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Interval-valued neutrosophic rough soft set based intelligent multi-criteria decision-making framework

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ABSTRACT: This paper introduces a new concept called Interval-Valued Neutrosophic Rough Soft Sets ($\mathcal{IVNRS\delta s}$) to address vagueness, imprecision, and uncertainty in complex decision-making situations. By integrating soft, rough, and interval-valued neutrosophic set theories, the framework provides a strong approach to deal with incomplete and indeterminate data. Fundamental operations such as intersection, union, complement, and innovative aggregation union operators, designed for multi-criteria decision-making (MCDM) applications, form the theoretical basis of $\mathcal{INNRSs}$. We also introduce an innovative $\mathcal{IVNRSSs}$ based MCDM model (β-model) to solve MCDM problems. The practical application of β-mode ℓ is demonstrated through a water quality assessment, where water samples are classified based on pollution scores into categories: Excellent, Safe, Moderate, Poor, and Highly Polluted, with corresponding degrees of contamination. A comparative analysis further validates the effectiveness of our proposed β-mode ℓ. Finally, our study concludes with key findings and recommendations for future research directions.

Key Words: Fuzzy set, neutrosophic set, soft set, rough set, interval-valued neutrosophic set.

Contents

1. Introduction

Environmental science, engineering, and medicine are just a few of the fields where modeling uncertainty and imprecision in decision-making processes has been shown to be a persistent challenge. The inability of classical set theories to adequately handle ambiguity and insufficient knowledge frequently calls for the creation of more complex mathematical frameworks ([1], [2]). In 1965, Zadeh made a revolutionary contribution by introducing fuzzy set theory, which made it possible to represent membership degrees as opposed to binary classifications [3]. Later, Atanassov's intuitionistic fuzzy sets added membership and non-membership degrees to this framework to reflect a wider range of uncertainty [4]. The notions of lower and upper approximations in rough set theory, which Pawlak [5] presented in 1982, allowed classification with ambiguous or inadequate information, complementing fuzzy logic. Thereafter, several researchers have done much research in these fields ([6], [7]). Interval-valued neutrosophic rough sets (IVNRSs) [8], soft sets [9], neutrosophic sets ([10], [11]), and interval-valued neutrosophic sets [12] are examples of sophisticated hybrid models that result from the combination of fuzzy logic with rough sets.

In 2015, Broumi and Smarandache [8] showed how IVNRSs can be used to handle intricate decision-making processes. By addressing real-world decision-making issues, these models provide strong MCDM tools. To address the twin problems of indeterminacy and incompleteness in decision-making, this paper presents a novel framework called $\beta - model$. By integrating the advantages of rough set theory, soft

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set theory, and neutrosophic sets, the suggested $\beta - model$ improves the evaluation of environmental phenomena, especially water quality ([13], [14], [15]). Real-world examples of water quality evaluation illustrate the practical application of $\beta - model$, and it is in line with international water quality standards, including those established by the Central Pollution Control Board [16] and the World Health Organization [17].

Originality of the Study

This paper presents the $\mathcal{IVNRSS}$ based $\beta - model$, a novel method for dealing with uncertainty, indeterminacy, and incompleteness in decision-making that integrates neutrosophic, rough, and soft set theories. Among the major contributions are:

- The Aggregate Union Operator is a new tool that unifies lower and higher approximations, allowing for balanced assessments of falsehood, indeterminacy, and truth.
- Use in Water Quality Assessment: A useful framework that addresses data uncertainties and inconsistencies by grouping water samples into Excellent, Safe, Moderate, Poor, and Highly Polluted categories.
  By bridging theoretical and practical gaps, this research provides a strong and flexible methodology for environmental and MCDM difficulties.

Literature Review

Advanced mathematical frameworks have evolved in response to the shortcomings of conventional set theories in managing uncertainty ([1], [2]). Atanassov [4] expanded on Zadeh's fuzzy set theory [3], which offered a non-binary method to uncertainty modeling, by addressing both membership and non-membership degrees with intuitionistic fuzzy sets. By employing lower and upper approximations to categorize things given ambiguous information, Pawlak's rough set theory [5] offered a complementary viewpoint. Smarandache's introduction of neutrosophy ([10], [11]) offered a cohesive framework for truth, indeterminacy, and falsity, which sparked innovations such as rough neutrosophic sets [19] and neutrosophic soft sets [18]. While Hai-Long et al. [21] developed generalized interval neutrosophic rough sets for MCDM, Broumi and Smarandache [20] offered interval-valued neutrosophic soft rough sets (IVNSRSs). Later, Deli [22] introduced the idea of Interval-valued neutrosophic soft set and its applications in decision-making situations by proposing new operators for them. The disciplines of fuzzy, soft, rough, and neutrosophic sets have since been the subject of several studies by various scholars ([23], [24], [25]).

Recent developments in MCDM techniques have given useful tools for handling difficult decision-making issues in various industries, such as environmental management, healthcare, energy, and tourism. The versatility and resilience of decision models have been improved by incorporating fuzzy set theory with various mathematical frameworks, such as spherical fuzzy sets, hesitant fuzzy sets, and q-rung orthopair fuzzy sets. These developments have opened the door for assessments in practical applications that are more precise and trustworthy. For instance, Saha et al. [27] presented q-rung orthopair fuzzy aggregation operators for multi-attribute decision-making, proving their efficacy in complex decision environments, while Sahoo and Debnath [26] suggested a novel hybrid spherical fuzzy MCDM approach for choosing the best hydroelectric power plant source in India. Additionally, as demonstrated by Saha et al. [28], the use of fuzzy-based models has been essential in evaluating sustainable development goals, underscoring the increasing applicability of these strategies in tackling global issues. To determine the ideal sites for regenerative tourism activities, Sahoo and Debnath [29] applied the fuzzy MABAC technique, highlighting the significance of sustainable development and community resilience. Saha et al. [30] presented the usefulness of fuzzy-based methods in intricate decision-making contexts for supplier selection in healthcare supply chains. Fuzzy sets, soft sets, and neutrosophic techniques are effective methods for environmental management, according to recent studies ([31], [32], [33], [34]). These methods increase the precision of decision-making processes by presenting uncertainty at various levels of complexity. Mukherjee and Das ([35], [36]) made a noteworthy contribution by using fuzzy soft multisets in MCDM and proving their value in dealing with uncertainty. By combining FP and IFP intuitionistic multi-fuzzy N-soft sets with

MCDM models, Das and Granados ([37],[38]) enhanced fuzzy-based MCDM frameworks. Weighted fuzzy soft multisets were created by Das [39], and weighted average ratings were later added for improved group MCDM ([40],[41]). While their weighted hesitant bipolar-valued fuzzy soft set [43] increased the complexity of the decision model, Das et al. [42] created novel operations on fuzzy soft sets to address real-world choice problems.

Table 1: Evolution of decision-making models leading to *IVNRS*

Authors	**Year**	**Concept**	**Key Features**
Zadeh [3]	1965	Fuzzy Set	Introduced degree of membership (μ) to handle uncertainty.
Pawlak [5]	1982	Rough Set	Introduced lower and upper approximations to classify uncertainty.
Smarandache [10]	1998	Neutrosophic Set	Extended fuzzy set by adding truth, indeterminacy, and falsity memberships.
Molodtsov [9]	1999	Soft Set	A parameterized approach for decision-making under uncertainty.
Wang et al. [12]	2005	IVNS	Allowed interval-based representation of truth, indeterminacy, and falsity.
Maji [18]	2013	Neutrosophic Soft Set	Combined neutrosophic set and soft set to improve uncertainty modeling.
Broumi et al. [19]	2014	Rough Neutrosophic Set	Integrated rough sets into neutrosophic theory for approximation-based analysis.
Deli [22]	2017	IVNSS	Merged IVNS with soft set for better decision-making flexibility.
Broumi and Smarandache [20]	2015	IVNSRS	Integrated rough sets into IVNSS for approximation-based analysis.
Proposed	**2025**	$\mathcal{IVNRSS}$	Integrates IVNS, rough set, and soft set for enhanced decision-making in water quality assessment.

To tackle the twin problems of incompleteness and indeterminacy in data analysis, this research presents a novel framework called $\mathcal{IVNRSS}$. The suggested strategy offers a solid methodology ($\beta - model$) for managing uncertainty in MCDM by combining the advantages of rough set theory with IVNSs. The usefulness of $\beta - model$ is demonstrated by a case study on water quality evaluation, which highlights how they can be used to solve actual environmental monitoring problems. The rest of this paper is organized as follows: In Section 2, important terms and fundamental ideas about $\mathcal{IVNRSS}s$ are described. The mathematical structure and functions of $\mathcal{IVNRSS}s$ are introduced in Section 3. In Section 4, we introduce an innovative $\beta - model$ based on $\mathcal{IVNRSS}s$ to solve MCDM problems such as water quality evaluation. Section 5 compares the suggested $\beta - model$ with a popular method for evaluating and making decisions about water quality. Section 6 ends with important conclusions and suggestions for further research.

For the sake of presenting clarity, we utilize the following abbreviations throughout this article:

	list of Abbreviations
MCDM	Multi-criteria decision-making
IVNS	Interval-valued Neutrosophic set
IVNRS	Interval-valued neutrosophic rough set
IVNSS	Interval-valued neutrosophic soft set
$\mathcal{IVNRSS}$	Interval-valued neutrosophic rough soft set
$\beta - model$	Interval-valued neutrosophic rough soft set based MCDM model

	List of Symbols
$\mathfrak{U}$	Universal set
$\mathfrak{K}$	Parameter set
$\mathfrak{G}$	Mapping
$(\mathfrak{F}, \mathfrak{G})$	Soft set, IVNSS
$(\mathfrak{U}, \mathfrak{R})$	Approximation space
$[u]_{\mathfrak{R}}$	Equivalence class
$\underline{(\mathfrak{F}, \mathfrak{G})}_{\mathfrak{K}}$	Lower approximations
$\overline{(\mathfrak{F}, \mathfrak{G})}_{\mathfrak{K}}$	Upper approximations
$\mathcal{N}$	Neutrosophic set, IVNS
$T_{\mathcal{N}}(u)$	Truth Membership Function
$I_{\mathcal{N}}(u)$	Indeterminacy Membership Function
$F_{\mathcal{N}}(u)$	Falsity Membership Function

2. Preliminaries

Let $\mathfrak{U}$ stand for the initial universe, or universal set, which includes every potential object that is being considered. The set comprising all potential subsets of $\mathfrak{U}$ is known as the power set of $\mathfrak{U}$, or P($\mathfrak{U}$). A collection of parameters that characterize different facets or qualities of the elements that make up $\mathfrak{U}$ is represented by $\mathfrak{G}$.

