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Optimal Solutions of the Time-Fractional Fokker–Planck Equations

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ABSTRACT: This article investigates the comparative analysis of the time-fractional Fokker-plank equation (TFFPE), a mathematical model used in biological and physical sciences. In this work, we used two different methods for analytical solution of the model time-fractional Fokker-plank equation (TFFPE), namely optimal homotopy asymptotic method (OHAM) and optimal auxiliary fractional method (OAFM). The obtained results analyzed analytically and graphically to demonstrate the efficiency and applicability of the applied methods, as well as to investigate the effects of partial arrangement on the behavior of the solutions. Its results indicate that the used methods are effective and accurate for solving fractional order differential equations. These methods applied for the TFFPE model are simple and efficient, allowing us to fully recognize the analytical solutions to both linear and nonlinear fractional differential equations.

Key Words: TFFPE, OHAM, OAFM, fractional differential equations.

Contents

1. Introduction

Fokker and Planck used the Fokker-Planck equation extensively to explain the Brownian particle motion. The Time-fractional Fokker-plank equation (TFFPE) is a mathematical model which can be used in the physical and biological sciences [1]- [10]. The extent of its Markovian and continuous nature causes some randomized, undesirable processes and procedures to approximate. For nonlinear boundary value problems (BVPs), scientists and engineers have recently developed wide variety of approximation techniques that can be used in the physical and biological sciences. Since excellent work on this subject has been done by many scientists and mathematicians, when applied to nonlinear problems, homotopy emerges as a prominent and effective mathematical tool. For nonlinear problems, Watson created a series of possibly one homotopy process in 1986. The software programs HOMPACK90 and POLSYSPLP were introduced as a result of advancements in computer simulation, which made these challenges more accessible [11]- [13]. In mathematical physics, engineering, and science, the precise and clear solution of NPDEs is important. As each NPDE has an infinite number of solutions, it is difficult to identify the proper answer. Such challenges either require an analytical and precise solution in the literature or can be solved analytically and precisely by applying transformations established on the invariance group analysis method [14], the Lie infinitesimal criteria [15], symbolic computation [16], and the Backlund transformation [17]. By using transformation, all of these methods transformed complicated problems into simple ones. In literature, majority of techniques, such as the homotopy perturbation method (HPM) [21], the Adomian decomposition method (ADM) [19], the differential transform method (DTM), and the variational iterative method (VIM) [18], have been used to solve weakly NPDEs. The perturbation method

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was introduced to address the strongly NPDEs [22]-[24]. These methods include a small parameter that is difficult to identify. There have been several new analytical techniques introduced, including the artificial parameter method [25], the homotopy analysis method (HAM) [26], and the homotopy perturbation method (HPM) [27]. These techniques combine perturbation techniques with homotopy theory. OHAM [28]-[31] was recently presented by Vasile Marinca et al. for the resolution of nonlinear problems, trying to remove the dependency of perturbation methods on the supposition of small parameters and intensive computation. The concept and application have been expanded to include systems of equations by Ullah et al. [32]-[42]. In order to solve nonlinear issues, Herisanu created the optimal auxiliary function method (OAFM) [43] in 2018. This technique was presented to reduce the amount of computation and obtain an accurate solution at the first iteration. Since excellent work on this subject has been done by many scientists and engineers. This study aims to adapt the optimal homotopy asymptotic method (OHAM) and optimal auxiliary fractional method (OAFM) for P.D.E.s solution with fractional orders such as time-fractional Fokker-plank equation (TFFPE). Complex fractional partial differential equations can be solved with less efforts and short time with easy approach of using (OHAM) and (OAFM), which have been found to be reliable and efficient methods especially for complex type of PDEs with fractional order. This paper is organized into sections for simplicity's sake. In the first section, we introduced the history and some background on the subject. The formulation is discussed in the second section. In the third section, we solved a well-known model of the time-fractional Fokker-plank equation (TFFPE) for testing the accuracy and efficiency of methods. Graphs and tables are also discussed in the third section, and similarly, in the fourth and fifth sections, results and conclusions are obtained from the article.

2. Mathematical Implementation

In this section, we used two different techniques namely, OHAM and OAFM to find out the approximate solution of the well-known time-fractional Fokker-plank equation (TFFPE) with fractional order.

2.1 OHAM's Basic Mathematical Theory

Consider a partial differential equation

$$A(f(s,t)+W(s,t)=0, \qquad s\varepsilon\Omega \tag{1}$$

$$B(f,\frac{\partial f}{\partial s})=0, \qquad s\epsilon\Gamma \tag{2}$$

Here, A is called differential operators, where $f(s,t)$ is unidentified function, s and t represent spatial and temporal independent variables, individually, Γ is the boundary of Ω and $W(s,t)$ are identified as analytic functions. We can divided 'A' into two portions as;

$$A=L+N \tag{3}$$

L, contains the linear part of the while N contains the non-linear part of partial differential equation. Construct an optimal Homotopy, According to OHAM;

$$\alpha(s,t,q):\Omega\times[0,1]\rightarrow R$$

Satisfying, equation

$$h(\alpha(s,t,q),q)=(1-q)L(\alpha(s,t,q))+W(x,t)-h(q)A(\alpha(s,t,q))+W(s,t)=0, \tag{4}$$

$h(q)$, Which is the auxiliary functions are non-zero for $q\neq 0$ and $h(0)=0$.
Eq. (4) is known as optimal Homotopy equation.

Obviously, we have

$$q = 0 \Rightarrow h(\alpha(s,t,0),0) = L(\alpha(s,t,0)) + W(s,t) = 0, \tag{5}$$

$$q = 1 \Rightarrow h(\alpha(s,t,1),1) = h(1)A(\alpha(s,t,q)) + W(s,t) = 0, \tag{6}$$

Clearly, when $q = 0$ and $q = 1$, we get;

$$\alpha(s,t,o) = f_0(s,t), \alpha(s,t,1) = f(s,t), \tag{7}$$

Correspondingly. When q differs from 0 to 1, then solution $\alpha(s,t,q)$, approaches from $f_0(s,t)$, to $f(s,t)$, where $f_0(s,t)$, is attained from Eq.(4) for $q = 0$.

$$L(f_0(s,t)) + W(s,t) = 0, \qquad B(f_0, \frac{\partial f_0}{\partial s}) = 0. \tag{8}$$

Auxiliary functions $h(q)$,are selected in the form;

$$h(q) = qc_1 + q^2c_2 + q^3c_3 + ... + q^m c_m, \tag{9}$$

To acquire the approximate solutions, we develop $\alpha(s,t,q,c_i)$ by Taylor‘s series about P in the following way,

$$\alpha(s,t,q,c_i) = f_0(s,t) + \sum_{k\geq 1}^{\infty} f_k(s,t,c_i)q^k, \tag{10}$$

