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Study of Dynamical System and Topological Transitivity with Ideals

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ABSTRACT: Various expression of limit points of a set can be composed by the Kuratowski's ideal. Due to these expression, transitivity and non-wandering point may be comprised by the ideal. These will be different when ideal will be different types. Before considering the ideal version transitivity and non-wandering point, the paper discussed the equivalent definition of transitivity as well as non-wandering point. It is fact that the study of transitivity and non-wandering point contain dense set rigorously. Thus, for the study of ideal related transitivity and non-wandering point, Remark 6.9(2) of [23] (published in Filomat, Vol. 27, Issue 2) is countered through this paper.

Key Words: transitivity, ideal-transitivity, complete metric space, dense orbit, I-dense orbit.

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1. Introduction

Let X be a space (metric space or topological space). Then a continuous map $f : X \to X$ together with the X (i.e., (X, f)) is called a dynamical system. As we know $f \circ f(i.e., f^2)$ is also continuous when f is continuous. This point of view, the characteristics of the set $\{x_0, f(x_0), f^2(x_0), f^3(x_0), ...\}$ when $x_0 \in X$ is an important phenomenon. In literature, $\{x_0, f(x_0), f^2(x_0), f^3(x_0), ...\}$ is called the orbit [7] of x_0. If it is dense in X, then it is called dense orbit [20] of x_0. For the study of the dense property of the set $\{x_0, f(x_0), f^2(x_0), f^3(x_0), ...\}$ where $x_0 \in X$, $f^n(U) \cap V \neq \varnothing$, for any nonempty open sets U and V in X is remarkable condition. This was introduced in literature by the name of topological transitive [20]. The authors Kolyada and Snoha have studied it at various angles. For this study, non-wandering point [8] is another noteworthy part. For further studies, the papers [3,4,6,7,8,28,29,34,35] and others are relevant. In view of the above mentioned points, we are investigating the answer of the following questions through this paper:

- Does any set characterize the topological transitivity?
- For any codense ideal **I**, are the collection of dense sets and collection of **I** dense sets coincident?
- Is the iteration $\{x_0, f(x_0), f^2(x_0), f^3(x_0), ...\}$ **I**-dense (respectively, $*$-dense) for any $x_0 \in X$?
- Under what condition(s) non-wandering points and ideal related non-wandering points are equal?

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To solve the above questions, we will give counter example for a result of [23].

2. Definitions and Notations

- $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}_+$ denote the set of reals, set of rationals and set of positive integers respectively.
- I be a countable set.
- (X,τ) (or simlpy X) be a topological space.
- $\mathcal{B}$ be a basis for the topological space X. Members of $\mathcal{B}$ are called basic open sets.
- Throughout this paper, we denote 'Cl' and 'Int' as the closure operator and interior operator respectively.
- If X is a topological space and $\mathbf{A} \subseteq X$, then $\mathbf{A}$ is said to be semi-open [22] (respectivey, preopen [24], b-open [5] and β-open [1]) if $\mathbf{A} \subseteq Cl(Int(\mathbf{A}))$ (respectively, $\mathbf{A} \subseteq Int(Cl(\mathbf{A}))$, $\mathbf{A} \subseteq Cl(Int(\mathbf{A})) \cup Int(Cl(\mathbf{A}))$ and $\mathbf{A} \subseteq Cl(Int(Cl(\mathbf{A})))$). The set of all semi-open sets (respectively, preopen sets, b-open sets and β-open sets) in a topological space (X,τ) is denoted as $SO(X)$ (respectively, $PO(X)$, $BO(X)$ and $\beta O(X)$), then the relations $\tau \subseteq SO(X) \subseteq BO(X) \subseteq \beta O(X)$ and $\tau \subseteq PO(X) \subseteq BO(X) \subseteq \beta O(X)$ hold. Let $\mathbf{A} \subseteq \mathbf{X}$, then the intersection of all semi (respectively, pre, b, β) open sets containing $\mathbf{A}$ is called semi (respectively, pre, b, and β) closure of $\mathbf{A}$. Throughout this paper, for a topological space X, we denote 'scl' (respectively, 'pcl', 'bcl' and 'βcl') as the semi (respectively, pre, b, and β) closure operator.
- An open set $\mathbf{O}$ in a topological space X is said to be regular open [35] if $\mathbf{O} = Int(Cl(\mathbf{O}))$.
- A subset $\mathbf{A}$ of a topological space X is called a dense set [35] in X if $Cl(\mathbf{A}) = X$. Equivalently, $\mathbf{A}$ is dense in X if and only if $\mathbf{A}$ intersects every nonempty open subset of X.
- For every pair of nonempty open sets U, V in X, there exists a positive integer n such that $f^n(U) \cap V \neq \varnothing$, then (X,f) is called topologically transitive [20]. We also say that f itself is topologically transitive if no misunderstanding can arise.
- A dynamical system (X,f) is called K-topologically transitive if for every pair of nonempty K-open sets A and B, there exists a positive integer n such that $f^n(A) \cap B \neq \varnothing$. In this case, K may be the semi-open set [22], preopen set [24], b-open set [5] or β-open set [1].
- A point $x \in X$ is said to be non-wandering if every neighbourhood U of x there exists a positive integer n such that $f^n(U) \cap U \neq \varnothing$ [8]. The collection of all non-wandering points of f will be denoted by $\Omega(f)$.
- A point $x \in X$ is said to be semi (respectively, pre, b, β) non-wandering if every semi (respectively, pre, b, β) open set U of x there exists a positive integer n such that $f^n(U) \cap U \neq \varnothing$.
- A point $x \in X$ "moves" its trajectory being the sequence $x, f(x), f^2(x), f^3(x), ...$ where f^n is the n^{th} iteration of f . The point $f^n(x)$ is the position of x after n units of time. The set of points of the trajectory of x under f is called the orbit of x, denoted by $orb_f(x)$(or simply $(O_f(x))$) [20].
- A dynamical system (X,f) has a dense orbit if there exists a point $x \in X$ such that the orbit $\{x, f(x), f^2(x), f^3(x), ...\}$ is dense in X [20].
- Ideal: According to Kuratowski [21], a collection $\mathbf{I} \subseteq 2^X$ (power set of X) is called an ideal on X if $\mathbf{I}$ is closed under hereditary property and finite additivity property. If $X \notin \mathbf{I}$, then $\mathbf{I}$ is called proper ideal. Due to the ideal $\mathbf{I}$ on a topological space (X,τ), Kuratowski's local function is, $A^* = \{x \in X : U_x \cap A \notin \mathbf{I}\ \forall U_x \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. Its associated set valued set function [30] (or complementary operator is ψ [32] and it is defined as, $\psi(A) = X \setminus (X \setminus A)^*$ [32], we will write simply $\psi_\tau(M)$ or $\psi_\tau^{\mathbf{I}}(M)$). $Cl^*(A) = A \cup A^*$ determines a Kuratowski closure operator for a topology $\tau^*(\mathbf{I}) = \{A \subseteq X : Cl^*(X \setminus A) = X \setminus A\}$ [17] or simply τ^*, called the $*$-topology,

