(t),(\mu_1x)(t),(\mu_2x)(t))-g(t,y(t),(\mu_1y(t),(\mu_2y)(t))|$$
$$\le L_1(t)|x-y|+L_2(t)|\mu_1x-\mu_1y|+L_3(t)|\mu_2x-\mu_2y|, \forall t\in\mathcal{J},\ x,y\in\mathbb{R};$$

(H_2) *$\phi(0)=0$ and there exists a positive constant L such that*

$$|\phi(x)-\phi(y)|\le L\|x-y\|_{3-\alpha},\quad \forall x,y\in\mathbb{R}.$$

Then, the problem (1.1)-(1.2) has a unique solution on $\mathcal{J}$, provided that

$$\Lambda = L\delta_1+\left(1+\frac{\varphi_1}{\alpha-2}+\frac{\varphi_2}{\alpha-2}\right)\Bigg(\delta_2e_1+\delta_3e_2+\delta_4e_3+\delta_5\sum_{i=1}^{m}|\omega_i|e_{4i}+\delta_6e_5$$
$$+\delta_7\sum_{j=1}^{n}|\sigma_j|e_{6j}+\delta_8e_7+T^{3-\alpha}e_8\Bigg)<1, \tag{3.5}$$

where $\delta_m, m=1,2,...,8$, are given in (3.2) and

$\varphi_1 = \sup_{t,s\in\mathcal{J}} |\Phi_1(t,s)|, \quad \varphi_2 = \sup_{t,s\in\mathcal{J}} |\Phi_2(t,s)|,$

$e_1 = \max\{|I^3 L_1(T)T^{\alpha-3}|, |I^3 L_2(T)T^{\alpha-2}|, |I^3 L_3(T)T^{\alpha-2}|\},$

$e_2 = \max\{|I^2 L_1(T)T^{\alpha-3}|, |I^2 L_2(T)T^{\alpha-2}|, |I^2 L_3(T)T^{\alpha-2}|\},$

$e_3 = \max\{|I L_1(T)T^{\alpha-3}|, |I L_2(T)T^{\alpha-2}|, |I L_3(T)T^{\alpha-2}|\},$

$e_{4i} = \max\{|I^\alpha L_1(\xi_i)\xi_i^{\alpha-3}|, |I^\alpha L_2(\xi_i)\xi_i^{\alpha-2}|, |I^\alpha L_3(\xi_i)\xi_i^{\alpha-2}|\},$

$e_5 = \max\{|I^{2\alpha-2} L_1(\eta_1)\eta_1^{\alpha-3}|, |I^{2\alpha-2} L_2(\eta_1)\eta_1^{\alpha-2}|, |I^{2\alpha-2} L_3(\eta_1)\eta_1^{\alpha-2}|\},$

$e_{6j} = \max\{|I^\alpha L_1(\zeta_j)\zeta_j^{\alpha-3}|, |I^\alpha L_2(\zeta_j)\zeta_j^{\alpha-2}|, |I^\alpha L_3(\zeta_j)\zeta_j^{\alpha-2}|\},$

$e_7 = \max\{|I^{2\alpha-1} L_1(\eta_2)\eta_2^{\alpha-3}|, |I^{2\alpha-1} L_2(\eta_2)\eta_2^{\alpha-2}|, |I^{2\alpha-1} L_3(\eta_2)\eta_2^{\alpha-2}|\},$

$e_8 = \sup_{t\in\mathcal{J}}\{|I^\alpha L_1(t)t^{\alpha-3}|, |I^\alpha L_2(t)t^{\alpha-2}|, |I^\alpha L_3(t)t^{\alpha-2}|\},$

I^α denotes the Riemann-Liouville fractional integral operator of order α and

$$I^n\phi(t) = \frac{1}{\Gamma(n)}\int_0^t (t-s)^{n-1}\phi(s)ds, \quad n = 1,2,3.$$