While

$$k = 1 = n = i = 1,2,3,4,....$$

Similarly putting Eq. (9)-(10) into Eq. (4) and by equating the coefficient of similar powers of q, we can find Zeroth order solution, which is given by Eq. (8), similarly the first and second order solution given by Eq. (11)- (12) correspondingly and the general governing equation for $u_k(s,t)$ are given by Eq.(13):

$$L(f_1(s,t)) - L(f_0(s,t)) = c_1L(f_0(s,t)) + N(f_0(s,t), \tag{11}$$

$$B(f_1, \frac{\partial f_1}{\partial s}) = 0,$$

$$L(f_2(s,t)) - L(f_1(s,t)) = c_1(L(f_1(s,t)) + N(f_1(s,t)) + c_2(L(f_0(s,t)) + N(f_0(s,t))), \tag{12}$$

$$B(f_2, \frac{\partial f_2}{\partial s}) = 0,$$

$$\begin{aligned} L(f_3(s,t)) - L(f_2(s,t)) = c_1(L(f_2(s,t)) + N(f_2(s,t)) + c_2(L(f_1(s,t)) + \\ N(f_1(s,t))) + c_3(L(f_0(s,t)) + N(f_0(s,t))), \end{aligned} \tag{13}$$

$$B(f_3, \frac{\partial f_3}{\partial s}) = 0,$$

$$L(f_k(s,t)) - L(f_{k-1}((s,t)) = \sum_{i=1}^{k} c_i(L(f_{k-i}(s,t)) + N(f_{k-i}(s,t) \tag{14}$$

$$k = 2,3,..., \qquad B(f_k, \frac{\partial f_k}{\partial s}) = 0,$$

It can be seen that the convergence of the series Eq. (10) depends upon the auxiliary constants $c_1, c_2, c_3, ...$, If it convergent at $q = 1$, one has

$$\alpha^r(s,t,c_i) = f_0(s,t) + \sum_{k\geq 1} f_k(s,t,c_i), \tag{15}$$

$$i = 1,2,...,m$$

Replacing Eq. (15) into Eq. (1), it outcomes the following expression for residuals

$$R(s,t,c_i) = L(\alpha^r(s,t,c_i)) + W(s,t) + N(\alpha^r(s,t,c_i)), \tag{16}$$

If $R(s,t,c_i) = 0$, then $alpha^r r(s,t,c_i)$, will be the precise solution to the problem. In nonlinear problems, specifically, it happens very rarely.
For the computation of auxiliary constant, $c_i, i = 1,2,...,m$ there are diverse approaches like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method. We will apply the Method of Least Squares which is described as under;

$$J(c_i) = \int_o^t \int_\Omega R_1^2(s,t,c_i)dsdt, \tag{17}$$

and

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = ...\frac{\partial J}{\partial c_m} = 0. \tag{18}$$

These constants will be used to obtain the approximate m^{th} order solution.The more general auxiliary functions,$h(q)$ can be optimally known by Eq. (18) and are helpful in error reduction, which depend on constants,$c_1, c_2, ...$,are useful for convergence.

2.2 OAFM analysis for P.D.E.s with fractional orders

Now let understand the formulation of OAFM to nonlinear ODE;

$$\frac{\partial^\alpha \Psi(s,t)}{\partial t^\alpha} = A(\Psi(s,t)) + W(s), \qquad \alpha>0 \tag{19}$$

Where mathematical expression,$\frac{\partial^\alpha}{\partial t^\alpha}$is called the Caputo/ Riemann-Liouville fractional derivative operator, where $A = L + N$ is said to be differential operator. Similarly the linear part is L,nonlinear part is N, where W is called source function, at this stage $\Psi(s)$is an unidentified function, the temporial independent variable is t while α is the parameter donating the fractional derivatives.
Here the initial conditions are

$$\begin{gathered} D_0^{\alpha-r}(s,0) = g_r(s), \qquad r = 0,1,2,...,w-1 \\ D_0^{\alpha-w}(s,0) = 0, \qquad w = [\alpha] \\ D_0^r(s,0) = h_r(s), \qquad while \quad r = 0,1,2,...,w-1 \\ D_0^w(s,0) = 0, \qquad w = [\alpha] \end{gathered}$$

Now selecting,

$$\Psi(s,t,G_k) = \Psi_0(s,t) + \Psi_1(s,t,G_k), \qquad k = 1,2,...,w \tag{20}$$

Using Eq. (21) in Eq. (19), we find the zeroth approximation, which is defined as;

$$\frac{\partial^\alpha(\Psi_0(s,t))}{\partial t^\alpha} - W(s) = 0, \tag{21}$$

$$\Psi_0(s,0) = g_r(s), \qquad r = 0,1,2,...,w-1$$

The first aproximation is found as,

$$\frac{\partial^\alpha(\Psi_1(s,t,G_k))}{\partial t^\alpha} + N(\Psi_0(s,t) + \Psi_1(s,t,G_k)) = 0, \tag{22}$$

$$\Psi_1(s,0) = h_r(s), \qquad r = 0,1,2,...,w-1$$

Since Eqs. (22)- (21), comprise the time fractional derivatives,
Hence by applying I^α operator, we get

$$\Psi_0(s,t) = I^\alpha[W(s)] = 0, \tag{23}$$

and

$$\Psi_1(s,t,G_k) = I^{\alpha}[N(\Psi_0(s,t) + \Psi_1(s,t,G_k))] = 0, \tag{24}$$

where the nonlinear term is expressed as;

$$N(\Psi_0(s,t) + \Psi_1(s,t,G_k)) = N(\Psi_0(s,t)) + \sum_{l=1}^{\infty} \Psi_1^l(t,G_k) N^l(\Psi_0(s,t)). \tag{25}$$

Eq. (26) can be written as,

$$L(\Psi_1(s,t,G_k)) + D_1((\Psi_0(s,t)),G_m)\Psi(N(\Psi_0(s,t))) + D_2(\Psi_0(s,t),G_n) = 0, \tag{26}$$

$$B(\Psi_1(s,t,G_k), \frac{d\Psi_1(s,t,G_k)}{d\xi})$$

where, $n = 1,2,...q,\quad$ and $m = q+1, q+2, ...w$

Convergence of the Method

As we know the, Method of Least Squares is used to conclude the ideal constants; so,

$$K(G_w) = \int_I R^2(s,G_w)ds, \tag{27}$$

Where I is called equation domain.

To find unidentified constants, we establish as

$$\partial_{G1}K = 0, \partial_{G2}K = 0, ...\partial_{Gs}K = 0, ... \tag{28}$$

For finding the approximated solution, we used the values of equations as;

$$\Psi(s,t) = \Psi_0(s,t) + \Psi_1(s,t) \tag{29}$$

2.3 Application of OHAM to time-fractional Fokker-Planck equation. Consider the following example of the model, the time-fractional Fokker-Planck equation, for the demonstration of the efficiency of the extended formulation of OHAM.

3. Numerical examples

To explain the efficiency and accuracy of the OHAM, we found an approximate solution to the time-fractional Fokker-Plank equation and compared it with the exact solution to the problem.

Example: 01 Consider time-fractional Fokker-Plank equation with initial equation and exact solution [44].

$$\frac{\partial^{\gamma} u}{\partial t^{\gamma}} - (1+s)\frac{\partial u}{\partial s} - (e^t s^2)\frac{\partial^2 u}{\partial s^2} = 0 \tag{30}$$

The exact solution for the no fractional case at $\gamma = 1$ becomes

$$u(s,t) = (s+1)e^t \tag{31}$$

with intial condition;

$$u(s,0) = 1 + s$$

To solve the given test example, firstly take initial condition as follow;

$$u_0(s,t) = (s+1) \tag{32}$$

To find the value of, u_1

$$u_1(s,t) = \frac{1}{\Gamma(1-a)} \int_0^t (t-r)^{a-1}(c_1 + sc_1)dr \tag{33}$$

From which we get,
Hence,

$$u_1(s,t) = \frac{t^a(1+s)c_1}{a\Gamma(1-a)} \tag{34}$$

$$\begin{aligned} u_2(s,t) = \frac{1}{\Gamma(1-a)} \int_0^t -(t-r)^{a-1}[c_1 + sc_1 \\ + \frac{t^{-1+a}(1+s)c_1}{\Gamma(1-a)} + \frac{t^{-1+a}(1+s)c_1^2}{\Gamma(1-a)} + c_2 + sc_2]dr \end{aligned} \tag{35}$$

$$u_2(s,t) = -\frac{t^{-1+a}(1+s)(t^a + t\Gamma(1-a))c_1 + t^a c_1^2 + t\Gamma(1-a)c_2}{a\Gamma(1-a)^2} \tag{36}$$

