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Chaotic Asymptotic in Solutions of the Conformable Fractional Hyperbolic Heat Transfer Equation

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ABSTRACT: We study for the first time in the literature on the subject, the chaotic behavior of the β-semigroup associated with the solution of the Cauchy problem for a hyperbolic conformal fractional heat equation.

Key Words: β-semigroups, hypercyclicity, Devaney chaos, chaotic β-semigroups, hyperbolic conformal fractional heat equation.

Contents

1. Introduction

Chaos is commonly associated with nonlinear phenomena, but it can also manifest in linear dynamical systems when the underlying space is infinite-dimensional. The study of chaos in finite-dimensional dynamical systems, encompassing discrete maps and ordinary differential equations, has seen significant development, resulting in crucial applications in physics, biology, chemistry, and engineering. Despite this progress, there was a prolonged absence of a chaos theory for fractional partial differential equation, which are fundamental in describing many significant natural phenomena in many applications. Various definitions of fractional derivatives have been introduced, including the Caputo and Riemann-Liouville definitions. For more information, we refer readers to the books [3,4]. The complexity of these definitions presents an unfortunate obstacle when it comes to real models. Nevertheless, a novel definition of the fractional derivative, called the conformable fractional derivative, has been introduced in [2]. This new derivative is simple and verifies all the properties of the usual derivative. Furthermore, it has a various advantage over other fractional derivatives in several aspects.

The study of β-semigroups introduced in [1] has been largely identified with fractional partial differential equations of parabolic and hyperbolic types involving conformable derivatives. It is now well-established that solutions to these equations can be expressed in termes of β-semigroups. These semigroups enable the resolution of the abstract Cauchy problem corresponding to a broader framework.

In this paper, we provide a new perspective on the chaotic bihavior of any β-semigroup which is the solution of a certain class of fractional partial differential equations with conformable derivatives, considering both Devaney and distributional chaos. The investigation will be conducted on Herzog type spaces, as detailed in [5]. Herzog's findings were subsequently refined in [6]. These spaces comprise analytic functions regulated by a parameter, facilitating control over their growth at infinity. Initially introduced for exploring the universality of solution operators for the heat equation, these spaces were the subject of examination by Chan and Shapiro in [8]. They explored the dynamics of the translation operator within spaces of analytic functions exhibiting slow growth, providing characterizations for when

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the derivative operator is bounded in these settings. Given that the derivative operator serves as the infinitesimal generator of the translation semigroup, it follows that the translation semigroup is uniformly continuous, and all its operators can be derived using the exponential formula, as elucidated in [9, Th. 3.7]. Noteworthy constructions and counterexamples have been presented within specific subspaces of analytic functions, as demonstrated in [10,11,12]. Godefroy and Shapiro also addressed the dynamics of shift operators within Hardy and Bergman spaces, as discussed in [7].

Our findings reveal a compelling interplay between chaos and stability, delineated by a critical parameter contingent on the specific equation under consideration. We derive precise conditions, incorporating both the equation's coefficients and the tuner, highlighting a noteworthy phenomenon seemingly connected to the tuning parameter's dependence within the underlying Herzog space. A comparable examination of the dichotomy between chaos and stability can be observed in [14,13].

This article is organized as follows: In Section 2, we revisit the definitions and tools needed to state the main result. More precisely, we present a useful spectral criterion for determining Devaney chaos in β-semigroups. Section 3 presents our main results (Theorem 3 and Theorem 4), which assert that the heat and wave equations admit chaos

2. Preliminairies

In this section, we give some notations, definitions and results on the conformable derivative and β-semigroup.