which is finer than τ. The member of $\tau^*(\mathbf{I})$ is called $*$-open set and the compliment of $*$-open set is called $*$-closed set. $A \subseteq X$ is $*$-open if and only if $Int^*(A) = A$ where $Int^*(A)$ is denoted as interior of A with respect to $*$-topology and $A \subseteq X$ is $*$-closed if and only if $Cl^*(A) = A$ where $Cl^*(A)$ is denoted as closure of A with respect to $*$-topology. A basis for the $*$-topology $\tau^*(\mathbf{I})$ is $\beta = \{O \setminus I_1 : O \in \tau, I_1 \in \mathbf{I}\}$ [17].

- For the condition $\tau \cap \mathbf{I} = \{\varnothing\}$ of an ideal topological space $(X, \tau, \mathbf{I})$, we say that $\mathbf{I}$ is τ-boundary [33](or codense [12]) and this type of space is called Hayashi-Samuel space [11].
- According to Njåstad, the ideal $\mathbf{I}$ is said to be compatible with τ, denoted by $\mathbf{I} \sim \tau$ [31] if the following holds for every $A \subseteq X$: if for all $x \in A$, there exists $O \in \tau(x)$ such that $O \cap A \in \mathbf{I}$, then $A \in \mathbf{I}$.
- For $\mathbf{I} \sim \tau$, $\psi(\psi(A)) = \psi(A)$ [13] and $\psi(A) \setminus A \in \mathbf{I}$ [13] for every $A \subseteq X$.
- A subset $\mathbf{A}$ of a topological space X with an ideal $\mathbf{I}$ is called $\mathbf{I}$-dense if and only if $\mathbf{A}^*(\mathbf{I}) = X$ [12]. Equivalently $\mathbf{A}$ is $\mathbf{I}$-dense in X if and only if for every nonempty open subset $\mathbf{O}$ of X, $\mathbf{A} \cap \mathbf{O} \notin \mathbf{I}$. Every $\mathbf{I}$-dense subset of a topological space is a dense subset. A subset $\mathbf{A}$ of a topological space (X, τ) with an ideal $\mathbf{I}$ is $\mathbf{I}$-dense if and only if $\psi(X \setminus \mathbf{A}) = \varnothing$.
- A subset $\mathbf{A}$ of a topological space (X, τ) with an ideal $\mathbf{I}$ is called $*$-dense if and only if $Cl^*(\mathbf{A}) = \mathbf{X}$ [14]. Equivalently $\mathbf{A}$ is $*$-dense in X if and only if for every nonempty $*$-open subset $\mathbf{O}$ of X, $\mathbf{A} \cap \mathbf{O} \neq \varnothing$. Every $\mathbf{I}$-dense subset of X is a $*$-dense but reverse may not be true [23].
- An ideal $\mathbf{I}$ on a topological space is called a completely codense if $PO(X) \cap \mathbf{I} = \{\varnothing\}$ [12].
- An ideal $\mathbf{I}$ is completely codense on (X, τ) if and only if each member of $\mathbf{I}$ is nowhere dense [12].
- A subset $\mathbf{A}$ of a topological space X with an ideal $\mathbf{I}$ is called $\mathbf{I}$-open if and only if $\mathbf{A} \subseteq Int(\mathbf{A}^*)$. The collection of all $\mathbf{I}$-open [2] sets in an ideal topological space $(X, \tau, \mathbf{I})$ is denoted by $\mathbf{I}O(X)$.
- Let $f : X \to Y$ be a function. If $\mathbf{I}$ is an ideal on X, then $f(\mathbf{I}) = \{f(\mathcal{I}_1) : \mathcal{I}_1 \in \mathbf{I}\}$ is also an ideal on Y [18]. Moreover $f^{\leftarrow}(\mathbf{I}) = \{A : A \subseteq f^{-1}(I) \in f^{-1}(\mathbf{I})\}$ is an ideal on X [16].
- Let X be a topological space and $f : X \to X$ be a function. A subset $\mathbf{A} \subseteq X$ is called invariant under f if $f(\mathbf{A}) \subseteq \mathbf{A}$ [20].
- Let X be a nonempty set. A filter [9] $\mathfrak{F}$ on X is a nonempty collection of nonempty subsets of X obeying the rules:
  1. $S \subseteq T$ and $S \in \mathfrak{F} \implies T \in \mathfrak{F}$;
  2. $S, T \in \mathfrak{F} \implies S \cap T \in \mathfrak{F}$.

  The study of filter can also be considered as a dual ideal and it's study is going so far (see [15], [19], [25], [26])
- Throughout this paper, we will write $\mathbf{I}$-space instead of ideal topological space and space instead of topological space if no misunderstanding can arise.

3. Equivalent definition of topological transitivity

This section will discuss that which collections of a topological space are not suitable for defining new topological transitivity. That is, we discuss equivalent definition of topological transitivity and non-wandering point.