Now,

$$u(s,t) = u_0(s,t) + u_1(s,t) + u_2(s,t) \tag{37}$$

$$u(s,t) = 1 + s + \frac{t^a(1+s)c_1}{a\Gamma(1-a)} - \frac{t^{a-1}(1+s)(t^a + t\Gamma(1-a))c_1 + t^a c_1^2 + t\Gamma(1-a)c_2}{a\Gamma(1-a)^2} \tag{38}$$

$$\begin{aligned} \partial_t u(s,t) = \frac{t^{-1+a}(1+s)c_1}{\Gamma(1-a)} - \\ \frac{t^{a-1}(1+s)(at^{-1+a} + \Gamma(1-a))c_1 + at^{a-1}c_1^2 + \Gamma(1-a)c_2}{a\Gamma(1-a)^2} - \\ \frac{(a-1)t^{a-2}(1+s)(t^a + t\Gamma(1-a)c_1 + t^a c_1^2 + t\Gamma(1-a)c_2)}{a\Gamma(1-a)^2} \end{aligned} \tag{39}$$

$$R = \frac{1}{\Gamma(1-a)} \int_0^t -(t-r)^{a-1}(\partial_t u(s,t))dr - (1+s)u(s,t)\partial_s u(s,t) - (e^t s^2)\partial_{ss} u(s,t) \tag{40}$$

$$\begin{aligned} R = \frac{t^{-2+2a}(1+s)(-1+2a)t^a c_1(1+c_1) + at\Gamma(1-a)c_2}{\alpha^2\Gamma(1-a)^3} - \\ (1+s)[(1+\frac{t^a c_1}{a\Gamma(1-a)}) - \frac{t^{-1+a}((t^a + t\Gamma(1-a))c_1 + t^a c_1^2 + t\Gamma(1-a)c_2}{a\Gamma(1-a)^2}] \\ ((1+s+\frac{t^a(1+s)c_1}{a\Gamma(1-a)}) - \frac{t^{-1+a}(1+s)((t^a + t\Gamma(1-a))c_1 + t^a c_1^2 + t\Gamma(1-a)c_2}{a\Gamma(1-a)^2}) \end{aligned} \tag{41}$$

Used Least Square Method for finding values of C_i,

$$J = \int_0^1 \int_0^1 R^2 ds dt \tag{42}$$

$$\begin{aligned}J = {} & 4.124570508056081 \times 10^{-7}(1.503186813727686 \times 10^{7} - 2228835.224727802c_1 - \\ & 2228540.16288397c_1^2 - 4128.369733556316c_1^3 - 13805.712537475934c_1^4 - \\ & 13936.656734249094c_1^5 - 4390.258127102639c_1^6 + 218.82352941176467c_1^7 \\ & +54.70588235294117c_1^8 - 9769362.178724483c_2 - 3387.189533785546c_1^2c_2 \\ & -5825.604700424672c_1^3c_2 - 540.8664323462881c_1^4c_2 + 2846.3231014392577c_1^5c_2 \\ & +948.7743671464192c_1^6c_2 + 2571732.951609002c_2^2 + 6189.643190159692c_1^2c_2^2 + \\ & 12379.286380319383c_1^3c_2^2 + 6189.643190159692c_1^4c_2^2 - \\ & 1028.1745026455726(3.4000000000000004c_1 \\ & (36.00000000000001 - 347.2000000000001c_2) + \\ & 3.4000000000000004c_1^2\left(36.00000000000001 - 347.2000000000001c_2\right) - \\ & 2.8000000000000007\left(85. - 124c_2\right)c_2^2) + 12.018162448859616(23998.464000000004c_1^3 \\ & +11999.232000000002c_1^4 + 302.40000000000003c_1\left(28.000000000000007 - 68.2c_2\right)c_2 \\ & +302.40000000000003c_1^2\left(39.68000000000001 + 28.000000000000007c_2 - 68.2c_2^2\right) + \\ & c_2^2\left(752.6400000000002 - 2079.0000000000005c_2 + 1636.8000000000002c_2^2\right)) + \\ & 4.031496459544175c_1(115.20000000000002c_1^2\left(45.000000000000014 - 248.c_2\right) + \\ & 57.60000000000001c_1^3\left(45.000000000000014 - 248.c_2\right) + \\ & c_2\left(806.4000000000005 - 3960.000000000002c_2 + 4464.000000000003c_2^2\right) + \\ & c_1\left(2592.0000000000014 - 13478.400000000003c_2 - 3960.000000000002c_2^2 + \right. \\ & \left. 4464.000000000003c_2^3\right)))\end{aligned} \tag{43}$$

By solving "J", we can get the values of constants;

$$c_1 = -0.4999999999999838, \qquad c_2 = 6.313616560095578$$

Now put values of these constant in equation (38), we get

$$u(s,t) = 1 + s - \frac{0.4999999999999838t^a(1+s)}{a\Gamma(1-a)} - \tag{44}$$

$$\frac{t^{-1+a}(1+s)\begin{pmatrix} 0.2499999999999838t^a + \\ 6.313616560095578t\Gamma(1-a) - \\ 0.4999999999999838\left(t^a + t\Gamma(1-a)\right) \end{pmatrix}}{a\Gamma^2(1-a)} \tag{45}$$

As we know that $t > 0,\ s \in \mathbb{R},\ 0 < \alpha \leq 1$, so by comparing with exact solution, [44]
The exact and the OHAM solutions are compared to determine the convergence of the method.

4. Figures and Tables

From Table 01, we can see that OHAM is closed converging to the exact solution and that the error is almost completely ignorable. Now for more comparison and finding accuracy and approximation, we find out some graphical analysis of the subject model for a better understanding of the solution of the method. So we draw the 3D and 2D graphs of the OHAM solution, i.e., $u(s,t)$, and the 2D and 3D graphs of the exact solution for comparison purposes with the help of Mathematica.

From the graphical analysis of both solutions of the model, it is very clear that OHAM has good convergence power and valuable accuracy for the closed-form solution.

Now, for a more accurate study of the method, we draw some 2D and 3D graphs of solutions with the help of Mathematica

Table 1: Comparison of Solution.

s	Exact Solution	OHAM Solution	Abs error
0.	1.0010005001667084	0.9985409395082848	0.002459560658423565
0.1	1.1011005501833793	1.0983950334591133	0.0027055167242659994
0.2	1.20120060020005	1.1982491274099416	0.0029514727901083226
0.3	1.301300650216721	1.2981032213607704	0.003197428855950646
0.4	1.4014007002333917	1.3979573153115987	0.003443384921792969
0.5	1.5015007502500626	1.4978114092624273	0.0036893409876352923
0.6	1.6016008002667335	1.5976655032132558	0.0039352970534776155
0.7	1.7017008502834043	1.6975195971640844	0.004181253119319939
0.8	1.8018009003000752	1.7973736911149127	0.004427209185162484
0.9	1.9019009503167459	1.897227785065741	0.004673165251004807
1.	2.0020010003334168	1.9970818790165696	0.00491912131684713

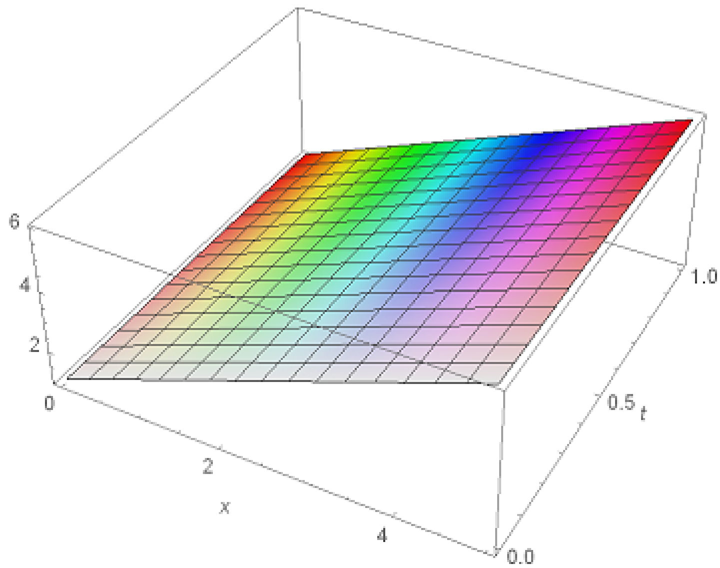

Figure 1: 3D plot of,OHAM Sol.