Definition 2.1 *[10], [11] A Neutrosophic Set is characterized as a structured object denoted by $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$, where $\mathfrak{U}$ is a non-empty fixed set. In this context, $T_{\mathcal{N}}(u), I_{\mathcal{N}}(u)$, and $F_{\mathcal{N}}(u)$ represent the degrees of membership, indeterminacy, and non-membership for each element $u \in \mathfrak{U}$, respectively. The values are constrained within the interval $]0^-, 1^+[$ for T (truth), I (indeterminacy), and F (falsity), adhering to the condition $0^- \leq T_{\mathcal{N}}(u) + I_{\mathcal{N}}(u) + F_{\mathcal{N}}(u) \leq 3^+$.*

Definition 2.2 *[12] An interval-valued neutrosophic set (IVNS) is a specialized form defined as $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$, where each membership function is represented by intervals: $T_{\mathcal{N}}(u) = \left[T^{\mathbb{L}}_{\mathcal{N}}(u), T^{\mathbb{U}}_{\mathcal{N}}(u)\right], I_{\mathcal{N}}(u) = \left[I^{\mathbb{L}}_{\mathcal{N}}(u), I^{\mathbb{U}}_{\mathcal{N}}(u)\right]$, and $F_{\mathcal{N}}(u) = \left[F^{\mathbb{L}}_{\mathcal{N}}(u), F^{\mathbb{U}}_{\mathcal{N}}(u)\right]$. These intervals indicate the range of values for each function, ensuring that $T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) \in \mathrm{Int}([0, 1])$. Here, $\mathrm{Int}([0, 1])$ refers to the set of all closed subintervals within the unit interval [0,1], providing a more flexible representation of uncertainty.*

Example 2.3 *Consider a decision-making scenario where an expert evaluates the suitability of different locations for installing a renewable energy system. Let $\mathfrak{U} = \{A, B, C\}$ be a set of three possible locations. or each location $u \in \mathfrak{U}$, we define its membership functions using interval values:*

1. **Truth Membership Function** $T_{\mathcal{N}}(u)$ *: Represents the degree of truth in the suitability of a location.*
2. **Indeterminacy Membership Function** $I_{\mathcal{N}}(u)$ *: Represents the degree of uncertainty in the assessment.*
3. **Falsity Membership Function** $F_{\mathcal{N}}(u)$ *: Represents the degree of falsity in the suitability of a location.*

Let's assign specific numerical values:

- *For location A:*
  - ◯ $T_{\mathcal{N}}(A) = [0.7, 0.9]$ *(Suitability is between 70% and 90%)*
  - ◯ $I_{\mathcal{N}}(A) = [0.1, 0.3]$ *(Uncertainty is between 10% and 30%)*
  - ◯ $F_{\mathcal{N}}(A) = [0.0, 0.2]$ *(Unsuitability is between 0% and 20%)*
- *For location B:*
  - ◯ $T_{\mathcal{N}}(B) = [0.5, 0.8]$
  - ◯ $I_{\mathcal{N}}(B) = [0.2, 0.4]$
  - ◯ $F_{\mathcal{N}}(B) = [0.1, 0.3]$
- *For location C:*
  - ◯ $T_{\mathcal{N}}(C) = [0.6, 0.85]$
  - ◯ $I_{\mathcal{N}}(C) = [0.15, 0.35]$
  - ◯ $F_{\mathcal{N}}(C) = [0.05, 0.25]$

- *Location A has the highest truth value range [0.7,0.9], indicating it is the most likely suitable choice.*
- *Location B has a lower truth value range [0.5,0.8], and higher uncertainty [0.2,0.4], meaning experts are less confident.*
- *Location C falls between A and B in suitability but has slightly lower uncertainty.*

The IVNS for the given example is written as follows:

$$\mathcal{N} = \{\langle u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u)\rangle : u \in \mathfrak{U}\}$$

where $\mathfrak{U} = \{A, B, C\}$ and the interval values for each membership function are:

$$\mathcal{N} = \left\{ \begin{array}{c} \langle A, [0.7, 0.9], [0.1, 0.3], [0.0, 0.2]\rangle, \\ \langle \mathrm{B}, [0.5, 0.8], [0.2, 0.4], [0.1, 0.3]\rangle, \\ \langle \mathrm{C}, [0.6, 0.85], [0.15, 0.35], [0.05, 0.25]\rangle \end{array} \right\}$$

This notation explicitly defines the interval-valued truth, indeterminacy, and falsity membership functions for each element in U, providing a structured representation of uncertainty in decision-making.

Definition 2.4 *[12] The complement of a given IVNS $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$, denoted by $\mathcal{N}^{\mathfrak{c}}$, is defined as $\mathcal{N}^{\mathfrak{c}} = \{< u, T_{\mathcal{N}}{}^{\mathfrak{c}}(u), I_{\mathcal{N}}{}^{\mathfrak{c}}(u), F_{\mathcal{N}}{}^{\mathfrak{c}}(u) >: u \in \mathfrak{U}\}$, where the mappings are derived as $T_{\mathcal{N}}{}^{\mathbb{C}}(u) = \left[F^{\mathbb{L}}_{\mathcal{N}}(u), F^{\mathbb{U}}_{\mathcal{N}}(u)\right], I_{\mathcal{N}}{}^{\mathbb{C}}(u) = \left[1 - I^{\mathbb{U}}_{\mathcal{N}}(u), 1 - I^{\mathbb{L}}_{\mathcal{N}}(u)\right]$, and $F_{\mathcal{N}}{}^{\mathbb{C}}(u) = \left[T^{\mathbb{L}}_{\mathcal{N}}(u), T^{\mathbb{U}}_{\mathcal{N}}(u)\right]$. The maximum of a IVNS is $< [1, 1], [0, 0], [0, 0] >$, denoted by 1_N and the minimum is $< [0, 0], [1, 1], [1, 1] >$, denoted by 0_N.*

Definition 2.5 *[12] A IVNS $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$ is considered to be a subset of another IVNS $\mathcal{K} = \{< u, T_{\mathcal{K}}(u), I_{\mathcal{K}}(u), F_{\mathcal{K}}(u) >: u \in \mathfrak{U}\}$, represented as $\mathcal{N} \subseteq \mathcal{K}$, if $T^{\mathbb{L}}_{\mathcal{N}}(u) \leq T^{\mathbb{L}}_{\mathcal{K}}(u), T^{\mathbb{U}}_{\mathcal{N}}(u) \leq T^{\mathbb{U}}_{\mathcal{K}}(u)$, $I^{\mathbb{L}}_{\mathcal{N}}(u) \geq I^{\mathbb{L}}_{\mathcal{K}}(u), I^{\mathbb{U}}_{\mathcal{N}}(u) \geq I^{\mathbb{U}}_{\mathcal{K}}(u)$, and $F^{\mathbb{L}}_{\mathcal{N}}(u) \geq F^{\mathbb{L}}_{\mathcal{K}}(u), F^{\mathbb{U}}_{\mathcal{N}}(u) \geq F^{\mathbb{U}}_{\mathcal{K}}(u), \forall u \in \mathfrak{U}$. This definition emphasizes the comparative relationships between neutrosophic sets, providing a clear framework for assessing inclusion.*

Example 2.6 *Let $\mathfrak{U} = \{A, B, C\}$ be a universal set of elements representing different locations for a renewable energy project. Consider two IVNSs N and K as follows:*

$$N = \left\{ \begin{array}{c} \langle A, [0.5, 0.7], [0.3, 0.5], [0.2, 0.4] \rangle \\ \langle \text{B}, [0.4, 0.6], [0.35, 0.55], [0.25, 0.45] \rangle, \\ \langle \text{C}, [0.3, 0.5], [0.4, 0.6], [0.3, 0.5] \rangle \end{array} \right\},$$

$$K = \left\{ \begin{array}{c} \langle A, [0.6, 0.8], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \langle \text{B}, [0.5, 0.7], [0.25, 0.45], [0.15, 0.35] \rangle, \\ \langle \text{C}, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4] \rangle \end{array} \right\}$$

Here, N is a subset of $K (N \subseteq K)$.

This example demonstrates how one IVNS can be a subset of another. The truth values in N are always lower than or equal to those in K, while the indeterminacy and falsity values in N are always greater than or equal to those in K. This aligns with the definition of subset inclusion in IVNS theory.

Definition 2.7 *[12] The Union of two IVNSs $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$ and $\mathcal{K} = \{< u, T_{\mathcal{K}}(u), I_{\mathcal{K}}(u), F_{\mathcal{K}}(u) >: u \in \mathfrak{U}\}$ is represented as $\mathcal{G} = \mathcal{N} \cup \mathcal{K}$. The truth, indeterminacy, and degree of nonmembership functions for $\mathcal{G}$ are described as $\forall u \in \mathfrak{U}$.*

$$\begin{array}{c} T^{\mathbb{L}}_{\mathcal{G}}(u) = \sup \left\{ T^{\mathbb{L}}_{\mathcal{N}}(u), T^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, T^{\mathbb{U}}_{\mathcal{G}}(u) = \sup \left\{ T^{\mathbb{U}}_{\mathcal{N}}(u), T^{\mathbb{U}}_{\mathcal{K}}(u) \right\} \\ I^{\mathbb{L}}_{\mathcal{G}}(u) = \inf \left\{ I^{\mathbb{L}}_{\mathcal{N}}(u), I^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, I^{\mathbb{U}}_{G}(u) = \inf \left\{ I^{\mathbb{U}}_{\mathcal{N}}(u), I^{\mathbb{U}}_{\mathcal{K}}(u) \right\} \\ F^{\mathbb{L}}_{\mathcal{G}}(u) = \inf \left\{ F^{\mathbb{L}}_{\mathcal{N}}(u), F^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, F^{\mathbb{U}}_{G}(u) = \inf \left\{ F^{\mathbb{U}}_{\mathcal{N}}(u), F^{\mathbb{U}}_{\mathcal{K}}(u) \right\}. \end{array}$$

Definition 2.8 *[12] The intersection of two IVNSs $\mathcal{N} = \{< u, T_{\mathcal{N}}(u), I_{\mathcal{N}}(u), F_{\mathcal{N}}(u) >: u \in \mathfrak{U}\}$ and $\mathcal{K} = \{< u, T_{\mathcal{K}}(u), I_{\mathcal{K}}(u), F_{\mathcal{K}}(u) >: u \in \mathfrak{U}\}$ is represented as $\mathcal{G} = \mathcal{N} \cap \mathcal{K}$. The truth, indeterminacy, and degree of nonmembership functions for $\mathcal{G}$ are described as $\forall u \in \mathfrak{U}$.*

$$\begin{array}{c} T^{\mathbb{L}}_{\mathcal{G}}(u) = \inf \left\{ T^{\mathbb{L}}_{\mathcal{N}}(u), T^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, T^{\mathbb{U}}_{\mathcal{G}}(u) = \inf \left\{ T^{\mathbb{U}}_{\mathcal{N}}(u), T^{\mathbb{U}}_{\mathcal{K}}(u) \right\} \\ I^{\mathbb{L}}_{\mathcal{G}}(u) = \sup \left\{ I^{\mathbb{L}}_{\mathcal{N}}(u), I^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, I^{\mathbb{U}}_{\mathcal{G}}(u) = \sup \left\{ I^{\mathbb{U}}_{\mathcal{N}}(u), I^{\mathbb{U}}_{\mathcal{K}}(u) \right\} \\ F^{\mathbb{L}}_{\mathcal{G}}(u) = \sup \left\{ F^{\mathbb{L}}_{\mathcal{N}}(u), F^{\mathbb{L}}_{\mathcal{K}}(u) \right\}, F^{\mathbb{U}}_{\mathcal{G}}(u) = \sup \left\{ F^{\mathbb{U}}_{\mathcal{N}}(u), F^{\mathbb{U}}_{\mathcal{K}}(u) \right\}. \end{array}$$

