Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.76015

A New Approach of Essential Pseudo Spectrum in Banach Space and Application to Transport Equation

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ABSTRACT: In the present paper we introduce and study the essential pseudo spectrum of bounded linear operator on Banach space. Beside that, we discuss some results of stability under Riesz operator and some properties of these pseudo spectrum. This paper also deals with the relationship between the essential pseudo spectrum and the pseudo Browder essential spectrum of bounded linear operator in Banach space. Finally, as an application, we apply these results to a transport equation.

Key Words: Pseudo spectrum, Pseudo essential spectrum, Pseudo essential Browder spectrum, Compact operator, Riesz operator and Transport operator.

Contents

1. Introduction

In order to deal with the issue of lack of information on the spectral analysis in several problems in science and engineering that involve non self adjoint operators, researchers have introduced the concept of pseudo spectrum. Indeed, the pseudo spectrum consists in determining and localizing the spectrum (eigenvalues) of an operator. Many works have been conducted employing the pseudo spectrum concept [3,4,5,7,8,12,17,18]. Especially in [17], L. N. Trefethen developed this concept for matrices and operators and used it to study interesting problems in mathematical physics. The concept of pseudo spectrum has been harnessed in several applications. As an example but not limited to, the pseudo spectrum (eigenvalues) is used to determine wether the flow over a wing is laminar or turbulent (aeronautics). In addition, eigenvalues can be used to determine the frequency response of an amplifier and the accuracy of national power system (in electrical engineering). Throughout this paper, X denotes a Banach space and we denote by $\mathcal{L}(X)$ the set of all bounded linear operators from X into X and we denote by $\mathcal{K}(X)$ the subspace of compact operators from X into X. For $A \in \mathcal{L}(X)$, let $N(A) \subset X$ and $R(A) \subset X$ represents respectively the null space and the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in X.

An operator $A \in \mathcal{L}(X)$ is a Fredholm operator if $R(A)$ is closed and both $\alpha(A)$, $\beta(A)$ are finite and we denote by $\Phi(X)$ the set of Fredholm operators. A complex number $\lambda \in \Phi_A(X)$ if $\lambda - A \in \Phi(X)$. For $A \in \Phi(X)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. It is clear that if $A \in \Phi(X)$ then $i(A) < \infty$. Let $R \in \mathcal{L}(X), R$ is said to be a Riesz operator if $\Phi_R(X) = \mathbb{C}\backslash\{0\}$. We denote by $\mathcal{R}(X)$ the set of Riesz operators and $\sigma_d(A)$ the discrete spectrum of A. Recall that for $A \in \mathcal{L}(X)$, the ascent, $\mathrm{a}(A)$, and the descent, $\mathrm{d}(A)$, are defined by

$$\mathrm{a}(A) = \inf\left\{n \geq 0 : \mathrm{N}\left(A^n\right) = \mathrm{N}\left(A^{n+1}\right)\right\}, \quad \mathrm{d}(A) = \inf\left\{n \geq 0 : \mathrm{R}\left(A^n\right) = \mathrm{R}\left(A^{n+1}\right)\right\}.$$

Submitted March 06, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* Primary: 47A53; 47A55 and Secondary: 47A13

If no such n exists, then $a(A)=\infty$ (resp. $d(A)=\infty$). An operator A is called Browder if satisfied $A\in\Phi(X), i(A)=0, \mathrm{a}(A)<\infty$ and $\mathrm{d}(A)<\infty$, and we denote by $\mathcal{B}(X)$ the set of Browder operator. For further information on the family of Fredholm operator, Fredholm perturbation, and Riesz operators we refer the reader to [10,11,15,16,9]. The Browder essential spectrum (resp. resolvent) of bounded linear operator A denoted by $\sigma_{\mathcal{B}}(A)=\sigma(A)\backslash\sigma_d(A)=\{\lambda\in\mathbb{C}, A-\lambda\notin\mathcal{B}(X)\}$ (resp. $\rho_B(A)=\mathbb{C}\backslash\sigma_{\mathcal{B}}(A)$).

Rakočević in [14] characterized the Browder essential spectrum for $A\in\mathcal{L}(X)$ by the following equality:

$$\sigma_{\mathcal{B}}(A)=\bigcap_{K\in\mathcal{K}(X), AK=KA}\sigma(A+K). \tag{1.1}$$

The definition of pseudo spectrum of $A\in\mathcal{L}(X)$ for every $\varepsilon>0$ is given by:

$$\sigma_{\varepsilon}(A):=\sigma(A)\cup\left\{\lambda\in\mathbb{C}:\left\|(\lambda-A)^{-1}\right\|>\frac{1}{\varepsilon}\right\}. \tag{1.2}$$

By convention, we write $\left\|(\lambda-A)^{-1}\right\|=\infty$ if $(\lambda-A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [5], Davies defined another equivalent of pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A\in\mathcal{L}(X)$, we have

$$\sigma_{\varepsilon}(A):=\bigcup_{\|D\|<\varepsilon}\sigma(A+D). \tag{1.3}$$

Inspired by the notion of pseudospectra F. Abdmouleh et al. defined in [2] the notion of pseudo Browder essential spectrum for bounded linear operators in the Banach space as follows:

$$\sigma_{\mathcal{B},\varepsilon}(A)=\sigma_{\mathcal{B}}(A)\cup\left\{\lambda\in\mathbb{C}:\|R_{\mathcal{B}}(\lambda,A)\|>\frac{1}{\varepsilon}\right\},$$

where $R_{\mathcal{B}}(\lambda,A)=\left((\lambda-A)|_{K_\lambda}\right)^{-1}(I-P_\lambda)+P_\lambda$, being P_λ the Riesz projection, K_λ a kernel of P_λ and R_λ a range of P_λ. By convention, we write $\|R_{\mathcal{B}}(\lambda,A)\|=\infty$ if $R_{\mathcal{B}}(\lambda,T)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma_B(A)$. The authors characterized the pseudo Browder essential spectrum for bounded linear operator in Banach space by

$$\sigma_{\mathcal{B},\varepsilon}(A)=\bigcup_{\|D\|<\varepsilon, AD=DA}\sigma_{\mathcal{B}}(A+D). \tag{1.4}$$

In this paper, and motivated by the notion of the the Browder essential spectrum, we introduce and study the notion of the essential pseudo spectrum of bounded linear operators A in the Banach space X which is denoted by $\sigma_{\varepsilon,e}(A)$, and given by

$$\sigma_{\varepsilon,e}(A)=\bigcap_{K\in\mathcal{K}(X), AK=KA}\sigma_{\varepsilon}(A+K).$$