Proof: For verifying the hypothesis of Banach's fixed point theorem, we consider a closed ball $\mathcal{S}_{\widehat{r}} = \{x \in C_{3-\alpha}(\mathcal{J},\mathbb{R}) : \|x\|_{3-\alpha} \le \widehat{r}\}$ with

$$\widehat{r} \ge (\overline{g}\ \overline{\delta})(1-\Lambda)^{-1}, \tag{3.6}$$

where $\sup_{t\in\mathcal{J}} |g(t,0,0,0)| = \overline{g}$,

$$\begin{aligned} \overline{\delta} &= \delta_2\frac{T^3}{6} + \delta_3\frac{T^2}{2} + \delta_4 T + \delta_5\sum_{i=1}^m |\omega_i|\frac{\xi_i^\alpha}{\Gamma(\alpha+1)} + \delta_6\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)} + \delta_7\sum_{j=1}^n |\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \delta_8\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{T^3}{\Gamma(\alpha+1)}, \end{aligned} \tag{3.7}$$

$\delta_m, m = 1,2,\dots,8$, are given in (3.2). Now, we establish that $\mathcal{G}\mathcal{S}_{\widehat{r}} \subset \mathcal{S}_{\widehat{r}}$, where $\mathcal{G} : \mathcal{S}_{\widehat{r}} \longrightarrow C_{3-\alpha}(\mathcal{J},\mathbb{R})$ is given by (3.1). By (H_1) and (H_2), we have

$$\begin{aligned} |g(t,x(t),(\mu_1x)(t),(\mu_2x)(t))| &\le |g(t,x(t),(\mu_1x)(s),(\mu_2x)(t)) - g(t,0,0,0)| + |g(t,0,0,0)| \\ &\le L_1(t)|x| + L_2(t)|\mu_1x| + L_3(t)|\mu_2x| + \overline{g}, \\ |\phi(x)| &\le |\phi(x)-\phi(0)| + |\phi(0)| \le L\|x\|_{3-\alpha}. \end{aligned} \tag{3.8}$$

For $x \in \mathcal{S}_{\widehat{r}}$, it follows by using (3.8) that

$$\begin{aligned} \|\mathcal{G}x\|_{3-\alpha} &= \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|M_1(t)\phi(x) + \int_0^T \Big[M_2(t)(\frac{(T-s)^2}{2} + M_3(t)(T-s) + M_4(t)\Big] \\ &\quad \times g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds \\ &\quad + M_5(t)\sum_{i=1}^m \omega_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds \\ &\quad + M_6(t)\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds \\ &\quad + M_7(t)\sum_{j=1}^n \sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds \\ &\quad + M_8(t)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))ds\Bigg|\Bigg\}\\
\leq\ & \sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[|M_1(t)|L\|x\|_{3-\alpha}+\int_0^T\Big[|M_2(t)|\frac{(T-s)^2}{2}+|M_3(t)|(T-s)\\
&+|M_4(t)|\Big]\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+|M_5(t)|\sum_{i=1}^m|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+|M_7(t)|\sum_{j=1}^n|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+|M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\Bigg]\Bigg\}\\
\leq\ & \sup_{t\in\mathcal{J}}|t^{3-\alpha}M_1(t)|L\|x\|_{3-\alpha}+\int_0^T\Big[\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_2(t)|\frac{(T-s)^2}{2}+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_3(t)|(T-s)\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_4(t)|\Big]\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_5(t)|\sum_{i=1}^m|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_7(t)|\sum_{j=1}^n|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\sup_{t\in\mathcal{J}}|t^{3-\alpha}M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\\
&+\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x|+L_2(s)|\mu_1x|+L_3(s)|\mu_2x|+\overline{g}\Big)ds\Bigg\}\\
\leq\ & \delta_1L\|x\|_{3-\alpha}+\int_0^T\Big[\delta_2\frac{(T-s)^2}{2}+\delta_3(T-s)+\delta_4\Big]\\
&\times\Bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\Bigg)ds\\
&+\delta_5\sum_{i=1}^m|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times\Bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\Bigg)ds
\end{aligned}
$$