Example 2.9 *Let $\mathfrak{U} = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ be a universal set representing different locations for a renewable energy project. Consider two IVNSs N and K, defined as follows:*

$$N = \left\{ \begin{array}{c} \langle A, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4] \rangle, \\ \langle \text{B}, [0.5, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \langle \text{C}, [0.3, 0.5], [0.4, 0.6], [0.3, 0.5] \rangle \end{array} \right\}$$

$$K = \left\{ \begin{array}{c} \langle A, [0.5, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \langle \text{B}, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4] \rangle, \\ \langle \text{C}, [0.2, 0.4], [0.5, 0.7], [0.4, 0.6] \rangle \end{array} \right\}$$

Thus, the union $G = N \cup K$ is:

$$G = \left\{ \begin{array}{c} \langle A, [0.5, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \langle B, [0.5, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \langle C, [0.3, 0.5], [0.4, 0.6], [0.3, 0.5] \rangle \end{array} \right\}$$

Thus, the intersection $H = N \cap K$ is:

$$H = \left\{ \begin{array}{c} \langle A, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4] \rangle, \\ \langle B, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4] \rangle, \\ \langle C, [0.2, 0.4], [0.5, 0.7], [0.4, 0.6] \rangle \end{array} \right\}$$

Definition 2.10 *[9] A soft set over a universe $\mathfrak{U}$ is defined as a couple $(\mathfrak{F}, \mathfrak{G})$, where $\mathfrak{F}$ is a function $\mathfrak{F} : \mathfrak{G} \to P(\mathfrak{U})$. This function associates each parameter in the set $\mathfrak{G}$ with a specific subset of $\mathfrak{U}$. In simpler terms, for each parameter $\mathfrak{p} \in \mathfrak{G}$, the mapping $\mathfrak{F}(\mathfrak{p})$ identifies a subset of elements in $\mathfrak{U}$ to which the parameter is applicable.*

Definition 2.11 *[5] For a subset $\mathfrak{B} \subseteq \mathfrak{U}$, the lower and upper approximations of $\mathfrak{B}$ concerning the approximation space $(\mathfrak{U}, \mathfrak{K})$ are denoted by $\underline{\mathfrak{B}}_{\mathfrak{K}}$ and $\overline{\mathfrak{B}}_{\mathfrak{K}}$, respectively. These are formally defined as follows:*

1. ***Lower approximation** $\underline{\mathfrak{B}}_{\mathfrak{K}}$: This consists of all elements $u \in \mathfrak{U}$ for which the equivalence class $[u]_{\mathfrak{K}}$ is entirely contained within $\mathfrak{B}$. Formally,*

$$\underline{\mathfrak{B}}_{\mathfrak{K}} = \{u \in \mathfrak{U} : [u]_{\mathfrak{K}} \subseteq \mathfrak{B}\} \text{ or } \cup \{[u]_{\mathfrak{K}} : [u]_{\mathfrak{K}} \subseteq \mathfrak{B}\}, u \in \mathfrak{U}.$$

2. ***Upper approximation** $\overline{\mathfrak{B}}_{\mathfrak{K}}$: This includes all elements $u \in \mathfrak{U}$ such that the intersection of $[u]_{\mathfrak{K}}$ with $\mathfrak{B}$ is nonempty. That is,*

$$\overline{\mathfrak{B}}_{\mathfrak{K}} = \{u \in \mathfrak{U} : [u]_{\mathfrak{K}} \cap \mathfrak{B} \neq \phi\} . \text{ or } \cup \{[u]_{\mathfrak{K}} : [u]_{\mathfrak{K}} \cap \mathfrak{B} \neq \phi\}, u \in \mathfrak{U}.$$

These approximations serve distinct purposes: the lower approximation $\underline{\mathfrak{B}}_{\mathfrak{K}}$ captures elements that can be confidently classified within $\mathfrak{B}$ based on available information, while the upper approximation $\overline{\mathfrak{B}}_{\mathfrak{K}}$ identifies elements that could potentially belong to $\mathfrak{B}$. When the lower and upper approximations coincide-that is, when $\underline{\mathfrak{B}}_{\mathfrak{K}} = \overline{\mathfrak{B}}_{\mathfrak{K}}$ the set $\mathfrak{B}$ is termed definable, as it can be precisely described using the equivalence classes induced by $\mathfrak{K}$. However, if $\underline{\mathfrak{B}}_{\mathfrak{K}} \neq \overline{\mathfrak{B}}_{\mathfrak{K}}, \mathfrak{B}$ is classified as a rough set, indicating some ambiguity or uncertainty in its boundaries.

3. Interval-Valued Neutrosophic Rough Soft Sets and their Basic Properties

In this section, we combine IVNSs, soft sets, and rough sets to introduce a novel concept of $\mathcal{IVNRSS}$. Over these sets, we also examine various operations, such as union, intersection, inclusion, and equality.

Definition 3.1 *Assume $\mathfrak{A}$ is a non-empty collection, and $\mathfrak{F}$ signifies a nonempty set of parameters. Let us assume $\mathrm{IVN}(\mathfrak{A})$ denotes the collection of all IVNSs and $\mathfrak{K}$ is a relation of equivalence established on $\mathfrak{A}$. Then an intervalvalued neutrosophic soft set (IVNSS) over $\mathfrak{A}$ is represented by a pair $(\mathfrak{F}, \mathfrak{G})$, where $\mathfrak{F}$ is a mapping described by $\mathfrak{F} : \mathfrak{G} \to \mathbb{IVNS}(\mathfrak{A})$. A IVNSS $(\mathfrak{F}, \mathfrak{G})$ in $\mathfrak{A}$ is defined as $\forall \mathfrak{p} \in \mathfrak{G}$,*

$$\mathfrak{F}(\mathfrak{p}) = \left\{ \left(u, \left\{ \begin{array}{c} T_{\mathfrak{F}(\mathfrak{p})}(u), \\ I_{\mathfrak{F}(\mathfrak{p})}(u), \\ F_{\mathfrak{F}(\mathfrak{p})}(u) \end{array} \right\} \right) : u \in \mathfrak{A} \right\}$$

where $T_{\mathfrak{F}(\mathfrak{p})}(u) = \left[T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(u), T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(u)\right], I_{\mathfrak{F}(\mathfrak{p})}(u) = \left[I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(u), I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(u)\right]$, and $F_{\mathfrak{F}(\mathfrak{p})}(u) = \left[F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(u)\right] \in \mathrm{Int}([0,1])(\mathrm{Int}([0,1])$ is the collection of all subintervals of $[0,1])$.

The lower and upper approximation $\underline{(\mathfrak{F}, \mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}, \mathfrak{G})}_{\mathfrak{K}}$ of $(\mathfrak{F}, \mathfrak{G})$ in the Pawlak approximation $(\mathfrak{A}, \mathfrak{K})$ are as follows:

$$\underline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}} = \left\{ \left(u, \left\{ \begin{array}{c} \left[\inf_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\sup_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \sup_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\sup_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \sup_{\mathfrak{y} \in [u]_{\mathfrak{K}}} \left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right] \end{array} \right. \right) : u \in \mathfrak{A} \right\}, \mathfrak{p} \in \mathfrak{G}.$$

$$\overline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}}=\left\{\left(u,\left\{\begin{bmatrix}\left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right],\\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right],\\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf^{\mathbb{L}}_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\end{bmatrix}\right\}\right):u\in\mathfrak{A}\right\},\mathfrak{p}\in\mathfrak{G}.\right\}$$

The relation $\mathfrak{K}$ denotes the equivalence relation associated with the IVNSS $(\mathfrak{F},\mathfrak{G})$. In this context, $[u]_{\mathfrak{K}}$ represents the equivalence class of the element u.

$$\left.\begin{array}{l}\left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{F(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\\ \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\\ \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\end{array}\right\}$$

Then $(\mathfrak{F},\mathfrak{G})_{\mathfrak{R}}$ is an IVNSS. Similarly, we have

$$\left.\begin{array}{l}\left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\in \mathrm{Int}([0,1])\end{array}\right\}.$$

Then $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}}$ is a IVNSS.

If
$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}}=\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}}$, i.e. $\underline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}}=\overline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}},\forall\mathfrak{p}\in\mathfrak{F}$, then $(\mathfrak{F},\mathfrak{G})$ is classified as a definable set; otherwise, $(\mathfrak{F},\mathfrak{G})$ is considered a $\boldsymbol{\mathcal{IVNRSS}}$ where $(\mathfrak{F},\mathfrak{G})=\left(\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}},\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}}\right)$.

Example 3.2 *Assume that a water regulatory agency monitors the quality of drinking water by evaluating specific water quality parameters such as pH, DO (dissolved oxygen), and BOD (Biological oxygen demand). The $\boldsymbol{\mathcal{IVNRSS}}$ framework is applied to assess water quality and classify it like Excellent, Safe, Moderate, Poor, and Highly Polluted while handling incomplete, inconsistent, and indeterminate data. Let the water samples $\mathfrak{U}=\{ws_1,ws_2,ws_3,ws_4\}$ represent four water samples, and the parameters being evaluated from the parameter set $\mathfrak{G}=\{\mathfrak{p}_1=pH,\mathfrak{p}_2=DO,\mathfrak{p}_3=BOD\}$. Each parameter will be evaluated using the IVNRS framework. The relation $\mathfrak{K}$ is defined as an equivalence relation based on the similarity of water samples in terms of their water quality.*
For each parameter $\mathfrak{p}\in\mathfrak{G}$, the IVNS at each water sample $ws\in\mathfrak{A}$ is defined as:

$$\mathfrak{F}(\mathfrak{p})=\left\{\left(ws,\left\{\begin{array}{c}T_{\mathfrak{F}(\mathfrak{p})}(ws),\\ I_{\mathfrak{F}(\mathfrak{p})}(ws),\\ F_{\mathfrak{F}(\mathfrak{p})}(ws)\end{array}\right\}\right):ws\in\mathfrak{A}\right\},$$

The intervals for truth (T), indeterminacy (I), and falsity (F) at each water sample are derived based on expert evaluations or historical data. Assume that the intervals for truth (T), indeterminacy (I), and falsity (F) at each water sample are shown in Table 2 according to expert assessments.

Table 2: Truth (T), indeterminacy (I), and falsity (F) at each water sample are derived based on expert evaluations

Water sample	p_1 (T, I, F)	p_2 (T, I, F)	p_3 (T, I, F)
ws_1	[0.7, 0.8], [0.1, 0.2], [0.0, 0.1]	[0.7, 0.9], [0.1, 0.2], [0.0, 0.1]	[0.6, 0.7], [0.2, 0.3], [0.0, 0.2]
ws_2	[0.5, 0.6], [0.3, 0.4], [0.2, 0.3]	[0.6, 0.8], [0.2, 0.3], [0.0, 0.1]	[0.2, 0.4], [0.3, 0.4], [0.3, 0.5]
ws_3	[0.6, 0.8], [0.0, 0.1], [0.0, 0.2]	[0.6, 1.0], [0.0, 0.1], [0.0, 0.1]	[0.6, 0.8], [0.1, 0.2], [0.0, 0.3]
ws_4	[0.6, 0.9], [0.3, 0.4], [0.2, 0.3]	[0.5, 0.6], [0.2, 0.3], [0.2, 0.3]	[0.3, 0.5], [0.3, 0.4], [0.3, 0.5]

Let IVNSS($\mathfrak{A}$) represent the collection of all IVNSSs over $\mathfrak{A}$ and let the equivalence relation $\mathfrak{R}$ be defined as $\mathfrak{A}/\mathfrak{R}=\{\{ws_1,ws_3\},\{ws_2,ws_4\}\}$ is established to classify water samples based on similarity in water quality characteristics, where $[\{ws_1\}]_{\mathfrak{R}}=\{ws_1,ws_3\}$ as they have similar water quality parameters (WQPs), $[\{ws_2\}]_{\mathfrak{R}}=\{ws_2\}$, and $\{[ws]_4\}_{\mathfrak{R}}=\{ws_3\}$.