Lemma 3.1 *Let (X, f) be a dynamical system and $\mathcal{B}$ be a basis of the topological space X. Then, the following are equivalent statements:*
(i) f is topologically transitive;
(ii) for every pair of nonempty open sets A and B, there exists a positive integer n such that $Cl(f^n(A)) \cap$

$B \neq \varnothing$;
(iii) for every pair of basic open sets $B_1, B_2 \in \mathcal{B}$, there exists a positive integer n such that $f^n(B_1) \cap B_2 \neq \varnothing$.

Lemma 3.2 *Let (X, f) be a dynamical system. Then, the following statements are equivalent:*
(i) f is topologically semi (respectively, pre, b, β) transitive;
(ii) for every pair of nonempty semi-open (respectively, preopen, b-open, β-open) sets A and B, there exists a positive integer n such that $scl(f^n(A)) \cap B$ (respectively, $pcl(f^n(A)) \cap B, bcl(f^n(A)) \cap B, \beta cl(f^n(A)) \cap B) \neq \varnothing$.

Lemma 3.3 *Let (X, f) be a dynamical system and $\mathcal{B}$ be a basis of the topological space X. Then, the following statements are equivalent:*
(i) x is a non-wandering point of X;
(ii) for every neighbourhood M of x, there exists a positive integer n such that $Cl(f^n(M)) \cap M \neq \varnothing$;
(iii) for every basic element B of $\mathcal{B}$ containing x, there exists a positive integer n such that $Cl(f^n(B)) \cap B \neq \varnothing$;
(iv) there exists $k \in \mathbb{Z}_+$ such that the filter $\mathfrak{F}$ containing the set $S = \{f^k(N_x) \cap N_x : N_x \in \aleph_x\}$ converges to x where $\aleph_x$ denotes the collection of all neighbourhoods containing x.

Proof: Here we only prove $(i) \implies (iv)$ and $(iv) \implies (i)$.
$(i) \implies (iv)$ Suppose x is a non-wandering point of X. Then, there exists $k \in \mathbb{Z}_+$ such that $f^k(U) \cap U \neq \varnothing$ for all open sets U containing x. Thus, the set $S = \{f^k(N_x) \cap N_x : N_x \in \aleph_x\}$ has finite intersection property and hence induced a filter $\mathfrak{F}$ containing $\aleph_x$. Then, $\mathfrak{F}$ converges to x.
$(iv) \implies (i)$ Suppose the filter $\mathfrak{F}$ containing the set $S = \{f^k(N_x) \cap N_x : N_x \in \aleph_x\}$ converges to x for some $k \in \mathbb{Z}_+$. Then, the set $S = \{f^k(N_x) \cap N_x : N_x \in \aleph_x\}$ has finite intersection property and hence $f^k(U) \cap U \neq \varnothing$ for all open sets U containing x. This proves that x is a non-wandering point of X. □

Lemma 3.4 *Let (X, f) be a dynamical system. Then, the following statements are equivalent:*
(i) x is a semi (respectively, pre, b, β) non-wandering point of X;
(ii) for every semi-open (respectively, preopen, b-open, β-open) set M containing x, there exists a positive integer n such that $scl(f^n(M)) \cap M (respectively, pcl(f^n(M)) \cap M, bcl(f^n(M)) \cap M, \beta cl(f^n(M)) \cap M) \neq \varnothing$.

4. Ideal version of topological transitivity

In this section, we have introduced ideal version of transitivity called topological ideal transitivity and investigate its various characterizations and features. To do this we will counter the Remark 6.9(2) of [23].

Definition 4.1 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space. A dynamical system (X, f) is called a topologically ideal transitive (or simply ideal transitive or $\mathbf{I}$-transitive) if for every pair of nonempty open sets A and B, there exists a positive integer n such that $f^n(A) \cap B \notin \mathbf{I}$.*

Example 4.1 *Suppose $X = \mathbb{R}$ (set of reals) endowed with the co-finite topology τ and $\mathbf{I} = \mathbf{I_c}$, ideal of all countable subsets of $\mathbb{R}$. Let us define a mapping $f : X \to X$ by $f(x) = x$, for all $x \in \mathbb{R}$. Then, f is $\mathbf{I}$-transitivity.*

Example 4.2 *Suppose $X = \{a, b, c, d\}$ endowed with the topology $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\}$ and $\mathbf{I} = \{\varnothing, \{d\}\}$. Let us define a mapping $f : X \to X$ by $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then, f is $\mathbf{I}$-transitive.*

Every $\mathbf{I}$-transitivity is topological transitivity. But the converse may not be true. We consider the following examples:

Example 4.3 *Suppose $X = \{a, b, c, d\}$ endowed with the topology $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\}$ and $\mathbf{I} = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}$. Let us define a mapping $f : X \to X$ by $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then, f is open, continuous and topologically transitive but not $\mathbf{I}$-transitive.*

Example 4.4 *Suppose $X = \mathbb{R}$ endowed with the topology $\tau = \{\varnothing, \mathbb{R}, (a,b), [a,b), [a,b]\}$ with $a < b$ and $a, b \in \mathbb{R}$ and $\mathbf{I} = 2^{[a,b]}$. Let us define a mapping $f : X \to X$ by $f(x) = x$, for all $x \in \mathbb{R}$. Then, f is topologically transitive but not $\mathbf{I}$-transitive.*

If $\mathcal{I} = \{\varnothing\}$, then $\mathbf{I}$-transitivity and topological transitivity becomes identical.
For the reverse inclusion we consider the following:

Theorem 4.1 *Let $\mathbf{I}$ be a codense ideal on a topological space (X, τ) and (X, f) be a dynamical system. If f is open and topologically transitive, then it is $\mathbf{I}$-transitive.*

Proof: By topological transitivity of f implies for every nonempty open sets U and V, there exists a positive integer k such that $f^k(U) \cap V \neq \varnothing$. Since f is an open mapping, then $f^k(U) \cap V \in \tau$. Hence, by codenseness of $\mathbf{I}$, $f^k(U) \cap V \notin \mathbf{I}$. Thus, f is $\mathbf{I}$-transitivity. This completes the proof. □

Theorem 4.2 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space. If (X, f) is $\mathbf{I}$-transitive, then the space $(X, \tau, \mathbf{I})$ is a Hayashi-Samuel space.*