From the study of the graphical analysis of both solutions, one point is very clear: the solution obtained by the Optimal Auxiliary Fractional Method (OAFM) is much closer to the exact solution of the problems. It is also noted from the graphical analysis that the solution found with the help of OAFM very quickly converged to the exact solution of the model. Now to explore approximation of both solutions obtained by OHAM and OAFM, we compare both solutions. Also, the absolute error found by these methods is compared, so that it becomes clearer which method is more reliable and precise than the other one.

From Table 3, we can see that the values obtained by both solutions are much closer to the closed form solution given in the problem. It can be seen that the absolute error of both methods is negligible. Now let the same model be solved by another method, i.e., the Optimal Auxiliary Fractional Method (OAFM);

The Model is,

$$\frac{\partial^\gamma u}{\partial t^\gamma} - (1+s)\frac{\partial u}{\partial s} - (e^t s^2)\frac{\partial^2 u}{\partial s^2} = 0 \tag{46}$$

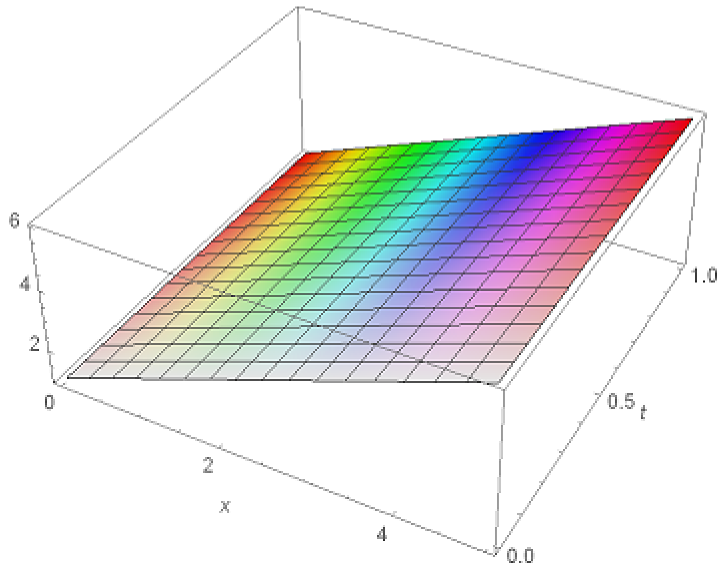

Figure 2: 3D plot of exact solution

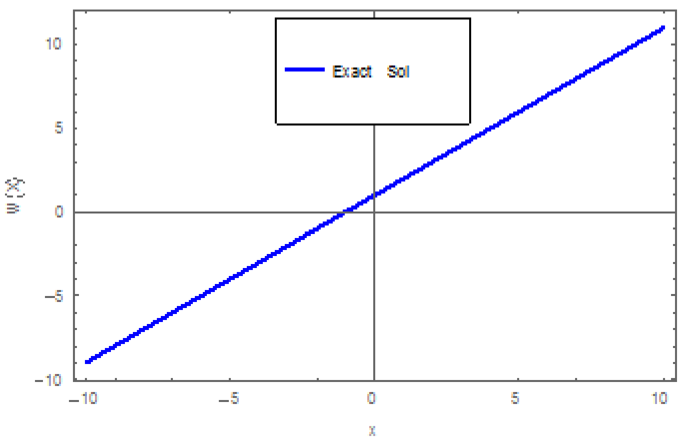

Figure 3: 2D plot of OHAM solution

The initial condition is,

$$u(s,0) = (1+s)$$

. First we take initial condition,

$$u_0(s,t) = 1+s \tag{47}$$

which gives us,

$$u_0 = 1+s$$

.
Now let consider,

$$NL = -(1+s)\,\partial_s u_0(s,t) - \left(e^t s^2\right)\partial_{ss}\left(u_0(s,t)\right) \tag{48}$$

$$NL = -1-s$$

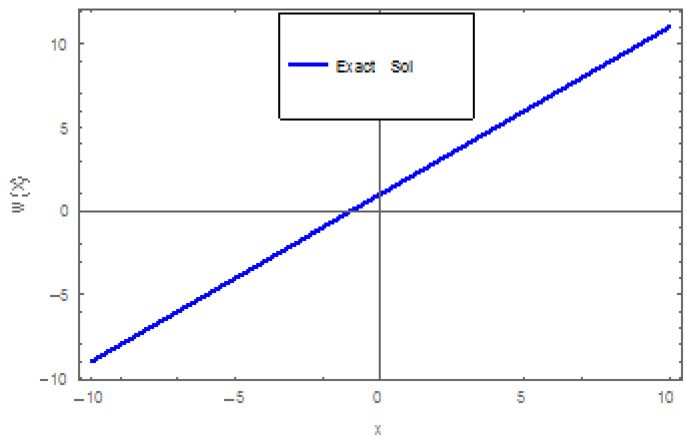

Figure 4: 2D plot of exact solution

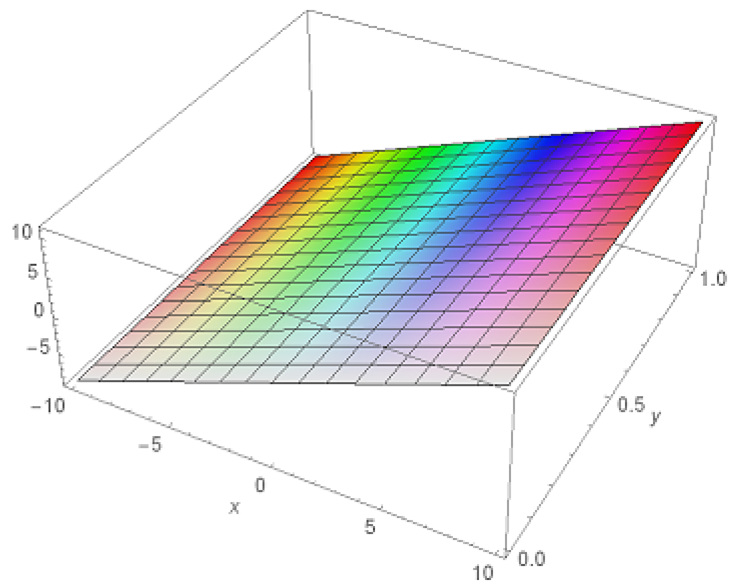

Figure 5: plot 3D[u(s,t),x,-10,10,y,0,1]

Let consider,

$$u_1 = \frac{1}{\Gamma(a)} \int_0^t (t-r)^{a-1} \left((A_1 NL + A_2)\right) \mathrm{d}r \tag{49}$$

From the given technique,

$$A_1 = c_1(1+s)^2 + c_2(1+s)^4 \tag{50}$$

$$A_2 = c_3(1+s)^6 + c_4(1+s)^8 \tag{51}$$

$$u_1 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left((A_1 NL + A_2)\right) \mathrm{d}r \tag{52}$$

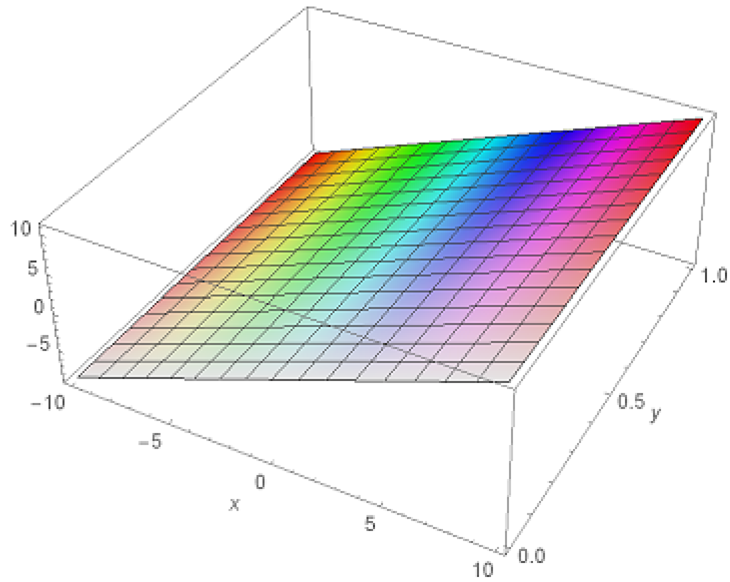

Figure 6: Plot 3D, [exact, x,-10, 10, y, 0, 1]