$$
\begin{aligned}
&+|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\\
&\times\bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1 s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\bigg)ds\\
&+|M_7(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times\bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1 s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\bigg)ds\\
&+|M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\\
&\times\bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1 s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\bigg)ds\\
&+\sup_{t\in\mathcal{J}}\bigg\{t^{3-\alpha}\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times\bigg(\Big(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1 s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 s^{\alpha-2}}{(\alpha-2)}\Big)\|x\|_{3-\alpha}+\overline{g}\bigg)ds\bigg\}\\
\leq\;&\bigg[L\delta_1+\Big(1+\frac{\varphi_1}{\alpha-2}+\frac{\varphi_2}{\alpha-2}\Big)\Big(\delta_2 e_1+\delta_3 e_2+\delta_4 e_3+\delta_5\sum_{i=1}^{m}|\omega_i|e_{4i}+\delta_6 e_5\\
&+\delta_7\sum_{j=1}^{n}|\sigma_j|e_{6j}+\delta_8 e_7+T^{3-\alpha}e_8\Big)\bigg]\widehat{r}+\overline{g}\bigg[\delta_2\frac{T^3}{6}+\delta_3\frac{T^2}{2}+\delta_4 T\\
&+\delta_5\sum_{i=1}^{m}|\omega_i|\frac{\xi_i^{\alpha}}{\Gamma(\alpha+1)}+\delta_6\frac{\eta_1^{2\alpha-2}}{\Gamma(2\alpha-1)}+\delta_7\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha+1)}\\
&+\delta_8\frac{\eta_2^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{T^3}{\Gamma(\alpha+1)}\bigg]\\
\leq\;&\Lambda\widehat{r}+\overline{g}\overline{\delta}. \qquad (3.9)
\end{aligned}
$$

Combining (3.9) with (3.6), we obtain

$$\|\mathcal{G}x\|_{3-\alpha}\leq\Lambda\widehat{r}+\overline{g}\;\overline{\delta}\leq\widehat{r},$$

which shows that $\mathcal{G}x\in\mathcal{S}_{\widehat{r}}$. Hence, $\mathcal{G}S_{\widehat{r}}\subset\mathcal{S}_{\widehat{r}}$ since $x\in\mathcal{S}_{\widehat{r}}$ is an arbitrary element.

Next, we verify that the operator $\mathcal{G}$ is a contraction. For that, let $x,y\in C_{3-\alpha}(\mathcal{J},\mathbb{R})$. Then, for any $t\in\mathcal{J}$, using (H_1) and the relation

$$
\begin{aligned}
&L_1(t)|x-y|+L_2(t)|(\lambda_1 x-\lambda_1 y|+L_3(t)|(\lambda_2 x-\lambda_2 y|\\
&\leq\Big(L_1(t)t^{\alpha-3}+L_2(t)\frac{\varphi_1 t^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 t^{\alpha-2}}{(\alpha-2)}\Big)\|x-y\|_{3-\alpha}, \qquad (3.10)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\|\mathcal{G}x-\mathcal{G}y\|_{3-\alpha}\\
\leq\;&\sup_{t\in\mathcal{J}}\bigg\{t^{3-\alpha}\bigg[|M_1(t)||\phi(x)-\phi(y)|+\int_0^{T}\Big[|M_2(t)|(\frac{(T-s)^2}{2}\\
&+|M_3(t)|(T-s)+|M_4(t)|\Big]|g(s,x(s),(\mu_1 x)(s),(\mu_2 x)(s))-g(s,y(s),(\mu_1 y)(s),(\mu_2 y)(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+|M_5(t)|\sum_{i=1}^{m}|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times|g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))-g(s,y(s),(\mu_1y)(s),(\mu_2y)(s))|ds\\
&+|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\\
&\times|g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))-g(s,y(s),(\mu_1y)(s),(\mu_2y)(s))|ds\\
&+|M_7(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&\times|g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))-g(s,y(s),(\mu_1y)(s),(\mu_2y)(s))|ds\\
&+|M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\\
&\times|g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))-g(s,y(s),(\mu_1y)(s),(\mu_2y)(s))|ds\\
&+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))-g(s,y(s),(\mu_1y)(s),(\mu_2y)(s))|ds\Bigg]\Bigg\}\\
\leq\ &\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[|M_1(t)|L\|x-y\|_{3-\alpha}+\int_0^{T}\Big[|M_2(t)|\frac{(T-s)^2}{2}+|M_3(t)|(T-s)+|M_4(t)|\Big]\\
&\times\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\\
&+|M_5(t)|\sum_{i=1}^{m}|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\\
&+|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\\
&+|M_7(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\\
&+|M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\\
&+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(L_1(s)|x-y|+L_2(s)|\mu_1x-\mu_1y|+L_3(s)|\mu_2x-\mu_2y|\Big)ds\Bigg]\Bigg\}\\
\leq\ &\delta_1L\|x-y\|_{3-\alpha}+\int_0^{T}\Big[\delta_2\frac{(T-s)^2}{2}+\delta_3(T-s)+\delta_4\Big]\\
&\times\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds\\
&+\delta_5\sum_{i=1}^{m}|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds\\
&+\delta_6\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds\\
&+\delta_7\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds\\
&+\delta_8\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds
\end{aligned}
$$