Let

$$(\mathfrak{F},\mathfrak{G})_{\mathfrak{R}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{l} (ws_1,[0.7,0.8],[0.1,0.2],[0.0,0.1]), \\ (ws_2,[0.5,0.6],[0.3,0.4],[0.2,0.3]), \\ (ws_3,[0.6,0.8],[0.0,0.1],[0.0,0.2]), \\ (ws_4,[0.6,0.9],[0.3,0.4],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{l} (ws_1,[0.7,0.9],[0.0,0.1],[0.0,0.1]), \\ (ws_2,[0.5,0.7],[0.2,0.3],[0.2,0.3]), \\ (ws_3,[0.8,1.0],[0.0,0.1],[0.0,0.1]), \\ (ws_4,[0.5,0.6],[0.2,0.3],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{l} (ws_1,[0.6,0.7],[0.2,0.3],[0.0,0.2]), \\ (ws_2,[0.2,0.4],[0.3,0.4],[0.3,0.5]), \\ (ws_3,[0.6,0.8],[0.1,0.2],[0.0,0.1]), \\ (ws_4,[0.3,0.5],[0.3,0.4],[0.3,0.5]) \end{array} \right\}\right) \end{array} \right\}$$

be an IVNSS on $\mathfrak{A}$. Then by definition, the lower and upper approximations for $(\mathfrak{F},\mathfrak{G})$ are calculated as:

$$\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{l} (ws_1,[0.6,0.8],[0.1,0.2],[0.0,0.2]), \\ (ws_2,[0.5,0.6],[0.3,0.4],[0.2,0.3]), \\ (ws_3,[0.6,0.8],[0.1,0.2],[0.0,0.2]), \\ (ws_4,[0.6,0.9],[0.3,0.4],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{l} (ws_1,[0.7,0.9],[0.0,0.1],[0.0,0.1]), \\ (ws_2,[0.5,0.7],[0.2,0.3],[0.2,0.3]), \\ (ws_3,[0.7,0.9],[0.0,0.1],[0.0,0.1]), \\ (ws_4,[0.5,0.6],[0.2,0.3],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{l} (ws_1,[0.6,0.7],[0.2,0.3],[0.0,0.2]), \\ (ws_2,[0.2,0.4],[0.3,0.4],[0.3,0.5]), \\ (ws_3,[0.6,0.7],[0.2,0.3],[0.0,0.2]), \\ (ws_4,[0.3,0.5],[0.3,0.4],[0.3,0.5]) \end{array} \right\}\right) \end{array} \right\}$$

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{R}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{l} (ws_1,[0.7,0.8],[0.0,0.1],[0.0,0.1]), \\ (ws_2,[0.5,0.6],[0.3,0.4],[0.2,0.3]), \\ (ws_3,[0.7,0.8],[0.0,0.1],[0.0,0.1]), \\ (ws_4,[0.6,0.9],[0.3,0.4],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{l} (ws_1,[0.8,1.0],[0.0,0.1],[0.0,0.1]), \\ (ws_2,[0.5,0.7],[0.2,0.3],[0.2,0.3]), \\ (ws_3,[0.8,1.0],[0.0,0.1],[0.0,0.1]), \\ (ws_4,[0.5,0.6],[0.2,0.3],[0.2,0.3]) \end{array} \right\}\right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{l} (ws_1,[0.6,0.8],[0.1,0.2],[0.0,0.1]), \\ (ws_2,[0.2,0.4],[0.3,0.4],[0.3,0.5]), \\ (ws_3,[0.6,0.8],[0.1,0.2],[0.0,0.1]), \\ (ws_4,[0.3,0.5],[0.3,0.4],[0.3,0.5]) \end{array} \right\}\right) \end{array} \right\}$$

Definition 3.3 *Assume $(\mathfrak{F}_1,\mathfrak{G})$ and $(\mathfrak{F}_2,\mathfrak{G})$ be* ***IVNRSS***, *$\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ are the lower and upper approximation of a* ***IVNRSS*** *$(\mathfrak{F}_1,\mathfrak{G})$ concerning the approximation space $(\mathfrak{U},\mathfrak{K})$ respectively, $\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ are the lower and upper approximation of a* ***IVNRSS*** *$(\mathfrak{F}_2,\mathfrak{G})$ concerning the approximation space $(\mathfrak{U},\mathfrak{K})$ respectively. The Union of these two* ***IVNRSS*** *$(\mathfrak{F}_1,\mathfrak{G})$ and $(\mathfrak{F}_2,\mathfrak{G})$ is represented as $(\mathfrak{F},\mathfrak{G}) = \left(\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}, \overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\right) = \left(\underline{(\mathfrak{F}_1,\mathfrak{G}) \cup (\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}, \overline{(\mathfrak{F}_1,\mathfrak{G}) \cup (\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}\right)$ The truth, indeterminacy, and degree of non-membership functions for $\underline{(\mathfrak{F}_1,\mathfrak{G}) \cup (\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}_1,\mathfrak{G}) \cup (\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ are described as $\forall u \in \mathfrak{U}$,*

$$\begin{gathered}
T^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{T^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},T^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
I^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{I^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},I^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{I^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
F^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{F^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},F^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{F^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}.\\
\textit{and}\\
T^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},T^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p}}}(u)=\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
I^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},I^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
F^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},F^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}.
\end{gathered}$$

Definition 3.4 *The intersection of two* $\boldsymbol{\mathcal{IVNRSS}}$ $(\mathfrak{F}_1,\mathfrak{G})$ *and* $(\mathfrak{F}_2,\mathfrak{G})$ *is represented as* $(\mathfrak{F},\mathfrak{G})=\left(\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}},\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\right)=\left(\underline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}},\overline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}\right)$. *The truth, indeterminacy, and degree of non-membership functions for* $\underline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *and* $\overline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *are described as* $\forall\ u\in\mathfrak{U}$.

$$\begin{gathered}
T^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{T^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},T^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
I^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{I^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},I^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{I^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
F^{\mathbb{L}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{F^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},F^{\mathbb{U}}_{\underline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{F^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}.\\
\textit{and}\\
T^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},T^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
I^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},I^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\\
F^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\},F^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cap\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}.
\end{gathered}$$

Theorem 3.5 *Assume* $(\mathfrak{F}_1,\mathfrak{G})$ *and* $(\mathfrak{F}_2,\mathfrak{G})$ *be* $\boldsymbol{\mathcal{IVNRSS}}$, $\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ *and* $\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *are the lower and upper approximation of a* $\boldsymbol{\mathcal{IVNRSS}}$ $(\mathfrak{F}_1,\mathfrak{G})$ *concerning the approximation space* $(\mathfrak{U},\mathfrak{K})$ *respectively,* $\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *and* $\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *are the lower and upper approximation of a* $\boldsymbol{\mathcal{IVNRSS}}\,(\mathfrak{F}_2,\mathfrak{G})$ *concerning the approximation space* $(\mathfrak{U},\mathfrak{K})$ *respectively. Then the following hold:*
(i) $\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\subseteq(\mathfrak{F}_1,\mathfrak{G})\subseteq\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$
(ii) $\overline{(\mathfrak{F}_1,\mathfrak{G})\cup(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cup\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}},\underline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cap\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$
(iii) $\overline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cap\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *and* $\underline{(\mathfrak{F}_1,\mathfrak{G})\cup(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cup\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$.
(iv) $(\mathfrak{F}_1,\mathfrak{G})\subseteq(\mathfrak{F}_2,\mathfrak{G})\Rightarrow\overline{(\mathfrak{F}_1,\mathfrak{G})_{\mathfrak{K}}}\subseteq\overline{(\mathfrak{F}_2,\mathfrak{G}}\Big)_{\mathfrak{K}}$ *and* $\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\subseteq\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$.

Proof: *(i) Assume*

$$(\mathfrak{F}_1,\mathfrak{G})=\left\{\left(\mathfrak{p},\left(\mathfrak{u},\left\{\begin{array}{c}\left[T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}),T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right],\\ \left[I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}),I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right],\\ \left[F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}),F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right]\end{array}\right\}\right):\mathfrak{u}\in\mathfrak{A}\right):\mathfrak{p}\in\mathfrak{G}\right\},$$

be a $\boldsymbol{\mathcal{IVNRSS}}$ *satisfying the conditions* $\left[T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u),T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right],\left[I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u),I^{\mathbb{U}}_{\mathfrak{F}_{1_{(\mathfrak{p})}}}(u)\right],$ *and* $\left[F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u),F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right]\in\mathrm{Int}([0,1]),\forall\ u\in\mathfrak{A},\mathfrak{p}\in\mathfrak{G}$.
Now, from the definition 3.1, $\forall u\in\mathfrak{U},\forall p\in\mathfrak{G}$

$$T^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} \leq T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \leq \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right) = T^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),$$
$$T^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T_{\mathfrak{F}_1}(\mathfrak{p})^{\mathbb{U}}(\mathfrak{y})\right\} \leq T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \leq \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} = T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),$$
$$I^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} \geq I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \geq \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} = I^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),$$
$$I^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} \geq I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \geq \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} = I^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(\mathfrak{u}),$$
$$F^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} \geq F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \geq \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} = F^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),$$
$$F^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u) = \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} \geq F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y}) \geq \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{y})\right\} = F^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u)$$

Thus

$$\left(\left[T^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u), T^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right], \left[I^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u), I^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right], \left[F^{\mathbb{L}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u), F^{\mathbb{U}}_{\underline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right]\right) \subseteq$$
$$\left(\left[T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right], \left[I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right], \left[F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right]\right) \subseteq$$
$$\left(\left[T^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u), T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right], \left[I^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u), I^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right], \left[F^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u), F^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u)\right]\right), \forall u \in \mathfrak{U}, \forall \mathfrak{p} \in \mathfrak{G}.$$