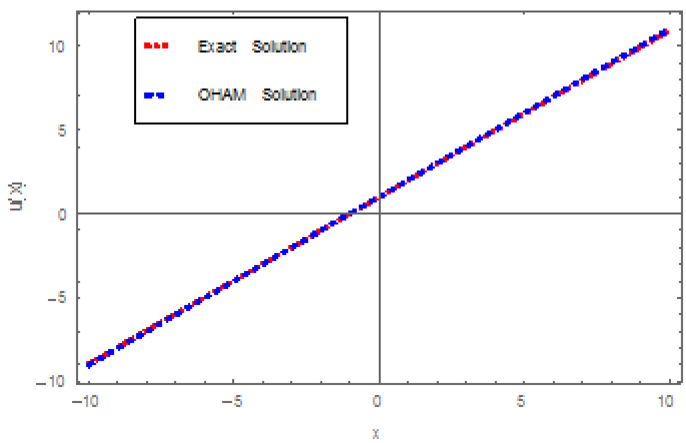

Figure 7: 2D Plot of comparison of Solutions.

$$u_1 = \frac{t^a (1+s)^3 \left(-c_1 + (1+s)^2 \left(-c_2 + (1+s)\left(c_3 + c_4(1+s)^2\right)\right)\right)}{a\Gamma(a)} \tag{53}$$

Now take,

$$u(s,t) = u_0(s,t) + u_1 \tag{54}$$

Put the values of, u_0 and u_1 in (52), we can get;

$$u(s,t) = (1+s) + \frac{t^a (1+s)^3 \left(-c_1 + (1+s)^2 \left(-c_2 + (1+s)\left(c_3 + c_4(1+s)^2\right)\right)\right)}{a\Gamma(a)} \tag{55}$$

$$\partial_t u(s,t) = \frac{t^{-1+a} (1+s)^3 \left(-c_1 + (1+s)^2 \left(-c_2 + (1+s)\left(c_3 + c_4(1+s)^2\right)\right)\right)}{\Gamma(a)} \tag{56}$$

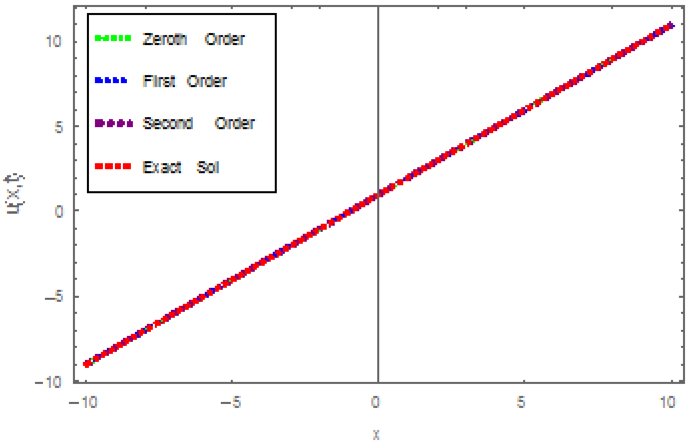

Figure 8: α,Variation while, $0<\alpha \leq 1$,

Table 2: Comparison of Solution.

s	Exact Solution	OAFM Solution	Abs error
0.	1.0001000050001667	1.0002534563121752	0.00015345131200850126
0.1	1.1001100055001836	1.1002903967997657	0.00018039129958213884
0.2	1.2001200060002	1.2003224484684567	0.00020244246825673606
0.3	1.3001300065002168	1.3003497517565028	0.00021974525628598585
0.4	1.4001400070002332	1.4003736670194276	0.0002336600191943372
0.5	1.50015000750025	1.5003965218506534	0.00024651435040334846
0.6	1.600160008000267	1.6004209817901156	0.00026097378984868413
0.7	1.7001700085002835	1.7004489487114267	0.0002789402111431638
0.8	1.8001800090003002	1.800479884088545	0.00029987508824480535
0.9	1.9001900095003166	1.9005084472533313	0.0003184377530147664
1.	2.0002000100003334	2.0005213316657753	0.0003213216654418538

$$R = \frac{1}{\Gamma(1-a)} \int_0^t (t-r)^{-a} \left(\partial_t u(s,t)\right) \mathrm{d}r - (1+s)\, \partial_s u_0(s,t) - \left(e^t s^2\right) \partial_{ss} \left(u_0(s,t)\right) \tag{57}$$

$$R = (-1-s) + \frac{(1+s)^3 \left(-c_1 + (1+s)^2 \left(-c_2 + (1+s)\left(c_3 + c_4 (1+s)^2\right)\right)\right)}{(1-a)\,\Gamma(1-a)\Gamma(a)} \tag{58}$$

$$where, 0 < \alpha \leq 1,$$

To find the values of C_i ,we used Least Square Method.

$$J = \int_0^1 \int_0^1 R^2 \mathrm{d}s\mathrm{d}t \tag{59}$$

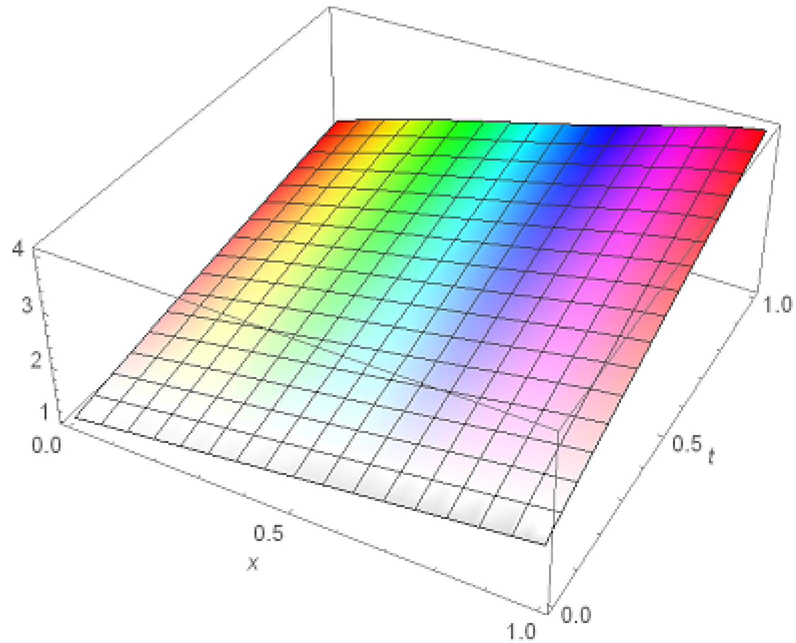

Figure 9: 3D plot of OAFM Sol.