$$
\begin{aligned}
& +\sup_{t\in\mathcal{J}}\left\{t^{3-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(L_1(s)s^{\alpha-3}+L_2(s)\frac{\varphi_1 s^{\alpha-2}}{(\alpha-2)}+L_3(s)\frac{\varphi_2 s^{\alpha-2}}{(\alpha-2)}\right)\|x-y\|_{3-\alpha}ds\right\}\\
\leq\ & \Lambda\|x-y\|_{3-\alpha},
\end{aligned}
$$

which, by the condition (3.5) (that is, $\Lambda<1$), shows that the operator $\mathcal{G}$ is a contraction. Thus, the hypothesis of Banach's fixed point theorem is verified and hence its conclusion implies that the operator $\mathcal{G}$ has a unique fixed point. Therefore, there exists a unique solution to the problem (1.1)-(1.2) on $\mathcal{J}$. The proof is finished. □

As a special case of Theorem 3.2, by taking $\Phi_1(t,s)=\dfrac{(t-s)^{p-1}}{\Gamma(p)}$, $\Phi_2(t,s)=\dfrac{(t-s)^{q-1}}{\Gamma(q)}$, $p,q>0$, in (1.1), we get a nonlinear fractional differential equation involving both Riemann-Liouville derivative and integral operators of the form

$$D^{\alpha}x(t)=g(t,x(t),I^{p}x(t),I^{q}x(t)). \tag{3.11}$$

Now we present a uniqueness result for fractional differential equation (3.11) subject to the boundary conditions (1.2).

Theorem 3.3 *Let $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous functions and $\phi\in C(\mathcal{J},\mathbb{R})$. In addition, (H_2) and the following condition hold:*

(H_3) *There exist positive functions $\widehat{w}_1(t)$, $\widehat{w}_2(t)$ and $\widehat{w}_3(t)$*

$$
\begin{aligned}
& |g(t,x(t),(I^px)(t),(I^qx)(t))-g(t,y(t),(I^py)(t),(I^qy)(t))|\\
\leq\ & \widehat{w}_1(t)|x-y|+\widehat{w}_2(t)I^p|x-y|+\widehat{w}_3(t)I^q|x-y|,\quad \forall\ t\in\mathcal{J},x,y\in\mathbb{R}.
\end{aligned} \tag{3.12}
$$

Then, the Riemann-Liuoville fractional differential equation (3.11) subject to the boundary conditions (1.2) has a unique solution on $\mathcal{J}$, provided that

$$
\begin{aligned}
\Lambda_1 =\ & L\delta_1+\left(1+\frac{\Gamma(\alpha-2)}{\Gamma(p+\alpha-2)}+\frac{\Gamma(\alpha-2)}{\Gamma(q+\alpha-2)}\right)\left(\delta_2k_1+\delta_3k_2+\delta_4k_3+\delta_5\sum_{i=1}^{m}|\omega_i|k_{4i}\right.\\
& \left.+\delta_6k_5+\delta_7\sum_{j=1}^{n}|\sigma_j|k_{6j}+\delta_8k_7+T^{3-\alpha}k_8\right)<1,
\end{aligned} \tag{3.13}
$$