Hence $\underline{(\mathfrak{F}_1, \mathfrak{G})}_{\mathfrak{K}} \subseteq (\mathfrak{F}_1, \mathfrak{G}) \subseteq \overline{(\mathfrak{F}_1, \mathfrak{G})}_{\mathfrak{K}}$

ii) Let $(\mathfrak{F}_1, \mathfrak{G}) = \left\{\left(\mathfrak{p}, \left(\mathfrak{u}, \left\{\begin{array}{l}\left[T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}), T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right],\\ \left[I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}), I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right],\\ \left[F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u}), F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(\mathfrak{u})\right]\end{array}\right\}\right) : \mathfrak{u} \in \mathfrak{A}\right) : \mathfrak{p} \in \mathfrak{G}\right\}$,
be a $\mathcal{IVNRSS}$ *satisfying the conditions* $\left[T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right], \left[I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right]$,
and $\left[F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(u)\right] \in \mathrm{Int}([0,1]), \forall u \in \mathfrak{A}, \mathfrak{p} \in \mathfrak{G}$.
$(\mathfrak{F}_2, \mathfrak{G}) = \left\{\left(\mathfrak{p}, \left(u, \left\{\begin{array}{l}\left[T^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right],\\ \left[I^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), I^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right],\\ \left[F^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right]\end{array}\right\}\right) : u \in \mathfrak{A}\right) : \mathfrak{p} \in \mathfrak{G}\right\}$,
be a $\mathcal{IVNRSS}$ *satisfying the conditions* $\left[T^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right], \left[I^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), I^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right]$,
and $\left[F^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(u)\right] \in \mathbb{I}nt([0,1]), \forall u \in \mathfrak{A}, \mathfrak{p} \in \mathfrak{G}$ *be two* $\mathcal{IVNRSS}$.

$$\overline{(\mathfrak{F}_1, \mathfrak{G}) \cup (\mathfrak{F}_2, \mathfrak{G})}_{\mathfrak{K}} = \left\{\left(\mathfrak{p}, \left(\mathfrak{u}, \left\{\begin{array}{l}\left[T^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), T^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right],\\ \left[I^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), I^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right],\\ \left[F^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), F^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right]\end{array}\right\}\right) : u \in \mathfrak{A}\right) : \mathfrak{p} \in \mathfrak{G}\right\}$$

where $\left[T^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), T^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right], \left[I^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), I^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right], \left[F^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u), F^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)\right] \in \mathrm{Int}([0,1])$.

Now
$\overline{(\mathfrak{F}_1, \mathfrak{G})}_{\mathfrak{K}} \cup \overline{(\mathfrak{F}_2, \mathfrak{G})}_{\mathfrak{K}} =$

$$\left\{\left(\mathfrak{p},\left(\mathfrak{u},\left\{\begin{array}{l}\left[\sup\left(T^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),T^{\mathbb{L}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right),\sup\left(T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),T^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right)\right],\\ \left[inf\left(I^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),I^{\mathbb{L}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right),inf\left(I^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),I^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right)\right],\\ \left[inf\left(F^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),F^{\mathbb{L}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right),inf\left(F^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}(u),F^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}(u)\right)\right]\end{array}\right\}\right):u\in\mathfrak{A}\right):\mathfrak{p}\in\mathfrak{G}\right\}$$

$$T^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\vee T^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\vee\left\{\sup\left\{T^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(T^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\vee T^{\mathbb{L}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$
$$T^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\vee T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\vee\left\{\sup\left\{T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\vee T^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$
$$I^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\inf\left\{I^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\wedge I^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\wedge\left\{\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\wedge T^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$
$$I^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{I^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\inf\left\{I^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\wedge I^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\wedge\left\{\inf\left\{T^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(T^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\wedge T^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$
$$F^{\mathbb{L}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\wedge F^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\wedge\left\{\inf\left\{F^{\mathbb{L}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(F^{\mathbb{L}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\wedge F^{\mathbb{L}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$
$$F^{\mathbb{U}}_{\overline{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}}(u)=\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_1\cup\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}=\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t)\wedge F^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}$$
$$=\left\{\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_1(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}\wedge\left\{\inf\left\{F^{\mathbb{U}}_{\mathfrak{F}_2(\mathfrak{p})}(t):t\in[u]_{\mathfrak{K}}\right\}\right\}=\left(F^{\mathbb{U}}_{\overline{\mathfrak{F}_1(\mathfrak{p})}}\wedge F^{\mathbb{U}}_{\overline{\mathfrak{F}_2(\mathfrak{p})}}\right)(u)$$

Hence $\overline{(\mathfrak{F}_1,\mathfrak{G})\cup(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cup\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *and similarly,* $\underline{(\mathfrak{F}_1,\mathfrak{G})\cap(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}=\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\cap\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ *for all* $u\in\mathfrak{A}$.
(iii) The proof for this property follows a similar reasoning to that in property (ii).
(iv) The proof is obvious. □

Definition 3.6 *If* $(\mathfrak{F},\mathfrak{G})=\left(\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}},\left(\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\right)\right.$ *is a* $\boldsymbol{\mathcal{IVNRSS}}$ *in* $(\mathfrak{U},\mathfrak{K})$ *then the complement of* $(\mathfrak{F},\mathfrak{G})$ *is a* $\boldsymbol{\mathcal{IVNRSS}}$ *denoted by* $(\mathfrak{F},\mathfrak{G})^{\mathfrak{C}}=\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}},\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ *where* $\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ *and* $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ *are the complements of* $\boldsymbol{\mathcal{IVNRSSs}}$ $\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ *and* $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ *respectively.*

$$\underline{\mathfrak{F}(\mathfrak{p})}^{\mathfrak{C}}_{\mathfrak{K}}=\left\{\left(u,\left\{\begin{array}{c}\left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right],\\ \left[\left(1-\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right),\left(1-\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right)\right\}\right],\\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\end{array}\right.\right):u\in\mathfrak{A}\right\},\mathfrak{p}\in\mathfrak{G}.$$

In this representation:

- $\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *and* $\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *denote the supremum (maximum) of the lower and upper membership degrees of* $\mathfrak{F}(\mathfrak{p})$ *over the equivalence class* $[u]_{\mathfrak{K}}$.
- $1-\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *and* $1-\sup_{\mathfrak{y}\in[u]_{\mathfrak{F}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *are the complements of the supremum (maximum) indeterminacy degrees.*
- $\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *and* $\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *denote the infimum (minimum) of the truth degrees of* $\mathfrak{F}(\mathfrak{p})$ *over* $[u]_{\mathfrak{K}}$.

$$\overline{\mathfrak{F}(\mathfrak{p})_{\mathfrak{K}}}^{\mathfrak{C}}=\left\{\left(u,\left\{\begin{array}{c}\left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right],\\ \left[\left(1-\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right),\left(1-\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right)\right],\\ \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right]\end{array}\right.\right):u\in\mathfrak{A}\right\},\mathfrak{p}\in\mathfrak{G}.$$

Here:

- $\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *and* $\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *denote the infimum (minimum) of the lower and upper membership degrees of* $\mathfrak{F}(\mathfrak{p})$ *within* $[u]_{\mathfrak{K}}$.
- $\left(1-\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right)$*, and* $(1-\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\})$ *are the complements of the infimum (minimum) indeterminacy degrees.*
- $\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *and* $\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}$ *indicate the supremum (maximum) of the truth degrees over the equivalence class* $[u]_{\mathfrak{K}}$.

Example 3.7 *Here, we investigate the idea of the complement of the* $\mathcal{IVNRSS}s$ *approximations, expanding on the analysis from Example 3.2. Our objective is to ascertain the upper and lower approximations of the complement* $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ *and* $\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$*, given the* $\mathcal{IVNRSS}s$ $(\mathfrak{F},\mathfrak{G})$ *and the equivalence relation* $\mathfrak{K}$ *over the universe* $\mathfrak{U}$*. The definition of* $\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$*, the lower approximation of the complement, is as follows:*

$$\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}=\left\{\begin{array}{l}\left(\mathfrak{p}_1,\left\{\begin{array}{l}(ws_1,[0.7,0.8],[0.8,0.9],[0.0,0.1]),\\(ws_2,[0.5,0.6],[0.6,0.7],[0.2,0.3]),\\(ws_3,[0.7,0.8],[0.9,1.0],[0.0,0.1]),\\(ws_4,[0.6,0.9],[0.6,0.7],[0.2,0.3])\end{array}\right\}\right),\\ \left(\mathfrak{p}_2,\left\{\begin{array}{l}(ws_1,[0.8,1.0],[0.9,1.0],[0.0,0.1]),\\(ws_2,[0.5,0.7],[0.7,0.8],[0.2,0.3]),\\(ws_3,[0.8,1.0],[0.9,1.0],[0.0,0.1]),\\(ws_4,[0.5,0.6],[0.7,0.8],[0.2,0.3])\end{array}\right\}\right),\\ \left(\mathfrak{p}_3,\left\{\begin{array}{l}(ws_1,[0.6,0.8],[0.7,0.8],[0.0,0.1]),\\(ws_2,[0.2,0.4],[0.6,0.7],[0.3,0.5]),\\(ws_3,[0.6,0.8],[0.8,0.9],[0.0,0.1]),\\(ws_4,[0.3,0.5],[0.6,0.7],[0.3,0.5])\end{array}\right\}\right)\end{array}\right\}$$

Similarly, the definition of $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$*, the upper approximation of the complement, is as follows:*

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}=\left\{\begin{array}{l}\left(\mathfrak{p}_1,\left\{\begin{array}{l}(ws_1,[0.6,0.8],[0.9,1.0],[0.0,0.2]),\\(ws_2,[0.5,0.6],[0.6,0.7],[0.2,0.3]),\\(ws_3,[0.6,0.8],[0.9,1.0],[0.0,0.2]),\\(ws_4,[0.6,0.9],[0.6,0.7],[0.2,0.3])\end{array}\right\}\right),\\ \left(\mathfrak{p}_2,\left\{\begin{array}{l}(ws_1,[0.7,0.9],[0.9,1.0],[0.0,0.1]),\\(ws_2,[0.5,0.7],[0.7,0.8],[0.2,0.3]),\\(ws_3,[0.7,0.9],[0.9,1.0],[0.0,0.1]),\\(ws_4,[0.5,0.6],[0.7,0.8],[0.2,0.3])\end{array}\right\}\right),\\ \left(\mathfrak{p}_3,\left\{\begin{array}{l}(ws_1,[0.6,0.7],[0.8,0.9],[0.0,0.2]),\\(ws_2,[0.2,0.4],[0.6,0.7],[0.3,0.5]),\\(ws_3,[0.6,0.7],[0.8,0.9],[0.0,0.2]),\\(ws_4,[0.3,0.5],[0.6,0.7],[0.3,0.5])\end{array}\right\}\right)\end{array}\right\}$$

In this upper approximation $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$*, the set includes additional elements from the equivalence classes of* $\mathfrak{K}$ *that partially meet the criteria of* $(\mathfrak{F},\mathfrak{G})$*'s complement, encompassing a broader scope of indeterminate and falsity values. This upper approximation reflects potential elements of* $(\mathfrak{F},\mathfrak{G})^{\mathfrak{C}}$ *based on the expanded intervals for each parameter, thereby handling cases where uncertainty and partial indeterminacy play a role.*

Together, $\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ and $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}^{\mathfrak{C}}$ illustrate how the $\boldsymbol{\mathcal{IVNRSS}}$ approach can manage and distinguish between complete and partial membership in a set's complement, providing nuanced approximations that can handle both high and low degrees of membership uncertainty. This dual approximation approach allows for robust decision-making applications where elements' inclusion or exclusion from a set is based on varying degrees of truth, indeterminacy, and falsity across intervals, adapting to the dynamic and often imprecise nature of real-world data.