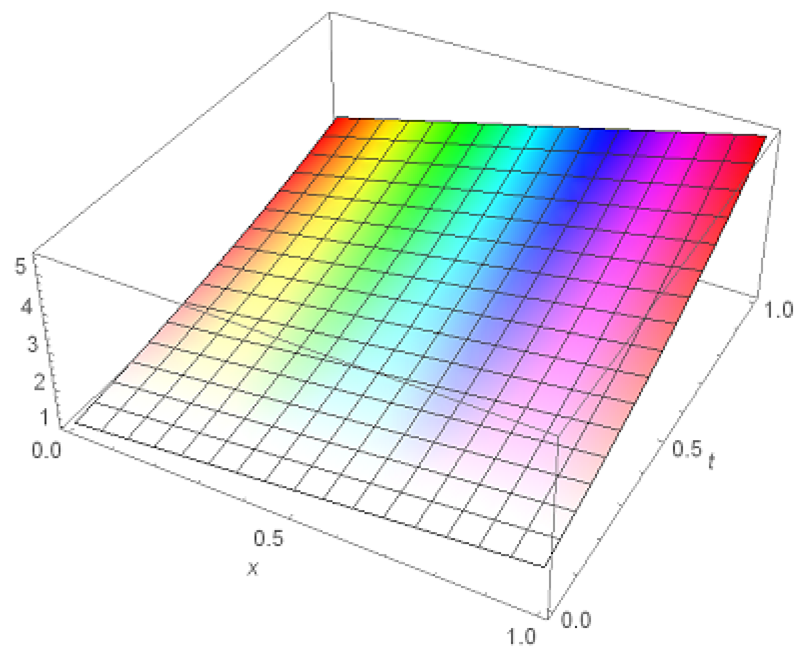

Figure 10: 3D plot of, exact solution.

$$\begin{aligned}
J = 2.3333333333333335 + 17.553780511133574{c_1}^2 + 180.04876230782605{c_2}^2 \\
-62.706517246570954c_3 + 609.6190873209363{c_3}^2 + \\
c_2\left(35.69177676331434 - 660.3400503302414c_3 - 2264.4376859362314c_4\right) \\
+c_1\left(12.197032374234977 + 109.86854398657006c_2 - 197.95688541768115c_3\right. \\
\left.-660.3400503302414c_4\right) - 201.25103417487713c_4 + 4227.0793512422015c_3c_4 \\
+7459.722537111404{c_4}^2
\end{aligned} \tag{60}$$

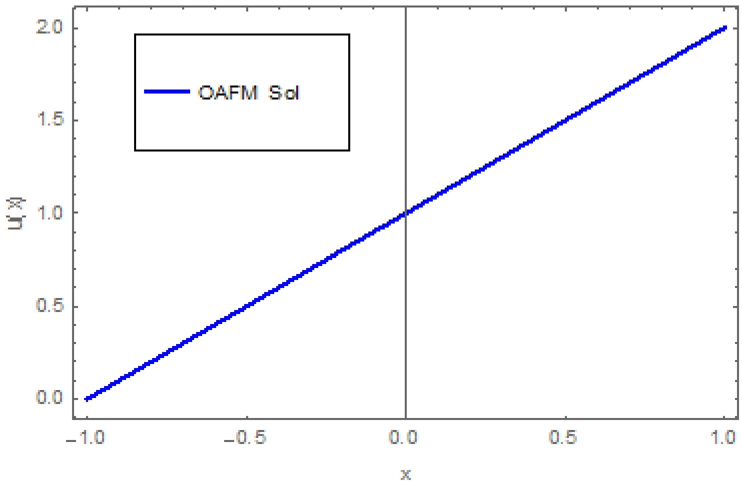

Figure 11: 2D plot of OAFM Sol.

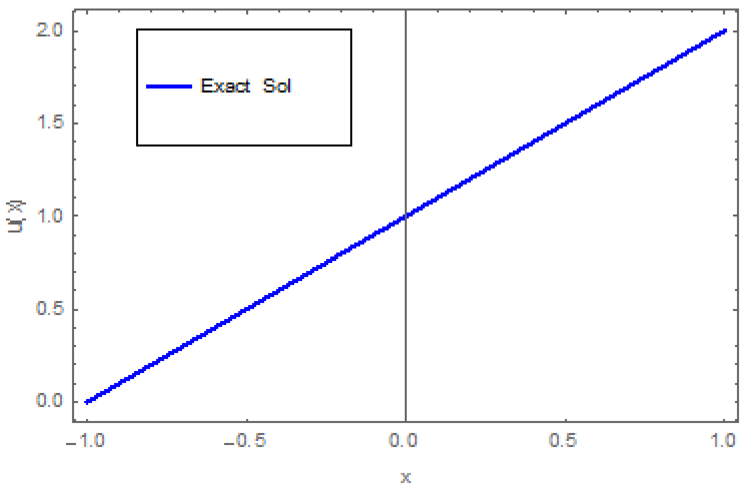

Figure 12: 2D plot of Exact Sol.

Put values of, C_i in (58),

$$c_1 = -2.1723055094044685$$
$$c_2 = 2.326960050050942$$
$$c_3 = 1.1924269486431436$$
$$c_4 = -0.0673239917243382$$

we get,

$$u(s,t) = (1+s)+ \frac{t^{\alpha}(1+s)^3(2.1723055094044685 + (1+s)^2(-2.326960050050942 + (1+s)(1.1251029569188)(1+s)^2}{\alpha\Gamma(\alpha)} \tag{61}$$

As we know that, $0 < \alpha \leq 1$, so for $\alpha = 1$,

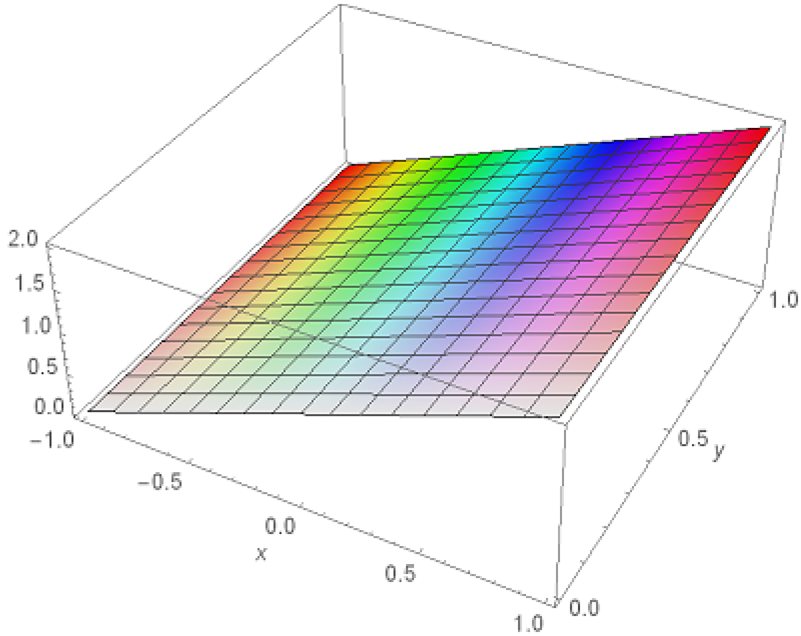

Figure 13: Plot 3D[OAFM,$x, -1, 1, y, 0, 1$]

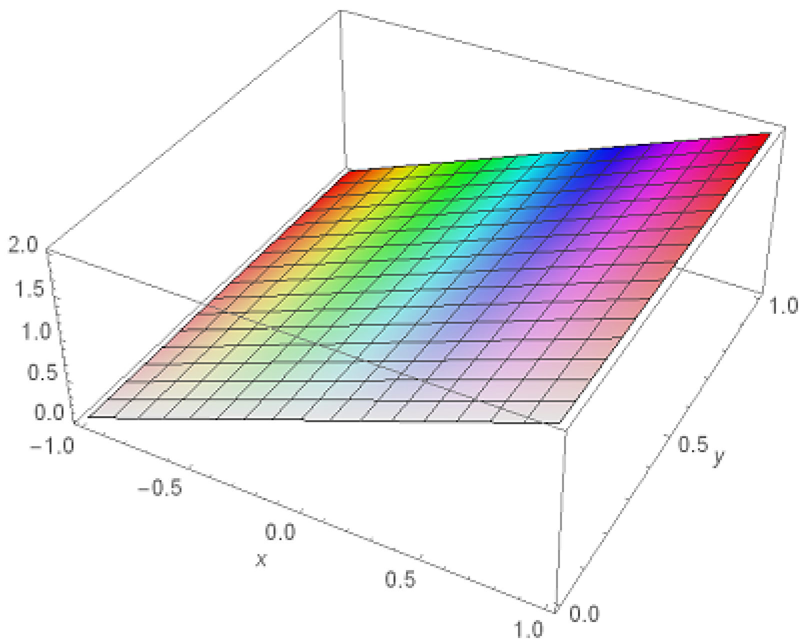

Figure 14: Plot 3D[Exact,$x, -1, 1, y, 0, 1$]