The first goal of this paper is to prove some properties of the essential pseudo spectrum for bounded operator in Banach space. One of the main problems, consists in characterizing the relation between the essential pseudo spectrum $\sigma_{\varepsilon,e}(A)$ and the pseudo Browder essential spectrum $\sigma_{\mathcal{B},\varepsilon}(A)$ for bounded linear operator A in the Banach space X. The second aim of this work, is to investigate the stability of the essential pseudo spectrum under Riesz operator perturbations satisfying some additional extra conditions. Moreover we characterize the relation between the pseudo Browder essential spectrum of the sum of two bounded linear operator and the pseudo Browder spectrum of each of these operator. Finally, we will apply the results described above to investigate the essential pseudo spectrum of the following integro-differential operator:

$$A_0\psi(x,\xi)=-\xi\frac{\partial\psi}{\partial x}(x,\xi)-\sigma(\xi)\psi(x,\xi)+\int_{-1}^{1}\kappa\left(x,\xi,\xi'\right)\psi\left(x,\xi'\right)d\xi' \quad =T_0\psi+K\psi,$$

with vacuum boundary conditions, where $x \in [-a, a], a > 0$, and $\xi \in [-1, 1]$. These operators describe the transport of particles (neutrons, photons, molecules of gas, etc. in a plane parallel domain with a width of $2a$ mean free paths. The function $\psi(x, \xi)$ represents the number (or probability) density of gas particles having the position x and the direction cosine of propagation ξ.(The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x -direction). The functions $\sigma(.)$ and $\kappa(., ., .)$ are called, respectively, the collision frequency and the scattering kernel.
We organize our paper in the following way: Section 2 contains the definition and the properties of the essential pseudo spectrum. One of the central questions consists in characterizing the relation between the essential pseudo spectrum and the pseudo Browder essential spectrum. Section 3 aims at characterizing as well as establishing the stability of the essential pseudo spectrum under Riesz operators, in addition we characterized the pseudo Browder essential spectrum of the sum of two bounded linear operator. Finally, Section 4 is devoted to an application that consists in applying the results obtained in Sections 2 and 3 to investigate the essential pseudo spectrum of a one-dimensional transport operator.

2. On Pseudo Essential Spectrum

Definition 2.1 *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. We define the essential pseudo spectrum of the operator A denoted by $\sigma_{\varepsilon,e}(A)$ as follows,*

$$\sigma_{\varepsilon,e}(A) = \bigcap_{K \in \mathcal{K}(X), AK = KA} \sigma_\varepsilon(A + K).$$

Proposition 2.1 *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$, then the following hold:*

(i) $\sigma_{\varepsilon,e}(A) \subset \sigma_\varepsilon(A)$.

(ii) $\bigcap_{\varepsilon > 0} \sigma_{\varepsilon,e}(A) = \sigma_{\mathcal{B}}(A)$.

(iii) *If $\varepsilon_1 < \varepsilon_2$ then $\sigma_B(A) \subset \sigma_{\varepsilon_1,e}(A) \subset \sigma_{\varepsilon_2,e}(A)$.*

(iv) *$\sigma_{\varepsilon,e}(A + K) = \sigma_{\varepsilon,e}(A)$ for all $K \in K(X)$ and $AK = KA$.*

Proof:

(i) Let $\lambda \in \sigma_{\varepsilon,e}(A)$ then by Definition 2.1 $\lambda \in \sigma_\varepsilon(A + K)$ for all $K \in K(X)$ and $AK = KA$. By choosing $K = 0$, one obtains $\lambda \in \sigma_\varepsilon(A)$.

(ii) Indeed one has:

$$\bigcap_{\varepsilon > 0} \sigma_{\varepsilon,e}(A) = \bigcap_{K \in \mathcal{K}(X), AK = KA} \sigma(A + K) = \sigma_{\mathcal{B}}(A).$$

(iii) Let $\lambda \in \sigma_B(A)$, then by Eq. (1.1), we obtain $\lambda \in \sigma(A + K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$. Since $\sigma(A + K) \subset \sigma_\varepsilon(A + K)$, then $\lambda \in \sigma_\varepsilon(A + K)$ for all $K \in \mathcal{K}(X)$ and $KA = AK$. Hence we get

$$\lambda \in \bigcap_{K \in \mathcal{K}(X), AK = KA} \sigma_\varepsilon(A + K) = \sigma_{\varepsilon,e}(A).$$

If $\varepsilon_1 < \varepsilon_2$ then $\sigma_{\varepsilon_1}(A + K) \subset \sigma_{\varepsilon_2}(A + K)$ for all $K \in \mathcal{K}(X)$ and $KA = AK$. Therefore,

$$\bigcap_{K \in \mathcal{K}(X), AK = KA} \sigma_{\varepsilon_1}(A + K) \subset \bigcap_{K \in \mathcal{K}(X), AK = KA} \sigma_{\varepsilon_2}(A + K).$$

(iv)

$$\sigma_{\varepsilon,e}(A + K) = \bigcap_{K' \in \mathcal{K}(X), (A+K)K' = K'(A+K)} \sigma_\varepsilon(A + K + K').$$

Let us choose $K_1 = K + K'$, we have $K_1 \in \mathcal{K}(X)$ and $AK_1 = A(K + K') = AK + AK' = KA + K'A = (K + K')A = K_1 A$. We infer that

$$\sigma_{\varepsilon,e}(A + K) = \bigcap_{K_1 \in \mathcal{K}(X), AK_1 = K_1 A} \sigma_\varepsilon(A + K_1) = \sigma_{\varepsilon,e}(A).$$

□

In the following theorem we present the relation between the essential pseudo spectrum and the pseudo essential Browder spectra for bounded linear operator A in the Banach space X.

Theorem 2.1 *Let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then*

(1) $\sigma_{\varepsilon,e}(A) \subset \sigma_{\mathcal{B},\varepsilon}(A)$.

(2) *For all $K \in \mathcal{K}(X)$, if $\|K\| < \varepsilon$, then $\sigma_{\mathcal{B},\varepsilon}(A) \subset \sigma_{\varepsilon,e}(A)$.*

Proof:

1. Let $\lambda \in \sigma_{\varepsilon,e}(A)$, then by Definition 2.1 we obtain

$$\lambda \in \sigma_{\varepsilon}(A+K) \quad \forall K \in \mathcal{K}(X), \; AK = KA.$$

By Eq. (1.2), we have $\lambda \in \sigma(A+K) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \left\|(A+K-\lambda I)^{-1}\right\| > \frac{1}{\varepsilon}\right\}$.
So there are two possible cases:

- Case 1: If $\lambda \in \sigma(A+K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$. Then

$$\lambda \in \bigcap_{K \in \mathcal{K}(X), AK=KA} \sigma(A+K) = \sigma_{\mathcal{B}}(A) \subset \sigma_{\mathcal{B},\varepsilon}(A).$$

- Case 2: If $\lambda \in \sigma_{\varepsilon,e}(A)$ and $\lambda \notin \sigma(A+K)$. Then we have

$$\|(A+K-\lambda I)^{-1}\| > \frac{1}{\varepsilon}, \quad \forall K \in \mathcal{K}(X), AK = KA. \tag{2.1}$$

We infer there exist a nonzero vector $f \in X$ such that $\|(A+K-\lambda I)\| \leq \varepsilon\|f\|$. Let $\psi \in X^*$ such that $\|\psi\| = 1$ and $\psi(f) = 1$. Let us define the rank one operator $D : X \longrightarrow X$ by $Dg = -\psi(g)\,(A+K-\lambda)\,f$. We see immediately that $\|D\| < \varepsilon$ and $AD = DA$. Furthermore $(A+D+K-\lambda)f = 0$, then $(A+D+K-\lambda)$ is not invertible. Therefore

$$\lambda \in \bigcup_{\|D\|<\varepsilon, AD=DA} \sigma_{\mathcal{B}}(A+D) = \sigma_{\mathcal{B},\varepsilon}(A).$$