Definition 2.1 *[2] Given a function $f : [0, \infty) \to \mathbb{R}$. Then the conformable fracfional derivative of f of order β is defined by*

$$d^{\beta} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\beta}) - f(t)}{\epsilon}$$

for all $t > 0$, $\beta \in (0, 1)$. If is β-differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} d^{\beta} f(t)$ exists, then define

$$d^{\beta} f(0) = \lim_{t \to 0} f^{(\beta)}(t).$$

The conformable derivative satisfies all the classical properties of derivative. Further, according to this derivative, the following statements are true, see

Proposition 2.1 *[2] Let $\beta \in (0, 1)$. Then*

1. $d^{\beta}(t^p) = pt^{p-\beta}$ *for all* $p \in \mathbb{R}$,
2. $d^{\beta}(\sin \frac{1}{\beta} t^{\beta}) = \cos \frac{1}{\beta} t^{\beta}$,
3. $d^{\beta}(\cos \frac{1}{\beta} t^{\beta}) = -\sin \frac{1}{\beta} t^{\beta}$,
4. $d^{\beta}(e^{\frac{1}{\beta} t^{\beta}}) = e^{\frac{1}{\beta} t^{\beta}}$

The β-fractional integral of a function f starting from $a \geq 0$, see

$$I^a_{\beta}(f)(t) = I^a_1(t^{\beta-1} f) = \int_a^t \frac{f(x)}{x^{1-\beta}} dx,$$

where the integral is the usual Riemann improper integral, and $\beta \in (0, 1)$. For more about higher conformable fractional integrals and derivatives in left and right senses and other basic concepts we refer to [2].

In [1], the authors gave a definition of fractional semigroups of operators associated with the conformable fractional derivative,

Definition 2.2 *[1] Let $\beta \in (0, a]$ for any $a > 0$. For a Banach space X. A family $\{T_\beta(t)\}_{t\geq 0} \subseteq \mathcal{L}(X, X)$ is called a fractional β-semigroup or β-semigroup of operators if:*

(i) $T_\beta(0) = I$,

(ii) $T_\beta(s+t)^{\frac{1}{\beta}} = T_\beta(s^{\frac{1}{\beta}})T_\beta(t^{\frac{1}{\beta}})$ *for all* $s, t \in [0, \infty)$.

Definition 2.3 *[1] An β-semigroup $T_\beta(t)$ is called a c_0-semigroup if, for each fixed $x \in X, T_\beta(t)x \to x$ as $t \to 0^+$.*

The conformable β-derivative of $T_\beta(t)$ at $t = 0$ is called the β-infinitesimal generator of the fractional β-semigroup $T_\beta(t)$, with domain equals

$$\left\{x \in X : \lim_{t\to 0^+} d^\beta T_\beta(t)x \textit{ exists}\right\}.$$

We will write A for such generator.

Theorem 2.1 *[15] Let X be an infinite-dimensional separable Banach space. Suppose that the sets*

$$X_0 = Span\left\{x \in X, \quad \exists\lambda > 0, \quad T_\beta(t)x = e^{\lambda\frac{t^\beta}{\beta}}x, \quad \forall t \geq 0\right\}$$

$$X_1 = Span\left\{x \in X, \quad \exists\lambda < 0, \quad T_\beta(t)x = e^{\lambda\frac{t^\beta}{\beta}}x, \quad \forall t \geq 0\right\}$$

$$X_p = Span\left\{x \in X, \quad \exists\lambda \in \mathbb{Q}, \quad T_\beta(t)x = e^{\pi\lambda i\frac{t^\beta}{\beta}}x, \quad \forall t \geq 0\right\}.$$

are dense in X, then $(T_\beta(t))_{t\geq 0}$ is chaotic.

3. Mains results

3.1. The hyperbolic heat transfer equation with conformable derivatives

In this section, we will study the chaotic behavior of fractional partial heat transfer equations of hyperbolic types involving conformable derivatives (FPHEHC) with respect to both time and space, such as the following

$$\begin{cases} \tau d_t^{2\beta}u(t,x) + d_t^\beta u(t,x) = \delta d_x^{2\beta}u(t,x) \\ u(0,x) = \phi_1(x) \\ d_t^\beta u(0,x) = \phi_2(x). \end{cases} \tag{3.1}$$

Here, $d^{2\beta}u = d^\beta(d^\beta u)$, where d^β is the conformable fracfional derivative of order β, δ, τ two real numbers.