Proof: Let $U(\neq \varnothing)$ be an open set in X. Then, for $\mathbf{I}$-transitivity of (X, f), for any $V(\neq \varnothing) \in \tau$, there exists a positive integer n such that $f^n(V) \cap U \notin \mathbf{I}$. This implies $U \notin \mathbf{I}$ (because if $U \in \mathbf{I}$, then $f^n(V) \cap U \in \mathbf{I}$) and hence $(X, \tau, \mathbf{I})$ is a Hayashi-Samuel space. This completes the proof. □

But the converse may not be true. we consider the following examples:

Example 4.5 *Suppose $X = \mathbb{R}$ endowed with the usual topology τ_u and consider an ideal $\mathbf{I} = 2^{\mathbb{R}\setminus\mathbb{Q}}$. Let us define a mapping $f : X \to X$ by $f(x) = x$, for all $x \in \mathbb{R}$. If we take, $U = (0,1)$ and $V = (1,2)$ be two open subsets of X, then for any positive integer n, $f^n(U) \cap V \in \mathbf{I}$. Thus, f is not $\mathbf{I}$-transitive but the space $(X, \tau, \mathbf{I})$ is a Hayashi-Samuel space.*

Example 4.6 *Suppose $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ (where $\mathbb{N}$ denotes the set of all natural numbers) endowed with the usual topology and consider an ideal $\mathbf{I} = \{\varnothing, \{0\}\}$. Let us define a mapping $f : X \to X$ by $f(\frac{1}{n}) = \frac{1}{(n+1)}$, $n = 1, 2, 3, ...$ and $f(0) = 0$. Clearly f is open and continuous. Take, $U = \{\frac{1}{2}\}, V = \{1\}$ be two open subsets of X. Then, for all $n \in \mathbb{N}$, $f^n(U) \cap V \in \mathbf{I}$. Thus, f is not $\mathbf{I}$-transitive but the space $(X, \tau, \mathbf{I})$ is a Hayashi-Samuel space.*

We learnt from various research papers that dense sets are important for discussing the properties of topological transitivity. In this point of view to discuss the properties of ideal version of topological transitivity, $\mathbf{I}$-dense sets play an important role. Thus we mentioned that if the collection of dense sets and the collection of $\mathbf{I}$-dense sets are equal, then the properties of $\mathbf{I}$-transitivity will not be interesting. But in [23], the authors considered "$\mathbf{I}$-dense, $*$-dense and dense are equivalent if the ideal $\mathbf{I}$ is codense". Followings are the counter examples:

Example 4.7 *Let us consider an ideal topological space $(X, \tau, \mathbf{I})$ where $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a, c, d\}\}$ and $\mathbf{I} = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}$. Here $\mathbf{I} \cap \tau = \{\varnothing\}$ and hence $\mathbf{I}$ is codense. Take $A = \{a, b, c\}$. Then, $Cl(A) = X$. This implies A is a dense subset of X. Also $A^* = \{b\} \neq X$ and $Cl^*(A) = A \cup A^* = \{a, b, c\}$. This shows that A is neither $*$-dense nor $\mathbf{I}$-dense. Thus, $\mathbf{I}$-dense, $*$-dense and dense are not equivalent though $\mathbf{I}$ is codense.*

Example 4.8 *Suppose $X = \mathbb{R}$ endowed with the co-finite topology τ and $\mathbf{I} = \mathbf{I_c}$, ideal of all countable subsets of $\mathbb{R}$. Then $\mathbf{I} \cap \tau = \{\varnothing\}$ and hence $\mathbf{I}$ is codense. Take $A = \mathbb{Q}$. Then, $Cl(A) = \mathbb{R}$. This implies A is a dense subset of X. Also $A^* = \varnothing \neq X$ and hence $Cl^*(A) = A \cup A^* = A \neq X$. This shows that A is neither $\mathbf{I}$-dense nor $*$-dense. Hence, $\mathbf{I}$-dense, $*$-dense and dense are not equivalent though $\mathbf{I}$ is codense.*

Above examples showed that $\mathbf{I}$-dense, $*$-dense and dense are not equivalent when $\mathbf{I}$ is codense.
Even if $\mathbf{I}$ is codense and $\mathbf{I} \sim \tau$, by the following example, we have reached that $\mathbf{I}$-dense, $*$-dense and dense are not equivalent:

Example 4.9 *Suppose $X = \mathbb{R}$ endowed with the topology $\tau = \{\varnothing, \mathbb{R}, \mathbb{Q} \cup \{i\}\}$ for some fixed $i \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathbf{I} = 2^{\mathbb{Q}}$. Then $\mathbf{I} \cap \tau = \{\varnothing\}$ and hence $\mathbf{I}$ is codense. Also $\mathbf{I} \sim \tau$. Take $A = \mathbb{Q}$. Then, $Cl(A) = \mathbb{R}$. This implies A is a dense subset of $\mathbb{R}$. Also $A^* = \varnothing \neq \mathbb{R}$ and hence $Cl^*(A) = A \cup A^* = A \neq \mathbb{R}$. This shows that A is neither $\mathbf{I}$-dense nor $*$-dense. Hence, $\mathbf{I}$-dense, $*$-dense and dense are not equivalent though $\mathbf{I}$ is codense and $\mathbf{I} \sim \tau$.*

In this regards, we shall give a rectification of the Remark **6.9**(**2**) of [23] in the next section.