2. Let $\lambda \in \sigma_{\mathcal{B},\varepsilon}(A)$, then by Eq. (1.4) there are two cases:

- Case 1: Let $\lambda \in \sigma_{\mathcal{B},\varepsilon}(A)$ and $\lambda \in \sigma_{\mathcal{B}}(A)$ we have

$$\lambda \in \sigma_{\mathcal{B}}(A) = \bigcap_{K \in \mathcal{K}(X), AK=KA} \sigma(A+K) \subset \bigcap_{K \in \mathcal{K}(X), AK=KA} \sigma_{\varepsilon}(A+K).$$

In this case we obtain

$$\sigma_{\mathcal{B},\varepsilon}(A) \subset \sigma_{\varepsilon,e}(A).$$

- Case 2: Let $\lambda \in \sigma_{\mathcal{B},\varepsilon}(A)$ and $\lambda \notin \sigma_{\mathcal{B}}(A)$ then we have $\lambda \in \left\{\lambda \in \mathbb{C}, \|R_{\mathcal{B}}(A,\lambda)\| > \frac{1}{\varepsilon}\right\}$. Suppose that $\lambda \notin \left\{\lambda \in \mathbb{C}, \|(A+K-\lambda I)^{-1}\| > \frac{1}{\varepsilon}\right\}$, then we obtain $\|(A+K-\lambda I)^{-1}\| \leq \frac{1}{\varepsilon}$. The operator $A-\lambda$ can read as follows:

$$A-\lambda = (A+K-\lambda)\left(I-(A+K-\lambda)^{-1}K\right).$$

The fact that, $\left\|(A+K-\lambda)^{-1}K\right\| \leq \|(A+K-\lambda)^{-1}\|\|K\| < 1$ implies that $I-(A+K-\lambda)^{-1}K$ is an invertible operator and

$$(A-\lambda)^{-1} = \left(I-(A+K-\lambda)^{-1}K\right)^{-1}(A+K-\lambda)^{-1}.$$

However, $KA = AK$, then

$$(I - (A+K-\lambda)^{-1} K)^{-1} = \sum_{n=0}^{+\infty} \left((-1)(A+K-\lambda)^{-1}\right)^n K^n.$$

Then $\left\|\left(I + (A+K-\lambda)^{-1}K\right)^{-1}\right\| < \frac{\varepsilon}{\varepsilon+\|K\|}$. Consequently

$$\|(A-\lambda)^{-1}\| \leq \frac{\varepsilon\|(A+K-\lambda)^{-1}\|}{\varepsilon+\|K\|} \leq \frac{1}{\varepsilon}.$$

Finally, we obtain that

$$\lambda \notin \left\{\lambda \in \mathbb{C} \text{ such that } \|R_{\mathcal{B}}(A,\lambda)\| > \frac{1}{\varepsilon}\right\}. \tag{2.2}$$

Since $\lambda \notin \sigma_{\mathcal{B}}(A)$ then by Eq. (2.2) we deduce that $\lambda \notin \sigma_{\mathcal{B},\varepsilon}(A)$.

□

3. Stability of the Pseudo Essential Spectrum

In the following theorem we examine the stability of the essential pseudo spectrum under Riesz operator perturbations.

Theorem 3.1 *Let $\varepsilon > 0, A \in \mathcal{L}(X)$ and $R \in \mathcal{R}(X)$ such that $R(A+K) = (A+K)R$ for all $K \in \mathcal{K}(X)$. If $\|R\| < \varepsilon$ then there exist $\varepsilon_0, \varepsilon_1$ such that $0 < \varepsilon_0 < \varepsilon < \varepsilon_1$ satisfying*

$$\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon_1,e}(A+R).$$

Proof: Let $\lambda \in \sigma_{\varepsilon,e}(A) = \bigcap_{K\in\mathcal{K}(X),AK=KA} \sigma_\varepsilon(A+K)$. If $\lambda \in \sigma(A+K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$, then

$$\lambda \in \bigcap_{K\in\mathcal{K}(X),AK=KA} \sigma(A+K) = \sigma_B(A).$$

First we prove that there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and $\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A)$.
For that let $\lambda \notin \left\{\lambda \in \mathbb{C}, \text{ such that } \|(A+K-\lambda I)^{-1}\| > \frac{1}{\varepsilon} \ \ \forall K \in \mathcal{K}(X)\right\}$, so $\|(A+K-\lambda I)^{-1}\| \leq \frac{1}{\varepsilon}$.
By writing $A+K+\lambda$ in the form

$$A+K+R-\lambda = (A+K-\lambda)\left(I + (A+K-\lambda)^{-1}R\right),$$

and due to the fact that $\|(A+K-\lambda I)^{-1}R\| \leq \|(A+K-\lambda I)^{-1}\|\|R\| < 1$, one gets that $\left(I+(A+K-\lambda)^{-1}R\right)$ is an invertible operator and its inverse is written as follows:

$$(A+K+R-\lambda)^{-1} = \left(I + (A+K-\lambda)^{-1}R\right)^{-1}(A+K-\lambda)^{-1}.$$

Using the fact that $R(A+K) = (A+K)R$, we have

$$\left(I + (A+K-\lambda)^{-1}R\right)^{-1} = \sum_{n=0}^{+\infty} \left((-1)(A+K-\lambda)^{-1}\right)^n.$$

Which implies that

$$(A+K+R-\lambda)^{-1} = \sum_{n=0}^{+\infty} \left((-1)(A+K-\lambda)^{-1}\right)^n R^n.$$

Since

$$\left\|\left(I + (A+K-\lambda)^{-1}R\right)^{-1}\right\| \leq \frac{\varepsilon}{\varepsilon - \|R\|},$$

this shows that $\left\|(A+K+R-\lambda)^{-1}\right\| \leq \frac{\|(A+K-\lambda)^{-1}\|\varepsilon}{\varepsilon-\|R\|}$. Consequently, we infer that

$$\left\|(A+K+R-\lambda)^{-1}\right\| \leq \frac{1}{\varepsilon-\|R\|}.$$

By taking $\varepsilon_0 = \varepsilon - \|R\|$ then $0 < \varepsilon_0 < \varepsilon$ and $\lambda \notin \sigma_{\varepsilon_0,e}(A+R)$, we conclude that, there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and

$$\sigma_{\varepsilon_0,e}(A+R) \subseteq \sigma_{\varepsilon,e}(A).$$

Hence we prove that there exists ε_1 such that $0 < \varepsilon < \varepsilon_1$ and $\sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon_1,e}(A+R)$. Let $\varepsilon_1 = \varepsilon + \|R\|$, then for $\lambda \notin \left\{\lambda \in \mathbb{C}, \left\|(A+K+R)^{-1}\right\| > \frac{1}{\varepsilon_1}\right\}$ implies $\left\|(A+K+R)^{-1}\right\| \leq \varepsilon_1^{-1}$. By writing $A+K-\lambda$ in the following form

$$A+K-\lambda = (A+K+R-\lambda)\left(I-(A+K+R-\lambda)^{-1}R\right), \tag{3.1}$$

and by the fact that, $\left\|(A+K+R-\lambda)^{-1}R\right\| < 1$ one gets that $I-(A+K+R-\lambda)^{-1}R$ is an invertible operator and using Eq.(4.1) we obtain,

$$(A+K-\lambda)^{-1} = \left(I-(A+K+R-\lambda)^{-1}R\right)^{-1}(A+K+R-\lambda)^{-1}.$$

However, $(R+K)A = A(R+K)$, then $\left\|\left(I+(A+K+R-\lambda)^{-1}R\right)^{-1}\right\| < \frac{\varepsilon_1}{\varepsilon_1-\|R\|}$. Consequently

$$\left\|(A+K-\lambda)^{-1}\right\| \leq \frac{1}{\varepsilon}.$$

Then one can conclude that, there exists ε_1 such that $0 < \varepsilon < \varepsilon_1$ satisfying

$$\sigma_{\varepsilon,e}(A+R) \subseteq \sigma_{\varepsilon_1,e}(A).$$

□

Inspired by the paper of F. Abdmouleh et al [1], the ensuing theorem devoted to the study of the pseudo Browder essential spectrum of the sum of two bounded linear operators by exhibiting its relation with the pseudo Browder spectrum of each of these operator.