Let $\rho > 0$. Consider the space

$$\mathcal{F}_{\rho,\beta} = \left\{\phi : \mathbb{R}_+ \to \mathbb{R}, x \to \sum_{n=0}^{\infty} \frac{a_n}{\beta^n n!}(x\rho)^{n\beta}, \quad (a_n) \in c_0(\mathbb{N})\right\}.$$

endowed with the norm

$$\|\phi\|_\beta = \sup_{n\in\mathbb{N}} \sup_{x\geq 0} \rho^{-n\beta}\, e^{-\rho^\alpha\frac{|x|^\alpha}{\alpha}} |\partial^{n\beta}\phi(x)|.$$

Lemma 3.1 *$(\mathcal{F}_{\rho,\beta}, \|.\|_\beta)$ is a Banach space.*

Proof: Observe that $\left|\frac{a_n}{\beta^n n!}(x\rho)^{n\beta}\right| = O_{n\to\infty}(\frac{(x\rho)^{n\beta}}{\beta^n n!})$, which implies the result. □

Theorem 3.1 *The spaces $(\mathcal{F}_{\rho,\beta}, \|.\|_\beta)$ and $(c_0(\mathbb{N}), \|.\|_\infty)$ are isometrically isomorphic.*

Proof: For each $(a_n) \in c_0(\mathbb{N})$, we have

$$\|a_n\|_\infty = \sup_n \rho^{-n\beta} \partial^{n\beta} \phi(0) \leq \sup_n \sup_x \rho^{-n\beta} e^{-\rho^\alpha \frac{|x|^\alpha}{\alpha}} \partial^{n\beta} \phi(x)| \leq \|a_n\|_\infty.$$

□

Corollary 3.1 *The topological dual of $\mathcal{F}_{\rho,\beta}$ is isometrically isomorphic to ℓ^1.*

Proof: We know that the space $\mathcal{F}_{\rho,\beta}$ is isometrically isomorphic to $c_0(\mathbb{N})$. Therefore, the topological dual of $\mathcal{F}_{\rho,\beta}$ is isometrically isomorphic to the dual of $c_0(\mathbb{N})$, which is ℓ^1. Hence, the topological dual of $\mathcal{F}_{\rho,\beta}$ is isometrically isomorphic to ℓ^1. □

This space is a Banach space of analytic functions, densely embedded in $C(\mathbb{R})$ with the topology of uniform convergence on compact sets of $\mathbb{R}$, as it encompasses all polynomials. Essentially, $\mathcal{F}_{\rho,\beta}$ represents a space of analytic functions with controlled growth extending to infinity. In fact, the pair $(\mathcal{F}_{\rho,\beta}, \|.\|_\beta)$ is isometrically isomorphic to $(c_0(\mathbb{N}), \|.\|_\infty)$.

Theorem 3.2 *The solution β-semigroup $\left\{e^{\frac{t^\beta}{\beta}A}\right\}_{t\geq 0}$ of is chaotic on $\mathcal{F}_{\rho,\beta} \oplus \mathcal{F}_{\rho,\beta}$ for each $\rho > \frac{\sqrt{\lambda(1+\tau\lambda)}}{\beta\delta^{1/2}}$.*

Proof:
The solution β-semigroup to the FPHEHC of the form given in (3.1) can be expressed in terms of its β-infinitesimal generator A. To do this, by setting $u_1 := u$, and $u_2 := u_t$ we arrive at the associated conformable derivative equation: The FPHEHC's solution semigroup, as outlined in (3.1), can be represented using its β-infinitesimal generator A. This involves defining u_1 as u and u_2 as $d_t^\beta u$, leading to the corresponding fractional equation. Puting $V = \begin{pmatrix} u \\ d_t^\beta u \end{pmatrix}$. Then, we get

$$d_t^\beta V = AV. \tag{3.2}$$

With $A = \begin{pmatrix} 0 & I \\ \frac{\delta}{\tau} d_x^{2\beta} & \frac{-1}{\tau} I \end{pmatrix}$.
To apply the eigenvalue criterion, let's look for the eigenvectors of A. For this, let

$$X = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

be a eigenvector associated to the eigenvalue $\lambda \in \mathbb{R}$. We get

$$AX = \lambda X \Leftrightarrow \begin{cases} \varphi_2 = \lambda \varphi_1 \\ \frac{\delta}{\tau} d_x^{2\beta} \psi_1 - \frac{1}{\tau} \psi_2 = \lambda \psi_2 \end{cases}.$$