Theorem 4.3 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. If f is $\mathbf{I}$-transitive, then for any nonempty open set U in X, $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $\mathbf{I}$-dense in X.*

Proof: Let f be an $\mathbf{I}$-transitive map and assume that $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is not $\mathbf{I}$-dense in X. Then, there exists a nonempty open set V in X such that $(\bigcup\limits_{n=0}^{\infty} f^n(U)) \cap V \in \mathbf{I}$. This implies $f^n(U) \cap V \in \mathbf{I}$ (by hereditary property of $\mathbf{I}$) for all $n \in \mathbb{Z}_+ \cup \{0\}$ where $\mathbb{Z}_+$ is the set of all positive integers. This leads a contradiction to the $\mathbf{I}$-transitive of f. Hence, $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $\mathbf{I}$-dense in X. This completes the proof. □

But the reverse may not be true. For this, let U and V be two nonempty open sets in X and assume that $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $\mathbf{I}$-dense in X. Then, $\bigcup\limits_{n=0}^{\infty} f^n(U) \cap V \notin \mathbf{I}$ i.e., $\bigcup\limits_{n=0}^{\infty} (f^n(U) \cap V) \notin \mathbf{I}$. This implies for all $m \in \mathbb{Z}_+ \cup \{0\}$, $f^m(U) \cap V$ may belongs to $\mathbf{I}$.
If we consider $\mathbf{I} = 2^{\mathbb{Z}_+} \setminus \{\mathbb{Z}_+\}$, then $\mathbf{I}$ becomes an ideal. Here, for all $n \in \mathbb{Z}_+$, $\{n\} \in \mathbf{I}$ but $\bigcup\limits_{n=1}^{\infty} \{n\} \notin \mathbf{I}$.

We know that $\mathbf{I}$-transitivity implies $\mathbf{I}$ is codense. Again in Hayashi-Samuel space, the collection of $\mathbf{I}$-dense sets and the collection of dense sets in the $*$-topology are equal [27]. Therefore, we get the following:

Corollary 4.1 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. If f is $\mathbf{I}$-transitive, then for any nonempty open set U in X, $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $*$-dense in X.*

Theorem 4.4 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. Then, f is $\mathbf{I}$-transitive if for any nonempty open set U in X, $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $\mathbf{I}$-dense in X and $\mathbf{I}$ is countably additive.*

Proof: let U and V be two nonempty open sets in X and assume that $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $\mathbf{I}$-dense in X. Then, $\bigcup\limits_{n=0}^{\infty} f^n(U) \cap V \notin \mathbf{I}$. This implies there exists $m \in \mathbb{Z}_+$ such that $f^m(U) \cap V \notin \mathbf{I}$, since $\mathbf{I}$ is countably additive. Hence, f is $\mathbf{I}$-transitive. This completes the proof. □

Corollary 4.2 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. Then, f is $\mathbf{I}$-transitive if for any nonempty open set U in X, $\bigcup\limits_{n=0}^{\infty} f^n(U)$ is $*$-dense in X and $\mathbf{I}$ is countably additive and codense.*

Theorem 4.5 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. If f is $\mathbf{I}$-transitive, then for any nonempty open set U in X, $\bigcup\limits_{n=0}^{\infty} f^{-n}(U)$ is $\mathbf{I}$-dense in X.*

Proof: Assume that f is $\mathbf{I}$-transitive. Since f is a continuous function and arbitrary union of open sets is open in a topological space, then $\bigcup\limits_{n=0}^{\infty} f^{-n}(U)$ is open. Again, since f is $\mathbf{I}$-transitive, $\bigcup\limits_{n=0}^{\infty} f^{-n}(U)$ is $\mathbf{I}$-dense in X. This completes the proof. □

Corollary 4.3 *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space and (X,f) be a dynamical system. If f is $\mathbf{I}$-transitive, then for any nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is $*$-dense in X.*

Theorem 4.6 *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space and (X,f) be a dynamical system. Then, f is topologically $f^k(\mathbf{I})$-transitive for some positive integer k if for every nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is $\mathbf{I}$-dense in X and $\mathbf{I}$ is countably additive.*

Proof: Let U and V be any two nonempty open sets in X. Since, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is $\mathbf{I}$-dense in X, then $\bigcup_{n=0}^{\infty} f^{-n}(U)\cap V \notin \mathbf{I}$ for every nonempty open sets V in X. This implies there exists a positive integer k such that $f^{-k}(U)\cap V \notin \mathbf{I}$ since $\mathbf{I}$ is countably additive. This implies, there exists a positive integer k such that $U\cap f^k(V) \notin f^k(\mathbf{I})$. Hence, f is topologically $f^k(\mathbf{I})$-transitive. This completes the proof. □

Corollary 4.4 *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space and (X,f) be a dynamical system. Then, f is topologically $f^k(\mathbf{I})$-transitive for some positive integer k if for any nonempty open set U in X, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is $*$-dense in X and $\mathbf{I}$ is countably additive and codense.*

Theorem 4.7 *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space and (X,f) be a dynamical system and $\mathbf{I}$-transitive. If $A\subseteq X$ is closed and $f(X\setminus A)\subseteq X\setminus A$, then either $A=X$ or $\psi(A^*(\mathbf{I}))=\varnothing$.*

Proof: Let f is $\mathbf{I}$-transitive, $A\subseteq X$ is closed and $f(X\setminus A)\subseteq X\setminus A$. Assume that $A\neq X$ and $\psi(A^*(\mathbf{I}))\neq\varnothing$. Put $U=X\setminus A$. Then U is open, and $f^n(U)\subseteq U$. Since $\psi(A^*(\mathbf{I}))\neq\varnothing$, then there exists an open set V in X such that $V\setminus A^*(\mathbf{I})\in\mathbf{I}$ i.e. $V\cap(X\setminus A^*(\mathbf{I}))\in\mathbf{I}$. Also for every subset A of X, $A^*(\mathbf{I})\subseteq Cl(A)$. Since A is closed, then $A^*(\mathbf{I})\subseteq Cl(A)=A$ and hence $V\cap(X\setminus A)\in\mathbf{I}$ (by hereditary property of $\mathbf{I}$). Further, we have, $U=X\setminus A$ and $f^n(U)\subseteq U$ for all $n\in\mathbb{Z}_+$. This implies, $V\cap f^n(U)\in\mathbf{I}$ for all $n\in\mathbb{Z}_+$ which contradicts the fact that f is $\mathbf{I}$-transitive. Hence, either $A=X$ or $\psi(A^*(\mathbf{I}))=\varnothing$. This completes the proof. □

Theorem 4.8 *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space. Then, the following statements are equivalent:*
(1) *f is $\mathbf{I}$-transitive;*
(2) *if $U\subseteq X$ is open and $f(U)=U$, then either $U=\varnothing$ or U is $\mathbf{I}$-dense in X;*
(3) *if $U\subseteq X$ is open and $f(U)=U$, then either $U=\varnothing$ or U is $*$-dense in X.*

Proof: (1) $\implies$ (2) Let f is $\mathbf{I}$-transitive, $U\subseteq X$ is open and $f(U)=U$. Assume that $U\neq\varnothing$ and U is not $\mathbf{I}$-dense in X. Then there exists a nonempty open set V in X such that $U\cap V\in\mathbf{I}$. Since $f(U)=U$, then $f^n(U)=U$ for all $n\in\mathbb{Z}_+$ and hence $f^n(U)\cap V\in\mathbf{I}$ for all $n\in\mathbb{Z}_+$ which contradicts the ideal transitivity of f. Hence, either $U=\varnothing$ or U is $\mathbf{I}$-dense in X in X.