Theorem 3.2 *Let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $\varepsilon > 0$. If for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$ we have $AD = DA$ and $A(B+D) \in \mathcal{R}(X)$, Then*

$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\} = \left[\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)\right]\backslash\{0\}.$$

Proof: For $\lambda \in \mathbb{C}$, we can write

$$(\lambda - A)(\lambda - B - D) = A(B+D) + \lambda(\lambda - A - B - D), \tag{3.2}$$

and

$$(\lambda - B - D)(\lambda - A) = (B+D)A + \lambda(\lambda - A - B - D). \tag{3.3}$$

Suppose that $\lambda \neq 0$ such that $\lambda \notin \sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)$. Then by Eq. (1.4),we obtain $(\lambda - A) \in \mathcal{B}(X)$ and $(\lambda - B - D) \in \mathcal{B}(X)$, for all $\|D\| < \varepsilon$ and $BD = DB$. As a consequence of [9, Theorem 7.9.2, page 276], we get

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}(X).$$

Using Eq. (4.2) we obtain $A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}(X)$. Since $A(B+D) \in \mathcal{R}(X)$ and $(B+D)A(\lambda - A - B - D) = (\lambda - A - B - D)(B+D)A$, then, by Equation (4.2) and Rakočević [14] we obtain $\lambda - A - B - D \in \mathcal{B}(X)$ $\forall D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$ and $(A+B)D = D(A+B)$. Applying Eq. (1.4), we get $\lambda \notin \sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\} \subset \left[\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)\right]\backslash\{0\}.$$

To prove the inverse, let $\lambda \notin \sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\}$, then from Eq. (1.4), we get

$$(\lambda - A - B - D) \in \mathcal{B}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (A+B)D = D(A+B).$$

Since $A(B+D) \in \mathcal{R}(X)$ and $(B+D)A \in \mathcal{R}(X)$, it follows from Rakočević [14] and Eqs. (4.2), (4.3) that $A(B+D)+\lambda(\lambda - A - B - D) \in \mathcal{B}(X)$ *and* $(B+D)A+\lambda(\lambda - A - B - D) \in \mathcal{B}(X)$. We apply Equations (4.2) and (4.3) we infer that

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \mathcal{B}(X).$$

Due to Equation (4.2) and [9, Theorem 7.9.2, page 276], we obtain

$$\lambda - A \in \mathcal{B}(X) \text{ and } (\lambda - B - D) \in \mathcal{B}(X)\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } BD = DB.$$

Then we have $\lambda \notin [\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)]\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\} = [\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)]\backslash\{0\}.$$

□

Corollary 3.1 *Under the assumptions of Theorem 2.1 and Theorem 4.2 we have,*

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} = [\sigma_{\mathcal{B}}(A) \cup \sigma_{e,\varepsilon}(B)]\backslash\{0\}.$$

4. Stability of the Pseudo Essential Spectrum

In the following theorem we examine the stability of the essential pseudo spectrum under Riesz operator perturbations.

Theorem 4.1 *Let $\varepsilon > 0, A \in \mathcal{L}(X)$ and $R \in \mathcal{R}(X)$ such that $R(A+K) = (A+K)R$ for all $K \in \mathcal{K}(X)$. If $\|R\| < \varepsilon$ then there exist $\varepsilon_0, \varepsilon_1$ such that $0 < \varepsilon_0 < \varepsilon < \varepsilon_1$ satisfying*

$$\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A) \subset \sigma_{\varepsilon_1,e}(A+R).$$

Proof: Let $\lambda \in \sigma_{\varepsilon,e}(A) = \bigcap_{K\in\mathcal{K}(X), AK=KA} \sigma_\varepsilon(A+K)$. If $\lambda \in \sigma(A+K)$ for all $K \in \mathcal{K}(X)$ and $AK = KA$, then

$$\lambda \in \bigcap_{K\in\mathcal{K}(X), AK=KA} \sigma(A+K) = \sigma_B(A).$$

First we prove that there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and $\sigma_{\varepsilon_0,e}(A+R) \subset \sigma_{\varepsilon,e}(A)$.
For that let $\lambda \notin \left\{\lambda \in \mathbb{C}, \text{ such that } \|(A+K-\lambda I)^{-1}\| > \frac{1}{\varepsilon}\ \ \forall K \in \mathcal{K}(X)\right\}$, so $\|(A+K-\lambda I)^{-1}\| \leq \frac{1}{\varepsilon}$.
By writing $A+K+\lambda$ in the form

$$A+K+R-\lambda = (A+K-\lambda)\left(I + (A+K-\lambda)^{-1}R\right),$$

and due to the fact that $\|(A+K-\lambda I)^{-1}R\| \leq \|(A+K-\lambda I)^{-1}\|\|R\| < 1$, one gets that $\left(I + (A+K-\lambda)^{-1}R\right)$ is an invertible operator and its inverse is written as follows:

$$(A+K+R-\lambda)^{-1} = \left(I + (A+K-\lambda)^{-1}R\right)^{-1}(A+K-\lambda)^{-1}.$$

Using the fact that $R(A+K) = (A+K)R$, we have

$$\left(I + (A+K-\lambda)^{-1}R\right)^{-1} = \sum_{n=0}^{+\infty}\left((-1)(A+K-\lambda)^{-1}\right)^n.$$

Which implies that

$$(A+K+R-\lambda)^{-1} = \sum_{n=0}^{+\infty}\left((-1)(A+K-\lambda)^{-1}\right)^n R^n.$$