We obtained the closed form solution of the given problem [44]

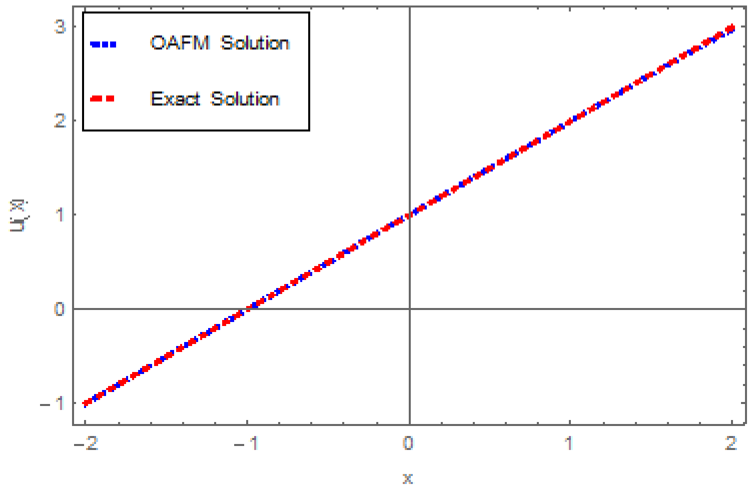

Figure 15: 2D Plot of comparison of Solutions.

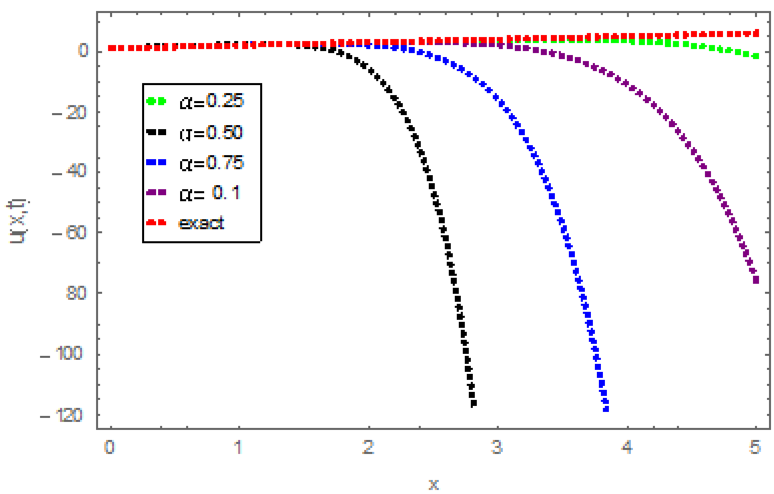

Figure 16: α,Variation while, $0<\alpha \leq 1$,

5. Results Analysis

The mathematical theories of OHAM and FOAFM offer a very precise solution to the time-fraction Fokker-Planck equation with the initial condition presented in Section 3.For the computational work, we used Mathematica software. The consequences found by the two methods are matched with the exact solutions to the problems given in Tables [1]-[2]. The techniques are effective and more correct than other analytical methods described in the literature. The absolute errors of the techniques for dissimilar numeral values are shown in Tables [1]-[2] and Figs. [7]-[15]. The solution is once again tested by comparing it with the exact solutions presented in 3D form in Figs. [1],[2], [9], and [10]. Similarly, the absolute errors found by OHAM and OAFM are matched with other approaches in the literature, and it is determined that these methods' consequences are more precise than the other technique.

Table 3: Comparison of Solutions of OHAM and OAFM.

s	Exact Sol	OHAM Sol	OAFM Sol	Abs error of OHAM	Abs error of OAFM
0.	1.00001	0.999948	1.00003	0.0000619947	0.0000219082
0.1	1.10001	1.09994	1.10004	0.0000681941	0.0000255587
0.2	1.20001	1.19994	1.20004	0.0000743936	0.0000285938
0.3	1.30001	1.29993	1.30004	0.0000805931	0.0000310311
0.4	1.40001	1.39993	1.40005	0.0000867926	0.0000330418
0.5	1.50002	1.49992	1.50005	0.000092992	0.0000349191
0.6	1.60002	1.59992	1.60005	0.0000991915	0.0000369984
0.7	1.70002	1.69991	1.70006	0.000105391	0.0000395192
0.8	1.80002	1.79991	1.80006	0.00011159	0.0000424137
0.9	1.90002	1.8999	1.90006	0.00011779	0.0000450096
1.	2.00002	1.9999	2.00007	0.000123989	0.0000456317

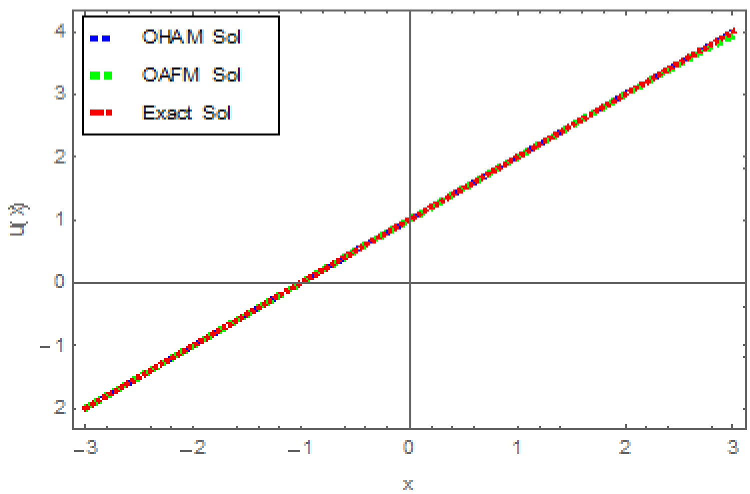

Figure 17: Comparison of OHAM, OAFM and Exact Solution.

6. Conclusion

The optimal homotopy asymptotic method and optimal auxiliary fractional method are used to solve the TFFPE. OHAM is applied to the TFFPE, and an approximate solution is obtained. Then, OAFM was applied to the same model, and a series solution for the first iteration was obtained. The accuracy of two methods is shown by comparing their results with those available in the literature, showing that these techniques are simple to apply and can provide accurate results with less computational effort not only for linear PDEs but also for nonlinear PDEs. The convergence of the methods is controlled by means of auxiliary functions. FOAFM does not have any constraints, which allows us to implement it in complex physical problems. Both methods are reliable and very easy for fractional-type problems. From the comparison of these techniques, we conclude that, these methods are simple in calculation and need less computational work. The accuracy and convergence is very strong. All the computational work has been done by Mathematica.

References

1. G. Baumann and F. Stenger, Fractional Fokker?Planck equation, Mathematics 5 (2017), 1?19.

2. X. N. Cao, J. L. Fu, and H. Huang, Numerical method for the time fractional Fokker?Planck equation, Adv. Appl. Math. Mech. 4 (2012), 848?863.

3. S. V. Dolgov, B. N. Khoromskij, and I. Oseledets, Fast solution of multi-dimensional parabolic problems in the TT/QTT formats with initial application to the Fokker?Planck equation, SIAM J. Sci. Comput. 34 (2012), A3016?A3038.

4. R. S. Dubey, B. S. Alkahtani, and A. Atangana, Analytical solution of space?time fractional Fokker?Planck equation by homotopy perturbation equation by Homotopy perturbation Sumudu transform method, Math. Probl. Eng. 2015 (2015), 780929.