(2) $\implies$ (1) Assume that the condition (2) holds. Let U be nonempty open subset of X with $f(U)=U$. Then, U is $\mathbf{I}$-dense in X. Then, for all nonempty open subsets V in X, $U\cap V\notin\mathbf{I}$. Assume that f is not $\mathbf{I}$-transitive, then there exists a nonempty open set V_1 in X such that $f^n(U)\cap V_1\in\mathbf{I}$ for all $n\in\mathbb{Z}_+$. This implies, $U\cap V_1\in\mathbf{I}$ for all $n\in\mathbb{Z}_+$ which contradicts the fact that $U\cap V\notin\mathbf{I}$ for all nonempty open subsets V in X. Hence, f is $\mathbf{I}$-transitive.
This completes the proof. □

Recall the following results:

Theorem 4.9 ([17]) *Let $(X,\tau,\mathbf{I})$ be an $\mathbf{I}$-space. Then, the following statements are equivalent:*
(1) *$\mathbf{I}$ is τ-boundary;*
(2) *$\psi(\varnothing)=\varnothing$;*

(3) *$X = X^*$;*
(4) *for every $A \in \tau$, $A \subseteq A^*$;*
(5) *if $I \in \mathbf{I}$, then $\psi(I) = \varnothing$.*

Theorem 4.10 ([13]) *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space. Then, the following statements are equivalent:*
(1) *$\mathbf{I} \sim \tau$;*
(2) *For every $A \subseteq X$, $A \setminus A^* \in \mathbf{I}$.*

Theorem 4.11 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space with $\mathbf{I} \sim \tau$ and (X, f) is a dynamical system. If f is $\mathbf{I}$-transitive, then $X \setminus (A \setminus A^*)$ is $\mathbf{I}$-dense (respectively, $*$-dense) for any $A \subseteq X$.*

Proof: Let f is $\mathbf{I}$-transitive. Then, by Theorem 4.2, the ideal $\mathbf{I}$ is codense i.e., $\mathbf{I}$ is τ-boundary . Since $\mathbf{I} \sim \tau$, then by Theorem 4.10, for every $A \subseteq X$, $A \setminus A^* \in \mathbf{I}$. Also since $\mathbf{I}$ is τ-boundary, then by Theorem 4.9, $\psi(A \setminus A^*) = \varnothing$. This implies, $X \setminus (X \setminus (A \setminus A^*))^* = \varnothing$ and hence $X = (X \setminus (A \setminus A^*))^*$. Thus, $X \setminus (A \setminus A^*)$ is $\mathbf{I}$-dense for every $A \subseteq X$. This completes the proof. □

Theorem 4.12 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space with $\mathbf{I} \sim \tau$ and (X, f) is a dynamical system. If f is $\mathbf{I}$-transitive, then $X \setminus (\psi(A) \setminus A)$ is $\mathbf{I}$-dense (respectively, $*$-dense) for any $A \subseteq X$.*

Proof: Let f is $\mathbf{I}$-transitive. Then, by Theorem 4.2, the ideal $\mathbf{I}$ is codense i.e., $\mathbf{I}$ is τ-boundary. Since $\mathbf{I} \sim \tau$, then $\psi(A) \setminus A \in \mathbf{I}$ [13] for every $A \subseteq X$. Since $\mathbf{I}$ is τ-boundary, $\psi(\psi(A) \setminus A) = \varnothing$. This implies, $X \setminus (\psi(A) \setminus A)$ is $\mathbf{I}$-dense for any $A \subseteq X$. This completes the proof. □

Theorem 4.13 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) be a dynamical system. If f is $\mathbf{I}$-transitive, then $X \setminus A^* \subseteq (X \setminus A)^*$ for any $A \subseteq X$.*

Proof: Let f is $\mathbf{I}$-transitive. Then, by Theorem 4.2, the ideal $\mathbf{I}$ is codense i.e., $\mathbf{I}$ is τ-boundary. Hence, $X = X^*$ by Theorem 4.9. Now, $A^* \cup (X \setminus A)^* = (A \cup (X \setminus A))^* = X^* = X$ [17] and hence $X \setminus A^* \subseteq (X \setminus A)^*$. Thus, $X \setminus A^* \subseteq (X \setminus A)^*$ for every $A \subseteq X$. This, completes the proof. □

Theorem 4.14 *Let $(X, \tau, \mathbf{I})$ be an $\mathbf{I}$-space and (X, f) is a dynamical system. If f is $\mathbf{I}$-transitive, then $\tau \subseteq \mathbf{I}O(X)$.*

Proof: Let f is $\mathbf{I}$-transitive. Then, by Theorem 4.2, $\mathbf{I} \cap \tau = \{\varnothing\}$. Let $A \in \tau$, then by Theorem 4.9, $A \subseteq A^*$. This shows that $A \subseteq Int(A) \subseteq Int(A^*)$ and hence $A \in \mathbf{I}O(X)$. Consequently, $\tau \subseteq \mathbf{I}O(X)$. This completes the proof. □

5. Non-wandering point with ideal

In this section, we have introduced ideal version of non-wandering points called ideal-non-wandering points and investigate its various characterizations and properties.