Implies that

$$\frac{\delta}{\tau} d_x^{2\beta} \psi_1 - \left(\frac{\lambda}{\tau} + \lambda^2 \right) \psi_1 = 0.$$

Now write $\psi_1 = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^n n!} x^{n\beta}$, we obtaind

$$\frac{\delta}{\tau} a_{n+2} - \left(\frac{\lambda}{\tau} + \lambda^2 \right) a_n = 0.$$

Then,

$$a_{2n+1} = a_1 R_\lambda^n \text{ and } a_n = a_0 R_\lambda^n,$$

where $R_\lambda = \lambda(\frac{1+\tau\lambda}{\delta})$. We define

$$\varphi_{1,\lambda,a_0,a_1}(x) = a_0 \sum_{n\leq 0} \frac{(R_\lambda)^n x^{2n}}{\beta^{2n}(2n)!} + a_1 \sum_{n\leq 0} \frac{(R_\lambda)^n x^{2n+1}}{\beta^{2n+1}(2n+1)!},$$

Clearly, this function is in $\mathcal{F}_{\rho,\beta}$ for all λ in, say, certain open disc of radius r centered at zero. If we set

$$f_{\lambda,a_0,a_1} = \left(\varphi_{1,\lambda,a_0,a_1}, \lambda\varphi_{1,\lambda,a_0,a_1}\right)^\intercal.$$

Then, we have

$$e^{\frac{t^\beta}{\beta}G}\varphi_1 = e^{\lambda\frac{t^\beta}{\beta}}\varphi_1,$$

for all $t \geq 0$. So that, if we prove that the sets

$$\mathcal{F}_0 := \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ 0 < \lambda < r,\ a_0, a_1 \in \mathbb{R}\right\}$$
$$\mathcal{F}_1 := \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ -r < \lambda < 0,\ a_0, a_1 \in \mathbb{R}\right\}$$
$$\mathcal{F}_p := \operatorname{span}\left\{f_{\lambda,a_0,a_1} :\ \lambda \in \pi i\mathbb{Q},\ |\lambda| < r,\ a_0, a_1 \in \mathbb{R}\right\}$$

are dense in $\mathcal{F}_{\rho,\beta} \oplus \mathcal{F}_{\rho,\beta}$, then the eigenvalue criterion asserts that the C_0-semigroup generated by A is chaotic on $\mathcal{F}_{\rho,\beta} \oplus \mathcal{F}_{\rho,\beta}$.

Since $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_p$ are linear subspaces of $\mathcal{F}_{\rho,\beta}$, it suffices to prove that they are weakly dense, that is: given $f \in \mathcal{F}^*_{\rho,\beta} \oplus \mathcal{F}^*_{\rho,\beta}$, if $\langle y, f\rangle = 0$ for all $y \in \mathcal{F}$ (where $\mathcal{F}$ is either $\mathcal{F}_0, \mathcal{F}_1$ or $\mathcal{F}_p$) then necessarily $f = 0$. In other words, given $g = ((\rho_n)_n, (\zeta_n)_n) \in \ell^1 \oplus \ell^1$, if

$$a_0 \sum_{n\geq 0} \mathcal{R}_\lambda^n \rho_{2n} + a_1 \sum_{n\geq 0} \mathcal{R}_\lambda^n \rho_{2n+1} + \lambda a_0 \sum_{n\geq 0} \mathcal{R}_\lambda^n \zeta_{2n} + \lambda a_1 \sum_{n\geq 0} \mathcal{R}_\lambda^n \zeta_{2n+1} = 0, \tag{3.3}$$

for all $a_0, a_1 \in \mathbb{R}$ and for all $0 < \lambda < r$ (respectively $-r < \lambda < 0, \mu = \pi i q$ with $q \in \mathbb{Q}$ and $|\lambda| < r$), then $\rho_n = \zeta_n = 0$ for all $n \geq 0$. Indeed, set $h(\lambda)$ as the left part of (3.3). Then $h(\lambda)$ is an entire function that vanishes on a subset of $\mathbb{C}$ with an accumulation point. Therefore, all coefficients of its power series should be 0 . The independent coefficient is $a_0\rho_0 + a_1\rho_1$, and this should be zero for any choice of $a_0, a_1 \in \mathbb{R}$, therefore