Theorem 3.8 *Assume $(\mathfrak{F}_1,\mathfrak{G})$ and $(\mathfrak{F}_2,\mathfrak{G})$ be $\boldsymbol{\mathcal{IVNRSS}}$, $\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}$ are the lower and upper approximation of a $\boldsymbol{\mathcal{IVNRSS}}$ $(\mathfrak{F}_1,\mathfrak{G})$ concerning the approximation space $(\mathfrak{U},\mathfrak{K})$ respectively, $\underline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ and $\overline{(\mathfrak{F}_2,\mathfrak{G})}_{\mathfrak{K}}$ are the lower and upper approximation of a $\boldsymbol{\mathcal{IVNRSS}}$ $(\mathfrak{F}_2,\mathfrak{G})$ concerning the approximation space $(\mathfrak{U},\mathfrak{K})$ respectively, Then $\underline{(\mathfrak{F}_1,\mathfrak{G})^{\mathfrak{C}}}_{\mathfrak{K}} = \left(\overline{(\mathfrak{F}_1,\mathfrak{G})}\right)_{\mathfrak{K}}^{\mathfrak{C}}, \overline{(\mathfrak{F}_1,\mathfrak{G})^{\mathfrak{C}}}_{\mathfrak{K}} = \left(\underline{(\mathfrak{F}_1,\mathfrak{G})}_{\mathfrak{K}}\right)^{\mathfrak{C}}$*

Proof: The proof is obvious. □

Definition 3.9 *If $(\mathfrak{F},\mathfrak{G}) = \left(\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}, \overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\right)$ is a $\boldsymbol{\mathcal{IVNRSS}}$ in $(\mathfrak{U},\mathfrak{K})$ then the aggregate union operator, represented by $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\Theta\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ is characterized in the following manner:*

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\Theta\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} = \left\{\left(\left(u, \left\{\begin{array}{l} \left[T^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), T^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right], \\ \left[I^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), I^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right], \\ \left[F^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), F^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right] \end{array}\right\}\right) : u \in \mathfrak{A}\right) : \mathfrak{p} \in \mathfrak{G}\right\}$$

$$= \left\{\left(\left(u, \left\{\begin{array}{l} \left[\begin{array}{l} \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} + \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} - \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \\ \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} + \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} - \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} \end{array}\right], \\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right] \end{array}\right\}\right) : u \in \mathfrak{A}\right) : \mathfrak{p} \in \mathfrak{G}\right\}.$$

The formula for the Aggregate Union Operator $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\Theta\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ involves the combination of three key components—truth (T), indeterminacy (I), and falsity (F):

Truth Component:
The truth values of the two approximations are added together, but their overlap is subtracted to avoid doubl counting:

$$\begin{aligned} \mathrm{T} &= \left[T^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), T^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right] \\ &= \left[\begin{array}{l} \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} + \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} - \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \\ \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} + \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} - \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\} \end{array}\right] \end{aligned}$$

This reflects the aggregation of truth information while considering the overlap between the two sets.

Indeterminacy Component:
The indeterminacy values of the two approximations are multiplied to ensure that the resulting uncertainty accounts for both sets' indeterminate information:

$$\mathrm{I} = \left[I^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), I^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right] =$$
$$\left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right].$$

This highlights the combined uncertainty from both approximations.

Falsity Component:
The falsity values are similarly combined by multiplying them, reflecting the false information present in both approximations:

$$\mathrm{F} = \left[F^{\mathbb{L}}_{\mathfrak{F}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u), F^{\mathbb{U}}_{\mathfrak{F}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)\right] =$$

$$\left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\},\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\cdot\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}}\left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right].$$

This ensures that the false elements are accounted for in the union of the two sets.
The primary goal of the Aggregate Union Operator is to combine the results from two different approximations $\left(\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} \text{ and } \underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\right)$ into a single, unified set that reflects the truth, indeterminacy, and falsity of the data in a balanced manner. This is especially useful in cases where data is uncertain or incomplete, as it captures both the definite and ambiguous aspects of the data.

Example 3.10 *Now consider the example, then we have*

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\Theta\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} = \left\{\begin{array}{l} \left(\mathfrak{p}_1,\left\{\begin{array}{l}(ws_1,[0.88,0.96],[0.00,0.02],[0.00,0.02]),\\(ws_2,[0.75,0.84],[0.09,0.16],[0.04,0.09]),\\(ws_3,[0.88,0.96],[0.00,0.02],[0.00,0.02]),\\(ws_4,[0.84,0.99],[0.09,0.16],[0.04,0.09])\end{array}\right\}\right),\\ \left(\mathfrak{p}_2,\left\{\begin{array}{l}(ws_1,[0.94,1.00],[0.00,0.03],[0.00,0.01]),\\(ws_2,[0.75,0.91],[0.04,0.09],[0.04,0.09]),\\(ws_3,[0.94,0.91],[0.00,0.01],[0.00,0.01]),\\(ws_4,[0.75,0.84],[0.04,0.09],[0.04,0.09])\end{array}\right\}\right),\\ \left(\mathfrak{p}_3,\left\{\begin{array}{l}(ws_1,[0.84,0.94],[0.02,0.06],[0.00,0.02]),\\(ws_2,[0.36,0.64],[0.09,0.16],[0.09,0.25]),\\(ws_3,[0.84,0.94],[0.02,0.06],[0.00,0.02]),\\(ws_4,[0.51,0.75],[0.09,0.16],[0.09,0.25])\end{array}\right\}\right)\end{array}\right\}$$

3.13 Advantages

- Comprehensive Representation: This operator allows for a more complete representation of data by considering both certain and uncertain elements in a unified framework.
- Handling of Uncertainty: The operator is specifically designed to address uncertainty in data, which is common in real-world applications.
- Flexibility: The Aggregate Union Operator can be applied to various types of data, including environmental, medical, and financial data, making it a versatile tool for many domains.

The Aggregate Union Operator provides a robust and flexible method for combining uncertain data from different approximations. Its ability to handle truth, indeterminacy, and falsity makes it an ideal tool for applications that involve incomplete, inconsistent, or ambiguous information. By incorporating this operator into decision-making processes, users can obtain more reliable and comprehensive results, ensuring better-informed decisions in complex scenarios.

4. Water quality assessment using $\mathcal{IVNRSS}$ and the aggregate union operator

In our study, the thresholds used for classifying water quality into Excellent, Safe, Moderate, Poor, and Highly Polluted categories were determined by a combination of expert judgment and standardized regulations. Expert judgment played a significant role in establishing the thresholds for key water quality parameters ([1], [2]). Environmental scientists, water quality specialists, and hydrologists familiar with the region's ecological and hydrological conditions were consulted. Their knowledge of local water quality issues, pollution sources, and the environmental sensitivity of the study area allowed for the development of thresholds that were practical and realistic for the specific context of the study ([31], [32]). For example,

experts may have determined that a dissolved oxygen level of 5mg/L is suitable for maintaining a healthy aquatic ecosystem in the study area, classifying it as "Good," while lower levels would indicate varying levels of pollution.
In addition to expert judgment, existing water quality standards provided a key foundation for the classification thresholds. The guidelines set by international bodies such as the World Health Organization [17] and the Central Pollution Control Board [16] in India were referenced to ensure that the classification system aligns with recognized criteria for safe and healthy water quality. These standards are based on extensive scientific research and are designed to safeguard public health and protect aquatic ecosystems.

4.1 Algorithm

In this section, we introduce the algorithm ($\beta - model$) developed for the water quality assessment, which is rooted in the principles of $\mathcal{IVNRSS}$. This $\beta - model$ is specifically tailored for MCDM applications, especially in environmental management scenarios where data is often uncertain, incomplete, and inconsistent. Traditional decision-making models often struggle with incomplete or inconsistent data, which is frequently encountered in real-world environmental studies due to the complex nature of water quality indicators. The $\beta - model$ is built on a combination of rough set theory [5], soft set theory [9], and IVNS [22], which are integrated within the $\mathcal{IVNRSS}$ framework. These set theories are known for their ability to manage and model uncertainty, vagueness, and indeterminacy in decision-making ([12-15], [23-43]). The $\mathcal{IVNRSS}$ framework allows for the consideration of multiple sources of uncertainty simultaneously, providing a more accurate representation of real-world data where exact values might not be available, or where there are inconsistencies and contradictions in the data.
This $\beta - model$ integrates $\mathcal{IVNRSS}$ and the aggregate union operator to assess water quality across multiple water samples. The steps are designed to classify water quality like Excellent, Safe, Moderate, Poor, or Highly Polluted by addressing uncertainty, indeterminacy, and falsity in the measurements of water quality parameters.

Step 1: Let us consider

- A set $\mathfrak{U} = \{u_1, u_2, u_3, \ldots, u_n\}$ representing the water samples.
- A set of water quality parameters $\mathfrak{G} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_m\}$, for each parameter $\mathfrak{p}_k \in \mathfrak{G}$, the intervals for truth, indeterminacy, and falsity (T, I, F) at each water sample are given.

Step 2: Define IVNSS
For each water quality parameter $\mathfrak{p}_k \in \mathfrak{G}$, define the IVNSS $(\mathfrak{F}, \mathfrak{G})$ for each water sample $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \ldots, \mathfrak{u}_n\}$. This includes truth, indeterminacy, and falsity intervals for each water sample:

$$\mathfrak{F}(\mathfrak{p}_k) = \left\{ \left(u, \left\{ \begin{array}{c} T_{\mathfrak{G}(\mathfrak{p}_k)}(u), \\ I_{\mathfrak{G}(\mathfrak{p}_k)}(u), \\ F_{\mathfrak{G}(\mathfrak{p}_k)}(u) \end{array} \right\} \right) : u \in \mathfrak{A} \right\}$$

Where:

- $T_{\mathfrak{F}(\mathfrak{p}_k)}(\mathfrak{u})$ is the truth interval for water sample $\mathfrak{u} \in \mathfrak{A}$ for parameter $\mathfrak{p}_k \in \mathfrak{G}$,
- $I_{\mathfrak{F}(\mathfrak{p}_k)}(\mathfrak{u})$ is the indeterminacy interval,
- $F_{\mathfrak{F}(\mathfrak{p}_k)}(\mathfrak{u})$ is the falsity interval.

Step 3: Establish the Equivalence Relation $\mathfrak{K}$
Define the equivalence relation $\mathfrak{K}$ to group water samples that have similar water quality characteristics (based on parameters $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_m$). In this context, $[u]_{\mathfrak{K}}$ represents the equivalence class of the element u.