Definition 5.1 *Let (X, f) be a dynamical system and $\mathbf{I}$ be an ideal on X. A point $x \in X$ is called an ideal non-wandering (or simply $\mathbf{I}$-non-wandering) if for every neighbourhood M containing x, there exists a positive integer n such that $f^n(M) \cap M \notin \mathbf{I}$. The set of all ideal-non-wandering points is denoted by $\Omega_{\mathbf{I}}(f)$.*

Every "**I**-non-wandering point" is a "non-wandering" point but the reverse may not be true. For the existence of ideal non-wandering points and reverse inclusion, we consider the following examples:

Example 5.1 *Consider an $\mathbf{I}$-space $(X, \tau, \mathbf{I})$ where $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathbf{I} = \{\varnothing, \{c\}\}$. Take, $a \in X$. Then, the neighbourhoods of a are $\{a\}, \{a, b\}$ and X. Let us define a mapping $f : X \to X$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then, for any neighbourhood M of a, $f^n(M) \cap M \notin \mathbf{I}$ for $n = 2$. Thus, 'a' is an $\mathbf{I}$-non-wandering point of X.*

Example 5.2 *Consider an* $\mathbf{I}$*-space* $(X,\tau,\mathbf{I})$ *where* $X=\{a,b,c\}$, $\tau=\{\varnothing,\{a\},\{b\},\{a,b\},X\}$ *and* $\mathbf{I}=\{\varnothing,\{a\},\{b\},\{a,b\}\}$. *Take,* $a\in X$. *Then, the neighbourhoods of* a *are* $\{a\},\{a,b\}$ *and* X. *Let us define a mapping* $f:X\to X$ *by* $f(a)=b$, $f(b)=a$ *and* $f(c)=c$. *Then, for any neighbourhood* M *of* a, $f^n(M)\cap M\neq\varnothing$ *for* $n=2$. *Thus, 'a' is a non-wandering point of* X. *But for all* $n\in\mathbb{N}$, $f^n(M)\cap M\in\mathbf{I}$ *when* $M=\{a\}$ *and* $M=\{a,b\}$. *Thus, 'a' is not an* $\mathbf{I}$*-non-wandering point of* X.

Example 5.3 *Consider an* $\mathbf{I}$*-space* $(X,\tau,\mathbf{I})$ *where* $X=\mathbb{N}$, *set of all natural numbers,* $\tau=\tau_u$, *usual topology on* $\mathbb{N}$ *and* $\mathbf{I}=2^{\mathbb{N}\setminus\mathbb{N}_o}$ *where* $\mathbb{N}_o$ *is the set of all odd natural numbers. Let us define a mapping* $f:X\to X$ *by* $f(k)=k$ *for all* $k\in X$. *Then, all odd natural numbers are non-wandering points of* $\mathbb{N}$ *but not* $\mathbf{I}$*-non-wandering point of* $\mathbb{N}$.

If a point $x\in X$ is a $\mathbf{I}$-non-wandering point (i.e., $x\in\Omega_{\mathbf{I}}(f)$), then no open set containing x (or neighbourhood of x) is a member of $\mathbf{I}$.
For the reverse inclusion, we consider the following:

Theorem 5.1 *Let* $\mathbf{I}$ *be a codense ideal on a space* X *and* (X,f) *be a dynamical system. If* f *is an open map, then any non-wandering point of* X *is an* $\mathbf{I}$*-non-wandering point.*

Proof: Let $x\in X$ be a non-wandering point. Then, for every nonempty open set M of x, there exists a positive integer k such that $f^k(M)\cap M\neq\varnothing$. Since f is open, then $f^k(M)\cap M\in\tau$. This implies $f^k(M)\cap M\notin\mathbf{I}$ since the ideal $\mathbf{I}$ is codense. Thus, x is an $\mathbf{I}$-non-wandering point. This completes the proof. □

Definition 5.2 *Let* (X,f) *be a dynamical system and* $\mathbf{I}$ *be an ideal on* X. *The dynamical system* (X,f) *is said to have* $\mathbf{I}$*-dense orbit with respect to the ideal* $\mathbf{I}$ *if there exists a point* $x\in X$ *such that* $A^*(\mathbf{I})=X$ *where* A *is an orbit of* x.

For the existence of $\mathbf{I}$-dense orbit, we consider the following examples:

Example 5.4 *Consider an* $\mathbf{I}$*-space* $(X,\tau,\mathbf{I})$ *where* $X=\{a,b\}$, $\tau=\{\varnothing,X,\{a\}\}$, *and* $\mathbf{I}=\{\varnothing,\{b\}\}$. *Let us define a mapping* $f:X\to X$ *by* $f(a)=f(b)=a$. *Then, the orbit of* b *is* $\mathbf{I}$*-dense.*

Example 5.5 *Suppose* $X=\{0\}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$ *(where* $\mathbb{N}$ *denotes the set of all natural numbers) endowed with the usual topology and consider an ideal* $\mathbf{I}=\{\varnothing,\{0\}\}$. *Let us define a mapping* $f:X\to X$ *by* $f(\frac{1}{n})=\frac{1}{(n+1)}$, $n=1,2,3,\ldots$ *and* $f(0)=0$. *Clearly,* f *is open and continuous. Then, the orbit of the point* $x=1$ *is* $\mathbf{I}$*-dense.*

Theorem 5.2 **([12])** *For an* $\mathbf{I}$*-space* $(X,\tau,\mathbf{I})$, *the following statements are equivalent:*
(1) $\mathbf{I}$ *is completely codense;*
(2) *Every dense set is* $\mathbf{I}$*-dense.*

This theorem is one of the rectification of the Remark 6.9(2) of [23].