5. H. Habenom and D. L. Suthar, Numerical solution for the time-fractional Fokker?Planck equation via shifted Chebyshev polynomials of the fourth kind, Adv. Differ. Eq. 2020 (2020), 1?16.

6. H. Habenom, D. L. Suthar, and M. Aychluh, Solution of fractional Fokker Planck equation using fractional power series method, J. Sci. Arts 48 (2019), 593?600.

7. S. Hesam, A. R. Nazemi, and A. Haghbin, Analytical solution for the Fokker?Planck equation by differential transform method, Sci. Iran. B 19 (2012), 1140 1145.

8. H. Risken, The Fokker?Planck equation methods of solution and applications,Springer, Berlin, Germany/New York, NY, 1989.

9. T. Sutradhar, B. K. Datta, and R. K. Bera, Analytical solution of the time fractional Fokker?Planck equation, Int. J. Appl. Mech. Eng. 19 (2014), 435?440.

10. Y. Yang, Y. Huang, and Y. Zhou, Numerical solutions for solving time fractional Fokker?Planck equations based on spectral collocation methods, J. Comput. Appl. Math. 339 (2018), 389?404.

11. L.T. Watson, Probabily one homotopies in computational sciences. Technical Report TR 00-03, comp. Sci. Virg. Tech. Vol. 1, 2000.

12. L.T. Watson, S.C. Billups, A.P. Morgan, HOMPACK: a suite of codes for globally convergent homotopy algorithms. ACM Trans. Math. Soft. 13 (1987) 281-310.

13. L.T. Watson, M. Sosonkina, R.C. Melvile, A.P.Morgan, H.F. Walker, Algorithm 777: HOMPACK 90: a suite of Fortran 90 codes for globally convergent homotopy algorithms. ACM Trans. Math. Soft. 23 (1997) 514-549.

14. M. Torrisi, R. Tracina, A. Valenti, A group analysis approach for a nonlinear differential system arising in diffusion phenomena. J. Math. Phys. 37 (9) (1996), 4758-4767.

15. P.G. Drzain, R.S. Johnson, An introduction discussion the theory of solution and its diverse applications. Camb. Uni. Pres. 1989.

16. B. Abdel-Hamid, Exact solutions of some nonlinear evolution equations using symbolic computations, Comp. Math. Appl. 40 (2000) 291-302.

17. G. Bluman, S. Kumei, on the remarkable nonlinear diffusion equations. J. Math. Phys. 21 (50 1019-1023.

18. LC. Chun, Fourier series based variational iteration method for a reliable treatment of heat equations with variable coefficients. Int. J. Non-linr. Sci. Num. Simu. 10 (2007) 1383-1388.

19. S.H. Chowdhury, A comparison between the modified homotopy perturbation method and adomian decomposition method for solving nonlinear heat transfer equations, J, Appl, Sci. 11 (8) (2011) 1416-1420.

20. H. Yaghoobi, M. Tirabi, The application of differential transformation method to nonlinear equation arising in heat transfer. Int. Comm. Heat Mass Transfer 38 (6) (2011) 815-820.

21. D.D. Ganji, The application of He?s homotopy perturbation method to nonlinear equation arising in heat transfer, Phy. Lett. A 355 (2006) 337-341.

22. R. Bellman, Perturbation techniques in Mathematics, Phys. and Engg. Holt, Rinehart and Winston, New York, 1964.

23. J.D. Cole, Perturbation Methods in Applied Mathematics, Blaisedell, Waltham, MA. 1968.

24. R.E. O?Malley, Introduction to Singular Perturbation. Acad. Pres. New York. 1974.

25. G.L. Liu, New research direction in singular perturbation theory: artificial Parameter approach and inverse perturbation technique, in: Conf. 7th Mod. Maths. Mech. 1997.

26. S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear Problems. PhD thesis, Shanghai Jiao Tong University; 1992.

27. V. Marinca, N.Herisanu, I.Nemes, Optimal homotopy asymptotic method with application to thin film flow. Central European Journal of Physics 2008; 6:64853.

28. N, Herisanu, V, Marinca, Explicit analytical approximation to large amplitude nonlinear osciallation of a uniform contilaver beam carrying an intermediate lumped mass and rotary inertia. Mecc, 45 847-855 2010.

29. V. Marinca, N.Herisanu, An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate. Applied Mathematics Letters 2009; 22:245?51.

30. V. Marinca, N.Herisanu, I.Nemes, A New Analytic Approach to Nonlinear Vibration of An Electrical Machine, Proceeding of the Romanian Academy 9(2008) 229-236.

31. H. Ullah, S. Islam, M. Idrees, R. Nawaz, Optimal Homotopy Asymptotic Method to doubly wave solutions of the coupled Drinfel?d-Sokolv-Wilson equations, Math. Prob. Engg. Arti. ID 362816, 8 pages.

32. H. Ullah, S. Islam, M. Idrees and M. Arif, Solution of Boundary Layer Problems with Heat Transfer by Optimal Homotopy Asymptotic Method. Abst. App. Anal.Volu. 2013 (2013), Arti. ID 324869, 10 pages.

33. H. Ullah, S. Islam, M. Idrees, M. Arif, Solution of boundary layer problems with heat treansfer by optimal homotopy asymptotic method. Abst. Appl. Anal. Volume (2013) Article ID 324869 10 pages.

34. H. Ullah, S. Islam, M. Idrees, R. Nawaz, Application of optimal homotopy asymptotic method to doubly wave solutions of the coupled Drinfel?d Sokolv-Wilson equations. Math. Prob. Engi. Volume (2013) Article ID 362816 8 pages.

35. H. Ullah, S. Islam, M. Idrees, R. Nawaz, M. Fiza, The flow past a rotating disk by optimal homotopy asymptotic method. World. Appl. Sci. J. 11 (2014) 1409-1414.

36. H. Ullah, S. Islam, M. Idrees, M. Fiza, An extenstion of the optimal homotopy asymptotic method to coupled Schrodinger-KdV equation. Int. J. Diff. Eqns. Volume 2014 Article ID 106934 12 pages.

37. H. Ullah, S. Islam, M. Idrees, M. Fiza, Solution of the differential-difference equation by optimal homotopy asymptotic method. Abst. Appl. Anal. Volume 2014 Aricle ID 520467 8 pages.

38. H. Ullah, S. Islam, M. Idrees, M. Fiza, The three dimensional flow past a stretching sheet by extended optimal homotopy asymptotic method. Sci. Int. 26 (2014) 567-576.

39. H. Ullah, S. Islam, I. Khan, S. Sharidna, M.Fiza, Formulation and implementation of optimal homotopy asymptotic method to coupled differential-difference equations. 10.1371/journal.pone.0120127. Plos One.

40. H. Ullah, S. Islam, I. Khan, S. Sharidan, M. Fiza, Approximate solution of the generalized coupled Hirota- Satsuma coupled KdV equation by extended optimal homotopy asymptotic method, Magnet research report Vol.2 (7). PP: 3022-3036.

41. H. Ullah, S. Islam, M. Fiza, The optimal homotopy asymptotic method with application to Inhomogeneous nonlinear wave equations. Sci.Int, 26(5), 1907-1913, 2014.

42. H. Ullah, S. Islam, I. Khan, L.C.C. Dennis, M. Fiza, Approximate solution of two dimensional nonlinear wave equation by optimal homotopy asymptotic method. Mathem. Prob. Engi. IF 1.082. In press.

43. B. Marinca, V. Marinca, Approximate analytical solutions for thin film flow of fourth grade fluid down a vertical cylinder, Proc. Roman. Acad. Ser. A. 19. (2018).

44. Mahdy, A. M. (2021). Numerical solutions for solving model time-fractional Fokker-Planck equation. Numerical Methods for Partial Differential Equations, 37(2), 1120-1135.

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