Step 4: Calculate Lower and Upper Approximations
For each parameter $\mathfrak{p}_k \in \mathfrak{G}$, calculate the lower and upper approximations of the IVNSS $(\mathfrak{F}, \mathfrak{G})$. These

approximations are used to classify the water samples into more certain or uncertain categories based on their water quality values:

Lower Approximation $\underline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}}$:

$$\underline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}} = \left\{ \left(u, \left\{ \begin{array}{l} \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right] \end{array} \right. \right) : u \in \mathfrak{A} \right\}, \mathfrak{p} \in \mathfrak{G}.$$

Upper Approximation $\overline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}}$:

$$\overline{\mathfrak{F}(\mathfrak{p})}_{\mathfrak{K}} = \left\{ \left(u, \left\{ \begin{array}{l} \left[\sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \sup_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{T^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{I^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right], \\ \left[\inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{L}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}, \inf_{\mathfrak{y}\in[u]_{\mathfrak{K}}} \left\{F^{\mathbb{U}}_{\mathfrak{F}(\mathfrak{p})}(\mathfrak{y})\right\}\right] \end{array} \right\} \right) : u \in \mathfrak{A} \right\}, \mathfrak{p} \in \mathfrak{G}.$$

Step 5: Apply Aggregate Union Operator
Obtain $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}\Theta\underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ using the aggregate union operator Θ. This operator aggregates the truth, indeterminacy, and falsity intervals to create a unified representation of the water quality for each water sample.

Step 6: Obtain the pollution-score $\boldsymbol{\mathcal{W}(u)}$ Compute the Pollution-Score $\mathcal{W}(u)$ for each water sample $u \in \mathfrak{A}$ using the formula

$$\mathcal{W}(u) = \frac{1}{|\mathfrak{G}|} \sum_{\mathfrak{p}\in\mathfrak{G}} \left(\frac{\left[\frac{T^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)+T^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)}{2}\right] - \left[\frac{F^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)+F^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)}{2}\right]}{1 + \left[\frac{I^{\mathbb{L}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)+I^{\mathbb{U}}_{\overline{\mathfrak{F}}\Theta\underline{\mathfrak{F}}(\mathfrak{p})}(u)}{2}\right]} \right)$$

Step 7: Decision Output If $\mathcal{W}(u)$ values near -1 indicate excellent water quality, while near +1 indicate poor water quality as mentioned in Figure 1.

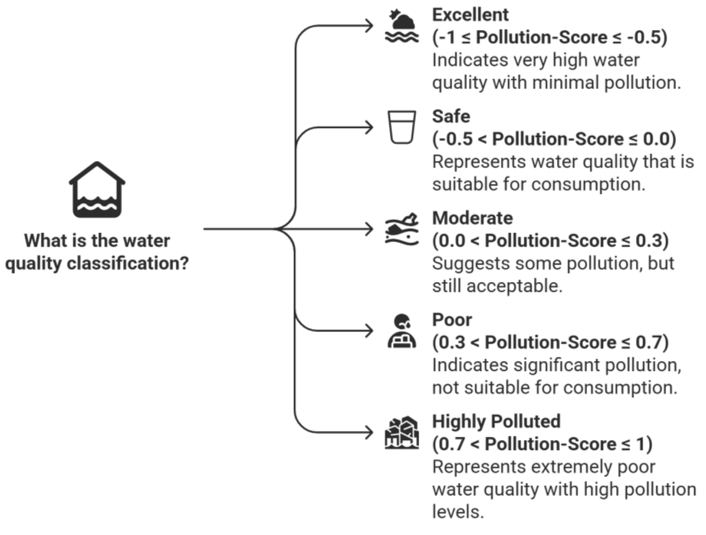

Figure 1: Decision Output

The detailed procedure for evaluating water quality using $\beta - model$ is shown in the flowchart in Figure 2. To determine the pollution score and ultimately the classification of the water quality, each step methodically identifies, computes, and assesses important components.

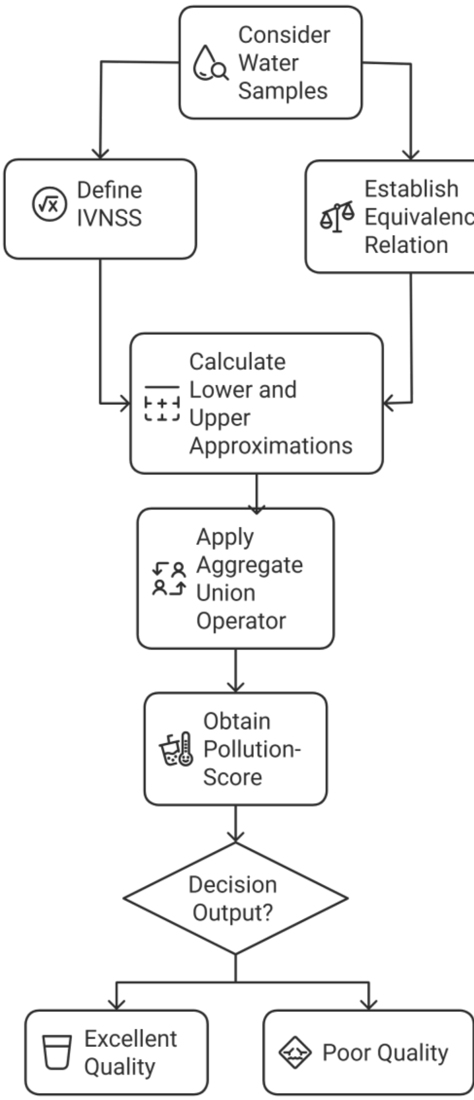

Figure 2: Water Quality Classification Algorithm ($\beta - model$)

4.2 Application Example

Assume that a water regulatory agency monitors the quality of drinking water by evaluating specific water quality parameters such as pH, DO, and BOD. The proposed $\beta - model$ is employed to assess water quality, classifying it such as excellent, safe, moderate, poor, and highly polluted, while effectively handling incomplete, inconsistent, and indeterminate data.

Let the water samples $\mathfrak{U} = \{ws_1, ws_2, ws_3, ws_4\}$ represent four water samples, and the parameters being evaluated from the parameter set $\mathfrak{G} = \{\mathfrak{p}_1 = pH, \mathfrak{p}_2 = DO, \mathfrak{p}_3 = BOD\}$. Each parameter will be evaluated using the IVNRS framework. The relation $\mathfrak{K}$ is defined as an equivalence relation based on the similarity of water samples in terms of their water quality.

For each parameter $\mathfrak{p} \in \mathfrak{G}$, the IVNS at each water sample $ws \in \mathfrak{A}$ is defined as:

$$\mathfrak{F}(\mathfrak{p}) = \left\{ \left(ws, \left\{ \begin{array}{l} T_{\mathfrak{F}(\mathfrak{p})}(ws), \\ I_{\mathfrak{F}(\mathfrak{p})}(ws), \\ F_{\mathfrak{F}(\mathfrak{p})}(ws) \end{array} \right\} \right) : ws \in \mathfrak{A} \right\},$$

The intervals for truth (T), indeterminacy (I), and falsity (F) at each water sample are derived based on expert evaluations or historical data.

Let IVNSS ($\mathfrak{A}$) represent the collection of all IVNSSs over $\mathfrak{A}$ and let the equivalence relation $\mathfrak{K}$ be defined as $\mathfrak{U}/\mathfrak{K} = \{\{sw_1, sw_3\}, \{sw_2\}, \{sw_4\}\}$ is established to classify water samples based on similarity in water quality characteristics, where

- $[ws_1]_{\mathfrak{K}} = \{ws_1, ws_3\}$, as they have similar WQPs.
- $[ws_2]_{\mathfrak{K}} = \{ws_2\}$ and $[ws_4]_{\mathfrak{K}} = \{ws_4\}$.

Let $$(\mathfrak{F}, \mathfrak{G}) = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{l} (ws_1, [0.7, 0.8], [0.1, 0.2], [0.0, 0.1]), \\ (ws_2, [0.5, 0.6], [0.3, 0.4], [0.2, 0.3]), \\ (ws_3, [0.6, 0.8], [0.0, 0.1], [0.0, 0.2]), \\ (ws_4, [0.6, 0.9], [0.3, 0.4], [0.2, 0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{l} (ws_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]), \\ (ws_2, [0.5, 0.7], [0.2, 0.3], [0.2, 0.3]), \\ (ws_3, [0.8, 1.0], [0.0, 0.1], [0.0, 0.1]), \\ (ws_4, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{l} (ws_1, [0.6, 0.7], [0.2, 0.3], [0.0, 0.2]), \\ (ws_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), \\ (ws_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1]), \\ (ws_4, [0.3, 0.5], [0.3, 0.4], [0.3, 0.5]) \end{array} \right\} \right) \end{array} \right\}$$ be a IVNSS on $\mathfrak{A}$.

Then by definition, the lower and upper approximations for $(\mathfrak{F}, \mathfrak{G})$ are calculated as:

$$\underline{(\mathfrak{F}, \mathfrak{G})}_{\mathfrak{K}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{l} (ws_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2]), \\ (ws_2, [0.5, 0.6], [0.3, 0.4], [0.2, 0.3]), \\ (ws_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2]), \\ (ws_4, [0.6, 0.9], [0.3, 0.4], [0.2, 0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{l} (ws_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]), \\ (ws_2, [0.5, 0.7], [0.2, 0.3], [0.2, 0.3]), \\ (ws_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]), \\ (ws_4, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{l} (ws_1, [0.6, 0.7], [0.2, 0.3], [0.0, 0.2]), \\ (ws_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), \\ (ws_3, [0.6, 0.7], [0.2, 0.3], [0.0, 0.2]), \\ (ws_4, [0.3, 0.5], [0.3, 0.4], [0.3, 0.5]) \end{array} \right\} \right) \end{array} \right\}$$

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{c} (ws_1, [0.7,0.8], [0.0,0.1], [0.0,0.1]), \\ (ws_2, [0.5,0.6], [0.3,0.4], [0.2,0.3]), \\ (ws_3, [0.7,0.8], [0.0,0.1], [0.0,0.1]), \\ (ws_4, [0.6,0.9], [0.3,0.4], [0.2,0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{c} (ws_1, [0.8,1.0], [0.0,0.1], [0.0,0.1]), \\ (ws_2, [0.5,0.7], [0.2,0.3], [0.2,0.3]), \\ (ws_3, [0.8,1.0], [0.0,0.1], [0.0,0.1]), \\ (ws_4, [0.5,0.6], [0.2,0.3], [0.2,0.3]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{c} (ws_1, [0.6,0.8], [0.1,0.2], [0.0,0.1]), \\ (ws_2, [0.2,0.4], [0.3,0.4], [0.3,0.5]), \\ (ws_3, [0.6,0.8], [0.1,0.2], [0.0,0.1]), \\ (ws_4, [0.3,0.5], [0.3,0.4], [0.3,0.5]) \end{array} \right\} \right) \end{array} \right\}$$

Now we obtain $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} \Theta \underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$ using the aggregate union operator Θ as

$$\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} \Theta \underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} = \left\{ \begin{array}{l} \left(\mathfrak{p}_1, \left\{ \begin{array}{c} (ws_1, [0.88,0.96], [0.0,0.02], [0.0,0.02]), \\ (ws_2, [0.75,0.84], [0.09,0.16], [0.04,0.09]), \\ (ws_3, [0.88,0.96], [0.0,0.02], [0.0,0.02]), \\ (ws_4, [0.84,0.99], [0.09,0.16], [0.04,0.09]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_2, \left\{ \begin{array}{c} (ws_1, [0.94,1.0], [0.0,0.03], [0.0,0.01]), \\ (ws_2, [0.75,0.91], [0.04,0.09], [0.04,0.09]), \\ (ws_3, [0.94,0.91], [0.0,0.01], [0.0,0.01]), \\ (ws_4, [0.75,0.84], [0.04,0.09], [0.04,0.09]) \end{array} \right\} \right), \\ \left(\mathfrak{p}_3, \left\{ \begin{array}{c} (ws_1, [0.84,0.94], [0.02,0.06], [0.0,0.02]), \\ (ws_2, [0.36,0.64], [0.09,0.16], [0.09,0.25]), \\ (ws_3, [0.84,0.94], [0.02,0.06], [0.0,0.02]), \\ (ws_4, [0.51,0.75], [0.09,0.16], [0.09,0.25]) \end{array} \right\} \right) \end{array} \right\}$$

Based on the above results of the Aggregate Union Operator $\overline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}} \Theta \underline{(\mathfrak{F},\mathfrak{G})}_{\mathfrak{K}}$, we compute the pollution-score $\mathcal{W}(sw_i)$ for each water sample $sw_i \in \mathfrak{U}$ as follows:

$$\mathcal{W}(sw_1) = 0.899, \mathcal{W}(sw_2) = 0.553, \mathcal{W}(sw_3) = 0.899, \mathcal{W}(sw_4) = 0.617$$

4.3 Results and Discussions

The decision output generated using the $\beta-model$ classifies the water samples based on their pollution levels. The classification results are as follows:

- Water sample ws_1 : Classified as Highly Polluted.
- Water sample ws_2 : Classified as Poor.
- Water sample ws_3 : Classified as Highly Polluted.
- Water sample ws_4 : Classified as Poor.