Since
$$\left\|\left(I+(A+K-\lambda)^{-1}R\right)^{-1}\right\|\leq\frac{\varepsilon}{\varepsilon-\|R\|},$$
this shows that $\left\|(A+K+R-\lambda)^{-1}\right\|\leq\frac{\left\|(A+K-\lambda)^{-1}\right\|\varepsilon}{\varepsilon-\|R\|}$. Consequently, we infer that
$$\left\|(A+K+R-\lambda)^{-1}\right\|\leq\frac{1}{\varepsilon-\|R\|}.$$
By taking $\varepsilon_0=\varepsilon-\|R\|$ then $0<\varepsilon_0<\varepsilon$ and $\lambda\notin\sigma_{\varepsilon_0,e}(A+R)$, we conclude that, there exists ε_0 such that $0<\varepsilon_0<\varepsilon$ and
$$\sigma_{\varepsilon_0,e}(A+R)\subseteq\sigma_{\varepsilon,e}(A).$$
Hence we prove that there exists ε_1 such that $0<\varepsilon<\varepsilon_1$ and $\sigma_{\varepsilon,e}(A)\subset\sigma_{\varepsilon_1,e}(A+R)$. Let $\varepsilon_1=\varepsilon+\|R\|$, then for $\lambda\notin\left\{\lambda\in\mathbb{C},\left\|(A+K+R)^{-1}\right\|>\frac{1}{\varepsilon_1}\right\}$ implies $\left\|(A+K+R)^{-1}\right\|\leq\varepsilon_1^{-1}$. By writing $A+K-\lambda$ in the following form
$$A+K-\lambda=(A+K+R-\lambda)\left(I-(A+K+R-\lambda)^{-1}R\right), \tag{4.1}$$
and by the fact that, $\left\|(A+K+R-\lambda)^{-1}R\right\|<1$ one gets that $I-(A+K+R-\lambda)^{-1}R$ is an invertible operator and using Eq.(4.1) we obtain,
$$(A+K-\lambda)^{-1}=\left(I-(A+K+R-\lambda)^{-1}R\right)^{-1}(A+K+R-\lambda)^{-1}.$$
However, $(R+K)A=A(R+K)$, then $\left\|\left(I+(A+K+R-\lambda)^{-1}R\right)^{-1}\right\|<\frac{\varepsilon_1}{\varepsilon_1-\|R\|}$. Consequently
$$\left\|(A+K-\lambda)^{-1}\right\|\leq\frac{1}{\varepsilon}.$$
Then one can conclude that, there exists ε_1 such that $0<\varepsilon<\varepsilon_1$ satisfying
$$\sigma_{\varepsilon,e}(A+R)\subseteq\sigma_{\varepsilon_1,e}(A).$$
□

The ensuing theorem devoted to the study of the pseudo Browder essential spectrum of the sum of two bounded linear operators by exhibiting its relation with the pseudo Browder spectrum of each of these operator.

Theorem 4.2 *Let $A,B\in\mathcal{L}(X)$ such that $AB=BA$ and $\varepsilon>0$. If for all $D\in\mathcal{L}(X)$ with $\|D\|<\varepsilon$ we have $AD=DA$ and $A(B+D)\in\mathcal{R}(X)$, Then*
$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\}=\left[\sigma_{\mathcal{B}}(A)\cup\sigma_{\mathcal{B},\varepsilon}(B)\right]\backslash\{0\}.$$

Proof: For $\lambda\in\mathbb{C}$, we can write
$$(\lambda-A)(\lambda-B-D)=A(B+D)+\lambda(\lambda-A-B-D), \tag{4.2}$$
and
$$(\lambda-B-D)(\lambda-A)=(B+D)A+\lambda(\lambda-A-B-D). \tag{4.3}$$
Suppose that $\lambda\neq0$ such that $\lambda\notin\sigma_{\mathcal{B}}(A)\cup\sigma_{\mathcal{B},\varepsilon}(B)$. Then by Eq. (1.4),we obtain $(\lambda-A)\in\mathcal{B}(X)$ and $(\lambda-B-D)\in\mathcal{B}(X)$, for all $\|D\|<\varepsilon$ and $BD=DB$. As a consequence of [9, Theorem 7.9.2, page 276], we get
$$(\lambda-A)(\lambda-B-D)\in\mathcal{B}(X).$$
Using Eq. (4.2) we obtain $A(B+D)+\lambda(\lambda-A-B-D)\in\mathcal{B}(X)$. Since $A(B+D)\in\mathcal{R}(X)$ and $(B+D)A(\lambda-A-B-D)=(\lambda-A-B-D)(B+D)A$, then, by Equation (4.2) and Rakočević [14]

we obtain $\lambda - A - B - D \in \mathcal{B}(X)$ $\forall D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$ and $(A+B)D = D(A+B)$. Applying Eq. (1.4), we get $\lambda \notin \sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\} \subset [\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)]\backslash\{0\}.$$

To prove the inverse, let $\lambda \notin \sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\}$, then from Eq. (1.4), we get

$$(\lambda - A - B - D) \in \mathcal{B}(X) \ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } (A+B)D = D(A+B).$$

Since $A(B+D) \in \mathcal{R}(X)$ and $(B+D)A \in \mathcal{R}(X)$, it follows from Rakočević [14] and Eqs. (4.2), (4.3) that $A(B+D) + \lambda(\lambda - A - B - D) \in \mathcal{B}(X)$ *and* $(B+D)A + \lambda(\lambda - A - B - D) \in \mathcal{B}(X)$. We apply Equations (4.2) and (4.3) we infer that

$$(\lambda - A)(\lambda - B - D) \in \mathcal{B}(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \mathcal{B}(X).$$

Due to Equation (4.2) and [9, Theorem 7.9.2, page 276], we obtain that

$$\lambda - A \in \mathcal{B}(X) \text{ and } (\lambda - B - D) \in \mathcal{B}(X) \ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon \text{ and } BD = DB.$$

Then we have $\lambda \notin [\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)]\backslash\{0\}$. Therefore

$$\sigma_{\mathcal{B},\varepsilon}(A+B)\backslash\{0\} = [\sigma_{\mathcal{B}}(A) \cup \sigma_{\mathcal{B},\varepsilon}(B)]\backslash\{0\}.$$

□

Corollary 4.1 *Under the assumptions of Theorem 2.1 and Theorem 4.2 we have,*

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} = [\sigma_{\mathcal{B}}(A) \cup \sigma_{e,\varepsilon}(B)]\backslash\{0\}.$$

5. Application to a Transport Operator

In this section we show the applicability of our sections' 2 and 3 results in order to study the essential the essential pseudo spectrum of the transport operator equation (1.1). To that end, we give some basic notations and definitions that are crucial to our work in this section. Let

$$X_p = L_p([-a,a] \times [-1,1], \mathrm{d}x\mathrm{d}\xi), \quad a > 0 \text{ and } p \in [1,\infty).$$

We also consider the boundary spaces:

$$\begin{aligned} X_p^o &:= L_p(\{-a\} \times [-1,0], |\xi|\mathrm{d}\xi) \times L_p(\{a\} \times [0,1], |\xi|\mathrm{d}\xi) \\ &:= X_{1,p}^o \times X_{2,p}^o \end{aligned}$$

and

$$\begin{aligned} X_p^i &:= L_p(\{-a\} \times [0,1], |\xi|\mathrm{d}\xi) \times L_p(\{a\} \times [-1,0], |\xi|\mathrm{d}\xi) \\ &:= X_{1,p}^i \times X_{2,p}^i, \end{aligned}$$

respectively equipped with the norms

$$\begin{aligned} \|\psi^o\|_{X_p^o} &= \left(\|\psi_1^o\|_{X_{1,p}^o}^p + \|\psi_2^o\|_{X_{2,p}^o}^p \right)^{\frac{1}{p}} \\ &= \left[\int_{-1}^0 |\psi(-a,\xi)|^p |\xi| \mathrm{d}\xi + \int_0^1 |\psi(a,\xi)|^p |\xi| \mathrm{d}\xi \right]^{\frac{1}{p}}, \end{aligned}$$

and

$$\begin{aligned} \|\psi^i\|_{X_p^i} &= \left(\|\psi_1^i\|_{X_{i,p}}^p + \|\psi_2^i\|_{X_{i,p}}^p \right)^{\frac{1}{p}} \\ &= \left[\int_0^1 |\psi(-a,\xi)|^p |\xi| \mathrm{d}\xi + \int_{-1}^0 |\psi(a,\xi)|^p |\xi| \mathrm{d}\xi \right]^{\frac{1}{p}}. \end{aligned}$$