where $\delta_m, m=1,2,...,8$, are given in (3.2),

$$k_1=\max\{|I^3\widehat{w}_1(T)T^{\alpha-3}|,|I^3\widehat{w}_2(T)T^{p+\alpha-3}|,|I^3\widehat{w}_3(T)T^{q+\alpha-3}\},$$

$$k_2=\max\{|I^2\widehat{w}_1(T)T^{\alpha-3}|,|I^2\widehat{w}_2(T)T^{p+\alpha-3}|,|I^2\widehat{w}_3(T)T^{q+\alpha-3}|\},$$

$$k_3=\max\{|I\widehat{w}_1(T)T^{\alpha-3}|,|I\widehat{w}_2(T)T^{p+\alpha-3}|,|I\widehat{w}_3(T)T^{q+\alpha-3}|\},$$

$$k_{4,i}=\max\{|I^{\alpha}\widehat{w}_1(\xi_i)\xi_i^{\alpha-3}|,|I^{\alpha}\widehat{w}_2(\xi_i)\xi_i^{p+\alpha-3}|,|I^{\alpha}\widehat{w}_3(\xi_i)\xi_i^{q+\alpha-3}|\},$$

$$k_5=\max\{|I^{2\alpha-2}\widehat{w}_1(\eta_1)\eta_1^{\alpha-3}|,|I^{2\alpha-2}\widehat{w}_2(\eta_1)\eta_1^{p+\alpha-3}|,|I^{2\alpha-2}\widehat{w}_3(\eta_1)\eta_1^{p+\alpha-3}|\},$$

$$k_{6,j}=\max\{|I^{\alpha}\widehat{w}_1(\zeta_j)\zeta_j^{\alpha-3}|,|I^{\alpha}\widehat{w}_2(\zeta_j)\zeta_j^{p+\alpha-3}|,|I^{\alpha}\widehat{w}_3(\zeta_j)\zeta_j^{q+\alpha-3}|\},$$

$$k_7=\max\{|I^{2\alpha-1}\widehat{w}_1(\eta_2)\eta_2^{\alpha-3}|,|I^{2\alpha-1}\widehat{w}_2(\eta_2)\eta_2^{p+\alpha-3}|,|I^{2\alpha-1}\widehat{w}_3(\eta_2)\eta_2^{q+\alpha-3}|\},$$

$$k_8=\sup_{t\in\mathcal{J}}\{|I^{\alpha}\widehat{w}_1(t)t^{\alpha-3}|,|I^{\alpha}\widehat{w}_2(t)t^{p+\alpha-3}|,|I^{\alpha}\widehat{w}_3(t)t^{q+\alpha-3}|\},$$

Proof: We omit the proof as it is similar to that of Theorem 3.2. □

4. Examples

In this section, we present examples illustrating the results obtained in the last section.

Example 4.1 *Let us consider the nonlinear fractional differential equation*

$$D^{\alpha}x(t) = g(t, x(t), (\mu_1 x)(t), (\mu_2 x)(t)), \quad t \in [0,1], \tag{4.1}$$

subject to the boundary conditions

$$\begin{cases} D^{\alpha-3}x(0^+) + a_1 D^{\alpha-3}x(T^-) = \phi(x), \\ D^{\alpha-2}x(0^+) + a_2 D^{\alpha-2}x(T^-) = v_1 I^{\alpha-2}x(\eta_1) + \sum_{i=1}^{m} \omega_i x(\xi_i), \\ D^{\alpha-1}x(0^+) + a_3 D^{\alpha-1}x(T^-) = v_2 I^{\alpha-1}x(\eta_2) + \sum_{j=1}^{n} \sigma_j x(\zeta_j). \end{cases} \tag{4.2}$$