Theorem 5.3 **([10])** *Let* X *be a compact Hausdorff space with a countable base and* $f:X\to X$ *be a continuous mapping. If* f *is a topologically transitive, then it has a dense orbit.*

Proposition 5.1 *Let* $\mathbf{I}$ *be a completely codense ideal on a compact Hausdorff space* X *with a countable base and* $f:X\to X$ *be a continuous mapping. If* f *is an* $\mathbf{I}$*-transitive, then it has an* $\mathbf{I}$*-dense orbit.*

Proof: Topological ideal transitivity of f implies f is topologically transitive. Since X is a compact Hausdorff space with countable base and f is a continuous mapping, then by Theorem 5.3, f has a dense orbit. Also since $\mathbf{I}$ is completely codense ideal, then by Theorem 5.2, dense orbit of f becomes $\mathbf{I}$-dense orbit. Hence, f has an $\mathbf{I}$-dense orbit. This completes the proof. □

Proposition 5.2 *Let $\mathbf{I}$ be an ideal of nowhere dense subsets of a compact Hausdorff space X with a countable base and $f : X \to X$ be a continuous mapping. If f is an $\mathbf{I}$-transitive, then it has an $\mathbf{I}$-dense orbit.*

Proof: The proof is straight forward and hence omitted. □

Proposition 5.3 *Let $\mathbf{I}$ be a completely codense ideal on a compact Hausdorff space X with a countable base and $f : X \to X$ be a continuous and open mapping. Then, f is an $\mathbf{I}$-transitive if and only if $\Omega_{\mathbf{I}}(f) = X$ and f has $\mathbf{I}$-dense (respectively, $*$-dense) orbit.*

Proof: Suppose f is an $\mathbf{I}$-transitive. Clearly, it has an $\mathbf{I}$-dense orbit, i.e., there exists $x_0 \in X$ such that $O_f(x_0)$ is $\mathbf{I}$-dense in X. Given that f is $\mathbf{I}$-transitive, this implies that $\mathbf{I}$ is codense. Further, f is topologically transitive, then for any neighbourhood M of x, $f^n(M) \cap M \neq \varnothing$. This implies that $f^n(M) \cap M \notin \mathbf{I}$ (since f is open) and hence $\Omega_{\mathbf{I}}(f) = X$.

Conversely, suppose f has an $\mathbf{I}$-dense orbit and $\Omega_{\mathbf{I}}(f) = X$ and let M, N be two nonempty open subsets of X. Let $x \in X$ have an $\mathbf{I}$-dense orbit. Since every $\mathbf{I}$-dense is dense, then the orbit of x will enter both M and N. Let m and n be the least positive integers such that $f^m(x) \in M$ and $f^n(x) \in N$. Assume $m < n$ and take $k = n - m$. Then, obviously $f^k(M) \cap N \neq \varnothing$ and hence $f^k(M) \cap N \notin \mathbf{I}$, since f is open and $\mathbf{I}$ is a completely codense ideal. Thus, f is an $\mathbf{I}$-transitive.
This completes the proof. □

Corollary 5.1 *Let $\mathbf{I} = \mathcal{N}$ be an ideal of nowhere dense subsets of a compact Hausdorff space X with a countable base and $f : X \to X$ be a continuous and open mapping. Then, f is an $\mathbf{I}$-transitive if and only if $\Omega_{\mathbf{I}}(f) = X$ and f has $\mathbf{I}$-dense (respectively, $*$-dense) orbit.*

We are ending this write up with some further scope of this research. One can define $\mathbf{I}$-transitive through generalized open sets viz semi-open, preopen set, b-open set and β-open set etc. by the following way: A dynamical system (X, f) is called K-topologically ideal transitive (or simply K-$\mathbf{I}$-transitive) if for every pair of nonempty K-open sets A and B, there exists a positive integer n such that $f^n(A) \cap B \notin \mathbf{I}$. In this case, K may be the semi-open set [22], preopen set [24], b-open set [5], β-open set [1]. As we know open set implies semi-open set, then semi $\mathbf{I}$-transitive $\implies$ $\mathbf{I}$-transitive and thus, we conclude the following:

$\beta - \mathbf{I} -$ transitive $\implies$ $b - \mathbf{I} -$ transitive $\implies$ Semi $- \mathbf{I} -$ transitive $\implies$ $\mathbf{I} -$ transitive $\implies$ Topologically transitive, and
$\beta - \mathbf{I} -$ transitive $\implies$ $b - \mathbf{I} -$ transitive $\implies$ Pre $- \mathbf{I} -$ transitive $\implies$ $\mathbf{I} -$ transitive $\implies$ Topologically transitive

However, reverse inclusion of the above relations may not be true in general. These have been followed by the following examples:

Example 5.6 *Suppose X is an infinite set endowed with the indiscrete topology τ and $\mathbf{I}$ is a proper ideal on X. Now, $SO(X) = \{\varnothing, X\}$ and $BO(X) = 2^X$. Let us define a mapping $f : X \to X$ by $f(x) = x$, for all $x \in X$. Then, f is semi-$\mathbf{I}$-transitive but not b-$\mathbf{I}$-transitive.*

Example 5.7 *Suppose $X = \{a, b, c\}$ endowed with the topology $\tau = \{\varnothing, X, \{a, b\}\}$ and $\mathbf{I} = \{\varnothing, \{c\}\}$. Now, $PO(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let us define a mapping $f : X \to X$ by $f(x) = x$, for all $x \in X$. Then, f is $\mathbf{I}$-transitive but not Pre-$\mathbf{I}$-transitive.*

Example 5.8 *Suppose $X = \{a, b, c\}$ endowed with the topology $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathbf{I} = \{\varnothing, \{c\}\}$. Now, $PO(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $BO(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let us define a mapping $f : X \to X$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then, f is Pre-$\mathbf{I}$-transitive but not b-$\mathbf{I}$-transitive.*

Example 5.9 *Suppose $X = \{a, b, c, d\}$ endowed with the topology $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\}$ and $\mathbf{I} = \{\varnothing, \{b\}\}$. Now, $SO(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let us define a mapping $f : X \to X$ by $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then, f is $\mathbf{I}$-transitive but not Semi-$\mathbf{I}$-transitive.*

6. Conclusion

Through this paper, we have introduced equivalent definition of topological transitivity and a new type of transitivity called topological ideal transitivity (or **I**-transitivity). These are related to earlier transitivities. Topological ideal transitivity implies topological transitivity. But reverse may not be true. Our several transitivity measures weaker as well as strong transitivity of the earlier transitivities.

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8. Conflicts of Interest

The authors declare that there is no conflict of interest.

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