Let $\mathcal{W}_p$ the space defined by:

$$\mathcal{W}_p=\left\{\psi\in X_p:\xi\frac{\partial\psi}{\partial x}\in X_p\right\}.$$

It is well-known that any function ψ in $\mathcal{W}_p$ possesses traces on the spatial boundary $\{-a\}\times(-1,0)$ and $\{a\}\times(0,1)$ which respectively belong to the space X_p^0 and X_p^i (see [6]). They are denoted, respectively, by ψ^0 and ψ^i.
Let H be the boundary operator

$$\begin{cases} H:X_p^o\longrightarrow X_p^i\\ H\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}H_{11}&H_{12}\\H_{12}&H_{22}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix},\end{cases}$$

where for all $H_{11}\in\mathcal{L}\left(X_{1,p}^o,X_{1,p}^i\right),H_{12}\in\mathcal{L}\left(X_{2,p}^o,X_{1,p}^i\right),H_{21}\in\mathcal{L}\left(X_{1,p}^o,X_{2,p}^i\right)$ and $H_{22}\in\mathcal{L}\left(X_{2,p}^o,X_{2,p}^i\right)$.
Now, we define the streaming operator T_H by

$$\begin{cases} T_H:\mathcal{D}(T_H)\subseteq X_p\longrightarrow X_p\\ \qquad\psi\longmapsto T_H\psi(x,\xi)=-\xi\frac{\partial\psi}{\partial x}(x,\xi)-\sigma(\xi)\psi(x,\xi),\\ \mathcal{D}(T_H)=\{\psi\in\mathcal{W}_p:\psi^o=H\psi^i\},\end{cases}$$

where $\sigma(.)\in L^\infty(-1,1),\psi^o=(\psi_1^o,\psi_2^o)^\perp$ and $\psi^i=\left(\psi_1^i,\psi_2^i\right)^\perp$ with $\psi_1^o,\psi_2^o,\psi_1^i$ and ψ_2^i given by

$$\begin{aligned} &\psi_1^i:\xi\in(0,1)\longmapsto\psi(-a,\xi),\\ &\psi_2^i:\xi\in(-1,0)\longmapsto\psi(a,\xi),\\ &\psi_1^0:\xi\in(-1,0)\longmapsto\psi(-a,\xi),\\ &\psi_2^0:\xi\in(0,1)\longmapsto\psi(a,\xi).\end{aligned}$$

Finally, K will denotes the following partially integral operator on X_p

$$\begin{cases} K:X_p\longrightarrow X_p\\ \qquad\psi\longmapsto\int_{-1}^1\kappa(x,\xi,\xi')\,\psi(x,\xi')\,\mathrm{d}\xi',\end{cases}$$

where the scattering kernel $\kappa(.,.,.)$ is a measurable function from $[-a,a]\times[-1,1]\times[-1,1]$ to $\mathbb{R}$. We can see that the operator K acts only on the variable ξ so x can be a parameter in $[-a,a]$. Hence we may consider K is a function $K:x\in[-a,a]\longmapsto K(x)\in\mathcal{Z}$ where $\mathcal{Z}:=\mathcal{L}\left(L_p([-1,1],d\xi)\right)$.
Let us now consider the resolvent equation for T_H is given by:

$$(\lambda-T_H)\,\psi=\varphi,$$

being φ an element of X_p and the unknown ψ must be sought in $\mathcal{D}(T_H)$. Let λ^* be the real defined by

$$\lambda^*:=\lim_{|\xi|\longrightarrow0}\inf\sigma(\xi).$$

For $\operatorname{Re}\lambda+\lambda^*>0$, the solution formally given by:

$$\begin{cases} \psi(x,\xi)=\psi(-a,\xi)e^{-\frac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}}\\ \qquad+\frac{1}{|\xi|}\int_{-a}^x e^{\frac{-(\lambda+\sigma(\xi))|x-\bar{x}|}{|\xi|}}\psi(x,\xi')\,\mathrm{d}x',\qquad 0<\xi<1,\\ \psi(x,\xi)=\psi(a,\xi)e^{-\frac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}}\\ \qquad+\frac{1}{|\xi|}\int_x^a e^{\frac{-(\lambda+\sigma(\xi))|x-\bar{x}|}{|\xi|}}\psi(x',\xi)\,\mathrm{d}x',\quad -1<\xi<0.\end{cases}\tag{5.1}$$

Similarly, $\psi(a,\xi)$ and $\psi(-a,\xi)$ are given by

$$\begin{cases} \psi(a,\xi) = \psi(-a,\xi)e^{\frac{-2a(\lambda+\sigma(\xi))}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{2a\frac{-(\lambda+\sigma(\xi))|a+x|}{|\xi|}}\psi(x,\xi)\mathrm{d}x', \quad 0<\xi<1, \\ \psi(-a,\xi) = \psi(a,\xi)e^{-\frac{2a(\lambda+\sigma(\xi))}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))|a-x|}{|\xi|}}\psi(x,\xi)\mathrm{d}x, \quad -1<\xi<0. \end{cases} \tag{5.2}$$

The following operators will be needed in the sequel

$$\begin{cases} M_\lambda : X_p^i \longrightarrow X_p^o, \ \ M_\lambda u := \left(M_\lambda^+ u, M_\lambda^- u\right) \quad \text{with} \\ \left(M_\lambda^+ u\right)(-a,\xi) := u(-a,\xi)e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))}, \qquad -1<\xi<0, \\ \left(M_\lambda^- u\right)(a,\xi) := u(a,\xi)e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))}, \qquad 0<\xi<1. \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \longrightarrow X_p, \quad B_\lambda = \chi_{(-1,0)}(\xi)B_\lambda^+ u + \chi_{(0,1)}(\xi)B_\lambda^- u \quad \text{with} \\ \qquad \left(B_\lambda^- u\right)(x,\xi) := u(-a,\xi)e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|}, \ 0<\xi<1, \\ \qquad \left(B_\lambda^+ u\right)(x,\xi) := u(-a,\xi)e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|}, \ 1<\xi<0. \end{cases}$$

$$\begin{cases} G_\lambda : X_p \longrightarrow X_p^0, G_\lambda u := \left(G_\lambda^+\varphi, G_\lambda^-\varphi\right) \ \text{with} \\ G_\lambda^+\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a-x|}\varphi(x,\xi)dx; \quad 0<\xi<1, \\ G_\lambda^-\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a+x|}\varphi(x,\xi)dx; \quad -1<\xi<0, \end{cases}$$

and

$$\begin{cases} C_\lambda : X_p \longrightarrow X_p, \quad C_\lambda\varphi = \chi_{(-1,0)}(\xi)C_\lambda^+\varphi + \chi_{(0,1)}(\xi)C_\lambda^-\varphi \quad \text{with} \\ C_\lambda^-\varphi := \frac{1}{|\xi|}\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x',\xi)\,\mathrm{d}x', \quad 0<\xi<1, \\ C_\lambda^+\varphi := \frac{1}{|\xi|}\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x',\xi)\,\mathrm{d}x', \quad -1<\xi<0, \end{cases}$$

where $\chi_{(-1,0)}(.)$ and $\chi_{(0,1)}(.)$ denote, respectively the characteristic functions of the intervals (-1,0) and (0,1) . The operators $M_\lambda, B_\lambda, G_\lambda$ and C_λ are bounded on their respective spaces. In fact, their norms are bounded above, respectively, by $e^{-2a(\operatorname{Re}\lambda+\lambda^*)}$, $[p\,(\operatorname{Re}\lambda+\lambda^*)]^{\frac{-1}{p}}$, $[p\,(\operatorname{Re}\lambda+\lambda^*)]^{\frac{-1}{q}}$ and $[p(\operatorname{Re}\lambda+\ \lambda^*)]^{-1}$ where q denotes the conjugate of p.
Now, Eq.(5.2) can read as follows in the space X_p^0 :

$$\psi^o = M_\lambda H\psi^o + G_\lambda\varphi.$$

Let λ_0 be the real defined by

$$\begin{cases} -\lambda^*, & \text{if } \|H\|\leq 1, \\ -\lambda^* + \frac{1}{2a}\log(\|H\|), & \text{if } \|H\| > 1. \end{cases}$$