Here, $\alpha = \dfrac{8}{3}$, $a_1 = \dfrac{1}{4}, a_2 = \dfrac{-3}{4}, a_3 = \dfrac{1}{2}, v_1 = -2, v_2 = -2, m = 3, n = 3, \omega_i = -3, i = 1,2,3, \sigma_j = -3, j = 1,2,3, \eta_1 = \dfrac{1}{4}, \eta_2 = \dfrac{1}{2}, \xi_1 = \dfrac{2}{3}, \xi_2 = \dfrac{3}{4}, \xi_3 = \dfrac{4}{5}, \zeta_1 = \dfrac{5}{6}, \zeta_2 = \dfrac{6}{7}, \zeta_3 = \dfrac{7}{8}, \mathcal{J} = [0,1], T = 1, \phi(x) = \dfrac{1}{30}\sin x$, *and*

$$g(t, x(t), (\mu_1 x)(t), (\mu_2 x)(t)) = \frac{e^x}{\sqrt{t^3+64}} + \int_0^t \frac{5^{(s-t)}}{300} x(s)ds + \int_0^t \frac{\cos(t-s)}{(t^2+120)} x(s)ds.$$

Using the given data, we find that $\delta_1 \approx 0.22937050, \delta_2 \approx 0.05734262, \delta_3 \approx 0.02303515, \delta_4 \approx 0.04927213, \delta_5 \approx 0.03071354, \delta_6 \approx 0.06142707, \delta_7 \approx 0.09854425, \delta_8 \approx 0.19708851, \psi_1 \approx 0.01666667$, $\psi_2 \approx 0.00833333$, $e_1 \approx 0.06136364, e_2 \approx 0.22500000, e_3 \approx 0.60000000, e_{4,1} \approx 0.02523240, e_{4,2} \approx 0.03736517$, $e_{4,3} \approx 0.04633361$, $e_5 \approx 0.00044079$, $e_{6,1} \approx 0.05308746$, $e_{6,2} \approx 0.05831409$, $e_{6,3} \approx 0.06246302$, $e_7 \approx 0.00044079$, $e_8 \approx 0.09748313$.
It is easy to check that (H_1) is satisfied with $L_1(t) = \frac{1}{\sqrt{t^3+64}}$, $L_2(t) = L_3(t) = 1$, the assumption (H_2) holds true with $L = 1/30$, and the condition (3.5) is satisfied as $\Lambda \approx 0.21234347 < 1$. As the hypotheses of Theorem 3.2 holds true, so its conclusion implies that the boundary value problem (4.1)-(4.2) has a unique solution on $[0,1]$.

Example 4.2 *Consider the nonlinear fractional integro-differential equation*

$$D^{\alpha}x(t) = g(t, x(t), (\mu_1 x)(t), (\mu_2 x)(t)), \quad t \in [0,1], \tag{4.3}$$

subject to the boundary conditions in (4.2), *where* $\alpha = 8/3, \phi(x) = \frac{1}{20}\sin x$, *and*

$$g(t, x(t), (\mu_1 x)(t), (\mu_2 x)(t)) = \frac{\tan^{-1} x(t)}{t+25} + \frac{1}{8}\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} x(s)ds + \frac{1}{40}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x(s)ds,$$

with $p = 4/5$, $q = 3/5$.

Using the given data, it is found $\delta_1 \approx 0.22937050, \delta_2 \approx 0.05734262, \delta_3 \approx 0.02303515, \delta_4 \approx 0.04927213, \delta_5 \approx 0.03071354, \delta_6 \approx 0.06142707, \delta_7 \approx 0.09854425$, $\delta_8 \approx 0.19708851, k_1 \approx 0.01340320$, $k_2 \approx 0.03564640, k_3 \approx 0.08522727, k_{4,1} \approx 0.00752976, k_{4,2} \approx 0.00990494, k_{43} \approx 0.01151017, k_5 \approx 0.00014082$, $k_{6,1} \approx 0.01265712$, $k_{6,2} \approx 0.01351453, k_{6,3} \approx 0.01417864, k_7 \approx 0.00014068, k_8 \approx 0.01934301$. Clearly, the condition (H_2) is satisfied with $L = 1/20$, and the assumption (H_3) holds true with $\varpi_1 = \frac{1}{(t+25)}, \varpi_2 = \frac{1}{8}, \varpi_3 = \frac{1}{40}$ and $\Lambda_1 \approx 0.17165453 < 1$. Thus, by the conclusion of Theorem 3.3, the equation (4.3) with the boundary conditions (4.2) has a unique solution on $[0,1]$.