These results indicate that samples ws_1 and ws_3 exhibit significantly high pollution levels, suggesting a considerable presence of contaminants that exceed safe thresholds for water quality. Conversely, samples ws_2 and ws_4 fall into the Poor category, indicating suboptimal water quality but with comparatively lower contamination levels than those classified as Highly Polluted.

The classification provided by the $\beta-model$ aligns with the expected trends observed in water quality assessment. Several key observations can be drawn from the results:

- Pollution Level Consistency: The designation of ws_1 and ws_3 as Highly Polluted implies that specific environmental or human-caused variables significantly contribute to pollution in particular locations or circumstances where these samples were taken.

- Moderate Degradation in Additional Samples: The samples ws_2 and ws_4, which are classified as Poor, show regions where there is noticeable but not severe deterioration in the quality of the water. Although the concentration of contaminants in these samples is lower than that of the Highly Polluted category, they may nevertheless be dangerous for direct eating.
- $\beta-model$'s Efficiency in Managing Uncertainty: Compared to traditional decision models, the use of $\beta-model$ successfully captures the imprecision and ambiguity in water quality metrics, guaranteeing a more trustworthy categorization. By integrating several uncertainty-handling strategies, the $\beta-model$ improves the robustness of decisions.

The categorization results highlight how important it is to use sophisticated decision-making frameworks such as $\beta - model$ when evaluating the quality of water. The approach offers a thorough and trustworthy classification that can help environmental organizations and governments make well-informed decisions for sustainable water management by skillfully managing imprecision and uncertainty.

4.4 Research Implications

The proposed $\mathcal{IVNRSS}$ framework offers significant implications for various domains, particularly in the fields of water resource management, environmental policy-making, and sustainable development. Below are key areas where this research can make a meaningful impact:

- **Water Resource Management:** The $\mathcal{IVNRSS}$ framework provides a structured methodology to handle uncertainty, indeterminacy, and incomplete data in water quality assessments. By computing pollution scores and categorizing river segments, the framework enables water resource managers to:
  - Prioritize high-risk areas for remediation.
  - Allocate resources efficiently for pollution control interventions.

Example 4.1 *Suppose a river is divided into five segments, and water quality indicators (e.g., Dissolved Oxygen, pH, and Turbidity) are evaluated using the* $\mathcal{IVNRSS}$ *model. If Segment A receives a pollution score of 0.82 (indicating high pollution), while Segment C scores 0.45 (moderate pollution), the model would prioritize Segment A for immediate remediation. A resource allocation model could then distribute intervention budgets proportionally based on the severity of pollution scores.*

Policy-Making: By offering a robust, data-driven approach to assessing water quality, the $\mathcal{IVNRSS}$ framework can inform policymakers on critical issues, such as:

- Regulating industrial discharges based on real-time water quality data.
- Implementing stricter pollution control measures in the most vulnerable regions (e.g., segments classified as **Poor**).

The framework aligns with global water quality standards, such as those outlined by the World Health Organization, helping governments ensure compliance and improve public health outcomes.

Example 4.2 *Using* $\mathcal{IVNRSS}$ *pollution-scores, policymakers could determine stricter regulations for industrial discharge. If real-time monitoring indicates that a specific industrial zone discharges effluents that increase a river segment's pollution score from 0.65 to 0.88, the regulatory body might impose stricter effluent limits or increase fines based on the severity index.*

Environmental Sustainability: By identifying specific areas and parameters contributing to poor water quality, this research supports initiatives to:

- Reduce pollution at its source, such as managing agricultural runoff or untreated wastewater.
- Encourage sustainable practices, like adopting eco-friendly farming methods or improving urban waste treatment infrastructure.

Example 4.3 *If agricultural runoff contributes* 40% *of the total pollution in a river segment, and reducing fertilizer use by* 20% *leads to a* 15% *drop in the pollution score, policymakers can quantify the effectiveness of sustainable agricultural practices and adjust environmental incentives accordingly.*

5. Comparative analysis

In this section, we compare the proposed model with an influential approach for water quality assessment and decision-making. Specifically, we examine the work of Patel and Chitnis [44], who modeled the dynamics of Sabarmati River water quality using fuzzy logic, considering the impacts of industrialization and climate change.

The comparative analysis includes a comparison of the fuzzy Water Quality Index (WQI) values derived from both the Patel-Chitnis model and the proposed model. In the Patel-Chitnis model, fuzzy data summaries of water quality parameters were processed to generate fuzzy WQI values for different water samples. The average fuzzy WQI values for the water samples at ws_1, ws_2, and ws_3 were found to be $\text{WQI}(ws_1) = 0.73195$, $\text{WQI}(ws_2) = 0.4778, \text{WQI}(ws_3) = 0.7722, \text{WQI}(ws_4) = 0.55145$. The decision output classifies each water sample as follows:

- Water sample ws_1 : Classified as Poor.
- Water sample ws_2 : Classified as Moderate.
- Water sample ws_3 : Classified as Poor.
- Water sample ws_2 : Classified as Moderate.

Both the Patel-Chitnis model and the proposed $\beta - model$ generate similar optimal choices and rankings (Table 3), but differences in the score values arise due to the distinct methods used. Our $\beta - model$ utilizes the pollution-score, whereas the Patel-Chitnis model employs fuzzy WQI values. One of the main innovations of our $\beta - model$ is the use of the pollution-score instead of the average score applied in the Patel-Chitnis model. The pollution score offers greater stability and feasibility in managing uncertainty. Additionally, the flexibility and adaptability of the pollution-score function $\mathcal{W}$ in our $\beta - model$ surpass those of the Patel-Chitnis model, making our $\beta - model$ more robust and applicable in various decision-making scenarios.

Table 3 Comparative water quality assessment for pollution rating

Models	Best optimal choice	Ranking according to good quality
Patel-model [44]	ws_2	$ws_2 \gg ws_4 \gg ws_1 \gg ws_3$
Proposed $\beta - model$	ws_2	$ws_2 \gg ws_4 \gg ws_1 \gg ws_3$

6. Conclusion and Future Directions

To address the issues of uncertainty, imprecision, and indeterminacy in decision-making processes, this study presented the $\mathcal{IVNRSS}$ framework, a novel approach. The $\mathcal{IVNRSS}$ framework offers a comprehensive and adaptable method for evaluating large datasets in various applications, including risk assessment, medical diagnostics, and environmental monitoring, by fusing the ideas of soft and rough set theories with IVNSs. The theoretical underpinnings of the suggested approach, including basic operations and innovative operators such as the aggregate union operator, were carefully investigated. Through a water quality evaluation, their usefulness was illustrated, demonstrating the framework's capacity to successfully identify water samples even in the face of imprecise, inconsistent, or ambiguous data. To overcome the main drawbacks of conventional decision-making models, the $\mathcal{IVNRSS}$ framework provides a dual capability to handle incompleteness and indeterminacy. It is therefore a noteworthy development in the field of MCDM. The study's real-world examples further support its applicability and highlight how it might improve the precision and dependability of decisions.

Limitations of the Study This study has several limitations that should be acknowledged. The water quality thresholds used for classification were based on expert judgment, which can be subjective and may vary across different regions or experts. Additionally, the study's data was limited by the temporal scope and availability of samples, which may not capture long-term or seasonal variations in water quality. The geographical focus on the study area restricts the generalization of the findings to other regions with different environmental conditions. Future research should address these limitations by expanding the dataset, refining thresholds, and including more parameters and regions for broader applicability.

Future Research Directions The suggested $\mathcal{IVNRSS}$ framework has proven to be successful in managing ambiguity, imprecision, and uncertainty in multi-criteria decision-making, especially in the evaluation of water quality. However, by incorporating sophisticated mathematical frameworks, there is a great chance to further improve its capabilities.

1. Extension to advanced set-theoretic frameworks: To improve decision-making in complex uncertainty, future research can investigate the use of Hyperrough Sets, Hypersoft Sets, Superhyperrough Sets, Superhypersoft Sets, Superhyperneutrosophic Sets, and Plithogenic Sets. Particularly in fields that demand extremely dynamic decision-support systems, these extensions may offer increased granularity in simulating interactions among several criteria.

2. Computational efficiency and optimization: It is possible to analyze and optimize the computational complexity of $\mathcal{IVNRSS}$ operations, such as intersection, union, complement, and aggregation operators. In large-scale datasets, sophisticated optimization algorithms like metaheuristic approaches (like genetic algorithms, particle swarm optimization, or deep reinforcement learning) may increase the effectiveness of real-time decision-making.

3. Use in various domains of decision-making: Future research could examine $\mathcal{IVNRSS}$ applications in other domains, including supply chain optimization, medical diagnosis, environmental monitoring, and financial risk assessment, even though this study concentrated on water quality assessment. These uses would provide a more thorough assessment of the model's resilience and flexibility.

4. Integration with AI and machine learning models: To improve automated decision-making in uncertain contexts, the $\mathcal{IVNRSS}$ framework could be used with deep learning models, fuzzy neural networks, and evolutionary computing. AI-driven methods could help with real-time decision assistance and increase the categorization accuracy of ambiguous datasets.

5. Hybrid decision-making models: To better manage multi-source uncertainty and contradictory information, future research should also investigate hybrid systems that combine Bayesian inference, Dempster-Shafer theory, and quantum-inspired decision models with $\mathcal{IVNRSS}$.

6. Real-world implementation and case studies: By using pilot studies, industry partnerships, and intelligent monitoring systems to apply the $\mathcal{IVNRSS}$ model in actual decision-making situations, its viability will be confirmed. This might involve employing IoT-based sensors in conjunction with $\mathcal{IVNRSS}$-based decision-making algorithms to evaluate water quality in real time.

Future research can enhance the theoretical underpinnings of the $\mathcal{IVNRSS}$ framework and broaden its applicability to intricate, high-uncertainty decision-making situations in many disciplines by tackling these directions.

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Declarations

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