It follows from the estimate of M_λ that, for $\operatorname{Re}\lambda > \lambda_0$, $\|M_\lambda H\| < 1$ and consequently

$$\psi^o = \sum_{n>0}\left(M_\lambda H\right)^n G_\lambda\varphi. \tag{5.3}$$

On the other hand, Eq.(5.1) can be rewritten in the form

$$\psi = B_\lambda H\psi^o + C_\lambda\varphi.$$

Replacing Eq. (5.3) into the last equation we get

$$\psi = \sum_{n\geq 0} B_\lambda H\left(M_\lambda H\right)^n G_\lambda \varphi + C_\lambda \varphi.$$

Finally, the resolvent set of the operator T_H contains $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \lambda_0\}$ and for $\operatorname{Re}\lambda > \lambda_0$ we have

$$(\lambda - T_H)^{-1} = \sum_{n\geq 0} B_\lambda H\left(M_\lambda H\right)^n G_\lambda + C_\lambda.$$

Theorem 5.1 *Let $\varepsilon > 0$ and H be a bounded boundary operator. Then there exists $C > 0$ such that*

$$\sigma_{\varepsilon,e}(T_H) \subset \left\{\lambda \in \mathbb{C} : Re\lambda < -\lambda^* + \varepsilon(1 + \varepsilon(1 + C\|H\|(p)^{\frac{-1}{p}})\right\}.$$

Proof: For $Re\lambda > \lambda_0$, the operator $(\lambda - T_H)^{-1}$ exists and is given by:

$$(\lambda - T_H)^{-1} = \sum_{n\leq 0} B_\lambda H(M_\lambda H)^n G_\lambda + C_\lambda,$$

so

$$(\lambda - T_H)^{-1} = B_\lambda H(I - M_\lambda H)^{-1} G_\lambda + C_\lambda.$$

Moreover, we have:

$$\|(\lambda - T_H)^{-1}\| \leq \|B_\lambda\|\|H\|\|(I - M_\lambda H)^{-1}\|\|G_\lambda\| + \|C_\lambda\|,$$

and since the operator $(I - M_\lambda H)^{-1}$ is uniformly bounded on the half plane $\{\lambda \in \mathbb{C}, Re\lambda > \lambda_0\}$. then for $Re\lambda > \lambda_0$ there exists $C > 0$ such that $\|(I - M_\lambda H)^{-1}\| < C$.
Furthermore we have:

$$\|(\lambda - T_H)^{-1}\| \leq \frac{\left(1 + C\|H\|(p)^{\frac{-1}{p}}\right)}{Re\lambda + \lambda^*}. \tag{5.4}$$

Let $\lambda \in \sigma_{\varepsilon,B}(T_H)$, it follows from Theorem 1.1 (i)

$$\|(\lambda - T_H)^{-1}\| > \frac{1}{\varepsilon}.$$

Using Eq.(5.4) we obtain $Re\lambda \leq -\lambda^* + \varepsilon\left(1 + C\|H\|(p)^{-\frac{1}{p}}\right)$. □

Theorem 5.2 *Let $\varepsilon >$ and $\lambda \in \mathbb{C}$ such that $Re\lambda + \lambda^* > 0$. Then there exists ε_0 satisfying $0 < \varepsilon_0 < \varepsilon$ such that:*

$$\{\lambda \in \mathbb{C},\ Re\lambda \leq \varepsilon_0 - \lambda^*\} \subset \sigma_{\varepsilon,e}(T_H) \subset \{\lambda \in \mathbb{C},\ Re\lambda \leq \varepsilon - \lambda^*\}.$$

Proof: For $Re\lambda > \lambda_0$, the operator $(\lambda - T_0)^{-1}$ exists and $(\lambda - T_0)^{-1} = C_\lambda$. In addition one has

$$\|(\lambda - T_0)^{-1}\| \leq \frac{1}{Re\lambda + \lambda^*}. \tag{5.5}$$

Let $\lambda \in \sigma_\varepsilon(T_0)$ then $\|(\lambda - T_0)^{-1}\| > \frac{1}{\varepsilon}$. Using, Eq.(5.5) we infer that

$$\frac{1}{Re\lambda + \lambda^*} > \frac{1}{\varepsilon} \text{ and we have } Re\lambda \leq \varepsilon - \lambda^*.$$

It is follows from Theorem 1.1(i)

$$\sigma_{\varepsilon,e}(T_H) \subset \{\lambda \in \mathbb{C},\ Re\lambda \leq \varepsilon - \lambda^*\}.$$

Finally, one has

$$\{\lambda \in \mathbb{C},\ Re\lambda \leq \varepsilon_0 - \lambda^*\} \subset \sigma_{e,\varepsilon}(T_H) \subset \sigma_{\varepsilon,e}(T_H).$$

□

In the following definition we give conditions on which the operator K is regular.

Definition 5.1 *A collision operator K is said to be regular if it satisfies the following conditions:*

$$(\mathcal{H}):\begin{cases} K \text{ is a measurable, i.e., } \{x\in[-a,a]: K(x)\in\mathcal{O}\} \text{ is measurable if } \mathcal{O}\subset\mathcal{Z} \text{ is open,}\\ \text{there exists a compact subset } T\subset\mathcal{Z}: K(x)\in T \text{ a.e.}\\ \text{and finally } K(x)\in\mathcal{K}\left(L_p([-1,1],\mathrm{d}\xi)\right) \text{ a.e.}\end{cases}$$

where $\mathcal{K}\left(L_p([-1,1],\mathrm{d}\xi)\right)$ denotes the set of all compact operators on $L_p([-1,1],\mathrm{d}\xi)$.