5. Stability Analysis

Let us first build the arguments for the Ulam–Hyers stability [39] of the problem (1.1)-(1.2).

For $\epsilon > 0$ and $t \in \mathcal{J}$, it is assumed that there exists $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ satisfying the following inequality with the boundary conditions (1.2)

$$\Big| D^{\alpha}u(t) - g(t, u(t), (\mu_1 u)(t)(\mu_2 u)(t)) \Big| \leq \epsilon. \tag{5.1}$$

Furthermore, $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ is a solution of the system of inequality (5.1) with the boundary conditions (1.2) if and only if there exists a function $\kappa \in C(\mathcal{J}, \mathbb{R})$ such that $|\kappa(t)| \leq \epsilon$, $t \in \mathcal{J}$ and

$$D^{\alpha}u(t) = g(t, u(t), (\mu_1 u)(t)(\mu_2 u)(t)) + \kappa(t).$$

Next, we consider a boundary value problem associated with (5.1) and the boundary conditions (1.2) as

$$\begin{cases} D^{\alpha}u(t) = g(t, u(t), (\mu_1 u)(t)(\mu_2 u)(t)) + \kappa(t), \quad t \in \mathcal{J}, \\ D^{\alpha-3}u(0^+) + a_1 D^{\alpha-3}u(T^-) = \phi(u), \\ D^{\alpha-2}u(0^+) + a_2 D^{\alpha-2}u(T^-) = v_1 I^{\alpha-2}u(\eta_1) + \sum_{i=1}^{m} \omega_i u(\xi_i), \\ D^{\alpha-1}u(0^+) + a_3 D^{\alpha-1}u(T^-) = v_2 I^{\alpha-1}u(\eta_2) + \sum_{j=1}^{n} \sigma_j u(\zeta_j). \end{cases} \tag{5.2}$$

Definition 5.1 The system (1.1) -(1.2) is called Ulam–Hyers stable if we can find $c_g > 0$, such that, for each solution $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of (5.2), there exists a unique solution $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of the system (1.1) satisfying

$$\|u - x\|_{3-\alpha} \leq c\epsilon, \quad t \in \mathcal{J}.$$

Definition 5.2 If there exists $\Psi \in C(\mathbb{R}^+, \mathbb{R}^+)$, with $\Psi(0) = 0$, such that, for each solution $u \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of (5.2), there exists a unique solution $x \in C_{3-\alpha}(\mathcal{J}, \mathbb{R})$ of the problem (1.1)–(1.2) satisfying

$$\|u - x\|_{3-\alpha} \leq \Psi(\epsilon), \quad t \in \mathcal{J}.$$

Then, the problem (1.1)–(1.2) is generalized Ulam–Hyers stable.

Theorem 5.1 *If the assumption (H_1)-(H_2) and the condition (3.5) are satisfied, then the problem (1.1)-(1.2) is Ulam–Hyers stable and hence generalized Ulam–Hyers stable in $C_{3-\alpha}(\mathcal{J}, \mathbb{R})$.*

Proof: By Lemma 2.2, the solution of (5.2) can be written as

$$\begin{aligned} u(t) \;=\; & M_1(t)\phi(u) \\ & + \int_0^T \Big[M_2(t)(\frac{(T-s)^2}{2} + M_3(t)(T-s) + M_4(t) \Big] \Big(g(s, u(s), (\mu_1 u)(s), (\mu_2 u)(s)) + \kappa(s) \Big) ds \\ & + M_5(t) \sum_{i=1}^{m} \omega_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \Big(g(s, u(s), (\mu_1 u)(s), (\mu_2 u)(s)) + \kappa(s) \Big) ds \\ & + M_6(t) \int_0^{\eta_1} \frac{(\eta_1 - s)^{2\alpha-3}}{\Gamma(2\alpha-2)} \Big(g(s, u(s), (\mu_1 u)(s), (\mu_2 u)(s)) + \kappa(s) \Big) ds \\ & + M_7(t) \sum_{j=1}^{n} \sigma_j \int_0^{\zeta_j} \frac{(\zeta_j - s)^{\alpha-1}}{\Gamma(\alpha)} \Big(g(s, u(s), (\mu_1 u)(s), (\mu_2 u)(s)) + \kappa(s) \Big)) ds \\ & + M_8(t) \int_0^{\eta_2} \frac{(\eta_2 - s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \Big(g(s, u(s), (\mu_1 u)(s), (\mu_2 u)(s)) + \kappa(s) \Big) ds \end{aligned}$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(g(s,u(s),(\mu_1 u)(s),(\mu_2 u)(s))+\kappa(s)\Big)ds.$$