$$\rho_0 = \rho_1 = 0.$$

Now, if $\lambda = \frac{-1}{\tau}$, then $R_\lambda = 0$ and we have

$$a_0\zeta_0 + a_1\zeta_1 = 0,$$

for all $a_0, a_1 \in \mathbb{R}$. This yields

$$\zeta_0 = \zeta_1 = \sigma_0 = \sigma_1 = 0.$$

Suppose that all $\rho_0 = \cdots = \rho_{2n-1} = 0$, $\zeta_0 = \cdots = \zeta_{2n-1} = 0$. If we divide $h(\lambda)$ by R_λ^n then we obtain an entire function that vanishes on a set with an accumulation point. Therefore, all its coefficients should be 0. The independent coefficient is $a_0\rho_{2n} + a_1\rho_{2n+1}$ with $a_0, a_1 \in \mathbb{R}$. A similar argument as before yields

$$\rho_{2n} = \rho_{2n+1} = 0.$$

Finally, taking $\lambda = \frac{-1}{\tau}$ we get

$$a_0\zeta_{2n} + a_1\zeta_{2n+1} = 0,$$

for any choice of a_0, a_1, and then

$$\zeta_{2n} = \zeta_{2n+1} = 0.$$

If we consider $\mathcal{F}_{\rho,\beta}$ just as the corresponding space containing only the real sequences, the hypercyclicity of $\left\{e^{\frac{t^\beta}{\beta}A}\right\}_{t\geq 0}$ can be deduced on $\mathcal{F}_{\rho,\beta} \oplus \mathcal{F}_{\rho,\beta}$. with a similar proof, avoiding the part of proving the density of $\mathcal{F}_{\rho,\beta}$. According to Thoerem 2.1, then $\left\{e^{\frac{t^\beta}{\beta}A}\right\}_{t\geq 0}$ is chaotic.

□

3.2. Conformable fractional wave equation

The conformable fractional wave equation can be expressed mathematically as follows:

$$\begin{cases} d_t^{2\beta} u(t,x) = \delta d_x^{2\beta} u(t,x) \\ u(0,x) = \phi_1(x) \\ d_t^{\beta} u(0,x) = \phi_2(x). \end{cases} \tag{3.4}$$

Here, δ represents the square of the wave propagation speed. The system can be represented by

$$d_t^{\beta} V = AV. \tag{3.5}$$

With $V = \begin{pmatrix} u \\ d_t^{\beta} u \end{pmatrix}$ and

$$A = \begin{pmatrix} 0 & I \\ \delta d_x^{2\beta} & 0 \end{pmatrix}.$$

Theorem 3.3 *The solution β-semigroup $\{e^{\frac{t^\beta}{\beta}A}\}_{t\geq 0}$ associated to A is chaotic on $\mathcal{F}_{\rho,\beta} \oplus \mathcal{F}_{\rho,\beta}$.*

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Proof: The proof follows the same steps as Theorem 3.2, but in this case, $R_\lambda = \frac{\lambda^2}{\alpha}$. For $\lambda = 0$, we have $a_0\rho_0 + a_1\rho_1 = 0$ for any choice of $a_0,\ a_1 \in \mathbb{R}$, implying

$$\rho_0 = \rho_1 = 0.$$

Dividing by λ results in $a_0\zeta_0 + a_1\zeta_1 = 0$ for all $a_0,\ a_1 \in \mathbb{R}$, leading to

$$\zeta_0 = \zeta_1 = 0.$$

Inductively, assuming $\rho_0 = \cdots = \rho_{2n?1} = \zeta_0 = \cdots = \zeta_{2n?1} = 0$, dividing $h(\lambda)$ by λ^{2n} gives $a_0\rho_{2n} + a_1\rho_{2n+1} = 0$ for every $a_0,\ a_1 \in \mathbb{R}$, resulting in

$$\rho_{2n} = \rho_{2n+1} = 0.$$

Similarly, we obtain

$$\zeta_{2n} = \zeta_{2n+1} = 0.$$

□

4. Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Author Contributions All authors contributed aqually to consctruct this work.

Data availability The data used to support the findings of this study are included in the references within the article.

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