Proposition 5.1 *[13, Lemma, 2.1] If the collision operator K is regular, then, for all $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda>-\lambda^*$, the operator $(\lambda-T_H)^{-1}K$ is compact on $X_p(1<p<\infty)$ and weakly compact on X_1.*

Now we are in position to state the main results of this section. Indeed:

Theorem 5.3 *Let $\varepsilon>0$, we assume that the operator K is regular on X_p. Then if $\|K\|<\lambda^*$ and $T_HK=KT_H$ one has*

$$\sigma_{\varepsilon,e}\left(A_H^{-1}\right)=\sigma_{\varepsilon,e}\left(T_H^{-1}\right).$$

Proof: Let $\bar{\lambda}$ be the leading eigenvalue of A_H. In [19], the strip $-\lambda^*<\bar{\lambda}\leq-\lambda^*+\|K\|$ contains at most isolated points of $\sigma\left(A_H\right)$. Now, using the fact that

$$\sigma\left(T_H\right)=\sigma C\left(T_H\right)=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\leq-\lambda^*\},$$

if $\|K\|<\lambda^*$, then $0\in\rho\left(T_H\right)\cap\rho\left(A_H\right)$. Since, $\lambda^*>\|K\|$ we have, $\left\|T_H^{-1}K\right\|\leq\frac{\|K\|}{\lambda^*}$, hence $\left\|T_H^{-1}K\right\|<1$. Therefore, one has

$$A_H^{-1}=T_H^{-1}+\sum_{n\geq1}\left[\left(T_H\right)^{-1}K\right]^nT_H^{-1}.$$

Let $\mathcal{K}_{\mathcal{T}}=\sum_{n\geq1}\left[\left(T_H\right)^{-1}K\right]^nT_H^{-1}$ then,

$$\sigma_{\varepsilon,e}\left(A_H^{-1}\right)=\sigma_{\varepsilon,e}\left(T_H^{-1}+\mathcal{K}_{\mathcal{T}}\right).$$

Under the condition of Definition 5.1, we infer that $\mathcal{K}_{\mathcal{T}}$ is compact on $X_p(1<p<\infty)$ and weakly compact on X_1. Moreover, we have :

$$\begin{aligned} T_H^{-1}\mathcal{K}_{\mathcal{T}} &= T_H^{-1}\textstyle\sum_{n\geq1}\left[\left(T_H\right)^{-1}K\right]^nT_H^{-1}\\ &= T_H^{-1}\left(I-\left(T_H\right)^{-1}K\right)^{-1}T_H^{-1}\\ &= \left((I-\left(T_H\right)^{-1}K)T_H\right)^{-1}T_H^{-1}\\ &= \left((T_H-\left(T_H\right)^{-1}KT_H)\right)^{-1}T_H^{-1}.\end{aligned}$$

Since $KT_H=T_HK$ then,

$$T_H^{-1}\mathcal{K}_{\mathcal{T}}=\left(T_H-K\right)^{-1}T_H^{-1}.$$

On the other hand we have

$$\begin{aligned} \mathcal{K}_{\mathcal{T}}T_H^{-1} &= \left(I-((T_H)^{-1}K)\right)^{-1}T_H^{-1}T_H^{-1}\\ &= \left(T_H(I-\left(T_H\right)^{-1}K)\right)^{-1}T_H^{-1}\\ &= \left(T_H-K\right)^{-1}T_H^{-1}.\end{aligned}$$

Which gives us $\mathcal{K}_{\mathcal{T}}T_H^{-1}=T_H^{-1}\mathcal{K}_{\mathcal{T}}$. Finally using Proposition 2.1(iv), we obtain

$$\sigma_{\varepsilon,e}\left(A_H^{-1}\right)=\sigma_{\varepsilon,e}\left(T_H^{-1}+\mathcal{K}_{\mathcal{T}}\right)=\sigma_{\varepsilon,e}\left(T_H^{-1}\right).$$

□

As a consequence to the previous result we deduce the following one.

Corollary 5.1 *Assume that the hypotheses of Theorem* 3.1 *are satisfied. Then, there exist* $\varepsilon' > 0$ *such that*

$$\sigma_{\varepsilon,e}(A_H^{-1}) \subset \left\{ \lambda \in \mathbb{C}, \quad Re\frac{1}{\lambda} \leq \varepsilon' - \lambda^* \right\}.$$

Theorem 5.4 *Assume that the hypotheses of Theorem* 3.1 *are satisfied. Then, there exist* $\varepsilon_1 > 0$ *such that*

$$\sigma_{\varepsilon,e}(A_H) \subset \{ \lambda \in \mathbb{C}, \quad Re\lambda \leq \varepsilon_1 - \lambda^* \}.$$

Proof: Let $\lambda \in \sigma_{\varepsilon,e}(A_H)$, then there exist $\varepsilon_1 > 0$ such that $\frac{1}{\lambda} \in \sigma_{\varepsilon_1,e}(A_H^{-1})$, and by applying Theorem 3.1 the proof is completed. □

References

1. F. Abdmouleh and A. Jeribi, Gustafson, Weidman, Kato, Wolf, Schechter, Browder, Rakočević and Schmoeger essential spectra of the sum of two bounded operators and application to transport operators. Math. Nachr: 284(2-3), 166-176, (2011).
2. F. Abdmouleh, A. Ammar and A. Jeribi, Pseudo-Browder essential spectra of linear operators and application to a transport equations, J. Comput. Theor. Transp. 44 (2015), 141-135.
3. F. Abdmouleh, B. Elgabeur, Pseudo Essential Spectra in Banach Space and Application to Operator Matrices, Acta Applicandae Mathematicae, vol. 181, (1), p. 7., https://doi. 10.1007/s10440-022-00527-5, (2022).
4. F. Abdmouleh , B. Elgabeur, On the pseudo semi-Browder essential spectra and application to 2 x 2 block operator matrices, Filomat, 37, no. 19, 6373-6386, https://doi.10.2298/FIL2433675E, (2024).
5. E. B. Davies, Spectral Theory and Differential Operators, Cambridge University Press, Cambridge, (1995).
6. Dautray, Robert, Lions, Jacques-Louis, Mathematical Analysis and Numerical Methods for Science and Technology, Tome9, Massons, Paris, (1988).
7. B. Elgabeur, A Characterization of Essential Pseudospectra Involving Polynomially Compact Operators, Filomat, vol 38, no. 33, 11675-11691, https://doi.10.2298/FIL2319373A, (2023).
8. D. Hinrichsen and A. J. Pritchard, Robust stability of linear evolution operators on Banach spaces, SIAM J. Control Optim. 32, 1503-1541, (1994).
9. R. Harte, Invertibility and singularity for bounded linear operators, Marcel Dekker, New York, 1988.
10. A. Jeribi and M. Mnif, Fredholm operators, essential spectra and application to transport equation. Acta Appl. Math. 89, 155-176, (2006).
11. T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Analyse Math., 6, 261-322, (1958).
12. H. J. Landau, On Szegï¿½'s eigenvalue distribution theorem and non-Hermitian kernels, J. Analyse Math. 28, 335-357, (1975).
13. M. Mokhtar-Kharroubi, Time asymptotic behaviour and compactness in transport theory, European J. Mech. B Fluids 11 (1), 39-68, (1992).
14. V. Rakočević, Semi-Browder operators and perturbations. Stud. Math. 112 (2), 131-137, (1997).
15. M. Schechter, Principles of Functional Analysis, second edition, American Mathematical Society, Providence, RI, (2002).
16. M. Schechter, Spectra of Partial Differential Operators, second edition, North-Holland Publishing Co., Amsterdam, (1986).
17. L. N. Trefthen, Pseudospectra of matrices, in "Numerical Analysis 1991 (Dundee, 1991), Longman Sci. Tech., Harlow, 234-266, (1992).
18. J. M. Varah, The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator", Thesis (Ph.D.), Stanford University, (1967).
19. I. Vidav, Existence and uniqueness of nonnegative eigenfunctions of the Boltzmann operator, J. Math. Anal. Appl. 22, 144-155, (1968).

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