Using $|\kappa|<\epsilon$ and (3.7), we get

$$\begin{aligned}
&\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg|u(t)-M_1(t)\phi(u)\\
&-\int_0^T\Big[M_2(t)\frac{(T-s)^2}{2}+M_3(t)(T-s)+M_4(t)\Big]g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\\
&-M_5(t)\sum_{i=1}^m\omega_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\\
&-M_6(t)\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\\
&-M_7(t)\sum_{j=1}^n\sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\\
&-M_8(t)\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\\
&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))ds\Bigg|\Bigg\}\le\overline{\delta}\epsilon.
\end{aligned}$$

In view of (H_1), (H_2), (3.5) and (3.10), we find that

$$\begin{aligned}
&\|u-x\|_{3-\alpha}=\sup_{t\in\mathcal{J}}\{t^{3-\alpha}|u(t)-x(t)|\}\\
&\le\overline{\delta}\epsilon+\sup_{t\in\mathcal{J}}\Bigg\{t^{3-\alpha}\Bigg[|M_1(t)||\phi(u)-\phi(x)|\\
&+\int_0^T\Big[|M_2(t)|\frac{(T-s)^2}{2}+|M_3(t)|(T-s)+|M_4(t)|\Big]\\
&\times|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\\
&+|M_5(t)|\sum_{i=1}^m|\omega_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\\
&+|M_6(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)}|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\\
&+|M_7(t)|\sum_{j=1}^n||\sigma_j\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\\
&+M_8(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,u(s),(\mu_1u)(s),(\mu_2u)(s))-g(s,x(s),(\mu_1x)(s),(\mu_2x)(s))|ds\Bigg]\Bigg\}\\
&\le\ \overline{\delta}\epsilon+\Lambda\|u-x\|_{3-\alpha},
\end{aligned}$$

which implies that

$$\|u-x\|_{3-\alpha}\le\frac{\overline{\delta}\epsilon}{1-\Lambda}.$$

Letting $c = c_g = \dfrac{\overline{\delta}}{1-\Lambda}$, we get $\|u - x\|_{3-\alpha} \leq c\epsilon$. Hence, the problem (1.1)–(1.2) is Ulam–Hyers stable. Moreover, it is generalized Ulam– Hyers stable as $\|u - x\|_{3-\alpha} \leq \Psi(\epsilon)$, with $\Psi(\epsilon) = c\epsilon, \quad \Psi(0) = 0$. This completes the proof. □

Example 5.1 *The problem* (4.1) *and* (4.3) *with the boundary conditions* (4.2) *are Ulam–Hyers stable, and generalized Ulam–Hyers stable since* $\Lambda \approx 0.21234347 < 1$, *and* $\Lambda_1 \approx 0.17165453 < 1$, *respectively.*

6. Conclusion

We have obtained the existence and uniqueness criteria for solutions of a nonlinear nonlocal Riemann–Liouville integral and multi-point boundary value problem. The nonlinearity in Eq. (1.1) involves classical integrals, while Eq. (3.11) involves Riemann–Liouville fractional integrals. Our results are useful in the given setting and yield several new results as special cases. For instance, our results correspond to the ones with nonlocal multi-point boundary conditions for $v_1 = v_2 = 0$, whereas the choice $\omega_i = \sigma_j = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ produces the results for nonlocal Riemann–Liouville integral boundary conditions.

Conflict of interest. The authors declare that they have no conflict of interest.

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