Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.76020

Spectral analysis of the quantum Cesàro operator over the sequence space ℓ_p $(1 < p < \infty)$

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ABSTRACT: In this article, we study the spectrum, fine spectrum and boundedness property of $q-$Cesàro Matrices $C_1(q)$, which is a lower triangular matrix with $0 < q < 1$, over the class ℓ_p $(1 < p < \infty)$, the p^{th} summable sequence space. We determine the approximate point spectrum, defect spectrum, compression spectrum and Goldberg classification of the operator on the class of sequence. We construct the infinite system of linear equations of the given matrix operator to derive the results. Appropriate examples are provided along with graphical representations to support our study.

Key Words: Spectrum, Cesàro operator, infinite matrices, $q-$analog.

Contents

1. Introduction

The theory of the spectrum of bounded linear operators on Banach and Hilbert spaces plays a significant role in various branches of mathematics due to its numerous applications. The fundamental principle of the modern spectral theorem is that, certain linear operators on infinite dimensional space can be represented as a infinite diagonal matrix form, from which we can determine the different spectrum of that matrix. The Spectrum of an operator can be classified as point spectrum, continuous spectrum and residual spectrum. These three disjoint parts together are known as the "fine spectrum".

In 1965, Brown et al. [14] published a paper in which they considered an operator which converts a sequence $(x_n)_{n\geq 1}$ into its sequence of averages $\left(\dfrac{x_1 + x_2 + \ldots + x_n}{n}\right)_{n\geq 1}$. This operator is called as Cesàro operator and it is denoted by C_1 which matrix representation is

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1/2 & 1/2 & 0 & \ldots \\ 1/3 & 1/3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In their paper, they proved that C_1 is a bounded operator on ℓ_2 space and also determined its spectrum. Several researchers have analyzed the spectrum and fine spectrum of this Cesàro matrix and its generalized form on different sequence spaces. In 1985, Reade [31] and Gonzalez [22] found out the spectrum of Cesàro matrix on the spaces c_0 and ℓ_p $(1 < p < \infty)$ respectively. Further, Okutoyi [26,27]; Basar and Akhmedov [2] and Tripathy and Saikia [37] studied the spectrum and fine spectrum of the Cesàro matrix C_1 over the

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Submitted March 07, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 47A10, 47B37, 15A18

sequence spaces bv_0, bv; c; bv_p $(1 \leq p < \infty)$ and $\overline{bv_0} \cap \ell_\infty$. The spectrum of the discrete generalized Cesàro operator was calculated by Rhaly [32], Durna and Yildirim [43] over the spaces ℓ_2 and ℓ_p $(1 < p < \infty)$ respectively. In recent studies, a number of research articles have been found related to $q-$analogs of some known linear operators. Properties of some $q-$Hausdorff matrices were studied by Rhoades and Akgun [1]. Bekar [11] in his thesis discussed about the summability methods of the $q-$analogs of Cesàro, Holder, Euler and the Hausdorff operator. Spectral study and boundness of $q-$Cesàro matrices with $0 < q < 1$ were done by Yildirim [42], Durna and Yildirim [38] and for their studies, they considered the sequence spaces c_0 and c respectively. The $q-$form of the Cesàro matrix $C_1(q)$, for $0 < q < 1$ is defined by

$$C_{nk}(q) = \begin{cases} \dfrac{q^{n-k}}{1+q+\ldots+q^n}, & 0 < k \leq n \\ 0, & n < k. \end{cases}$$

Here if we take limit $q \longrightarrow 1^-$, then we get the ordinary Cesàro operator C_1.

$q-$**Analog**: A q-analog of a number, a theorem , an identity or expression is a generalization that involves a new parameter q and reduces to the original number, theorem, identity, or expression in the limit as $q \longrightarrow 1^-$. In the 19^{th} century, the basic hypergeometric series became the first $q-$analog to be extensively studied. In recent research in many areas of Mathematics, such as combinatorics, approximation theory, and difference and integral equations, $q-$calculus have been used extensively.

The $q-$analog $[m]_q$ of m for $q \in (0,1)$ can be defined by

$$[m]_q = \begin{cases} \sum_{k=0}^{m-1} q^k, & m = 1,2,3,\ldots \\ 0, & m = 0. \end{cases}$$

One might observe that $[m]_q = m$ whenever $q \longrightarrow 1^-$. The $q-$analog $\binom{m}{k}_q$ of binomial coefficient $\binom{m}{k}$ can be defined by

$$\binom{m}{q}_q = \begin{cases} \dfrac{[m]_q!}{[m-k]_q![k]_q!}, & m \geq k \\ 0, & k > m. \end{cases}$$

Where, the $q-$analog of the factorial, i.e., $q-$factorial, is defined by

$$[m]_q! = \begin{cases} \prod_{k=1}^{m} [k]_q, & m = 1,2,3,\ldots \\ 1, & m = 0. \end{cases}$$

The $q-$analog of some specific binomials such as $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ m \end{pmatrix} = 1$, also $\begin{pmatrix} m \\ m-k \end{pmatrix}_q = \begin{pmatrix} m \\ k \end{pmatrix}_q$. For an in-depth study of quantum calculus, we refer to the book [23].

The spectrum of various difference operators and their $q-$analogs on various sequence spaces has been researched immensely. The initial study of the spectrum of the difference operator Δ, $(\Delta y_k = y_k - y_{k-1})$ was done by Altay and Basar [8], Kayaduman and Furkan [24], and Akhmedov and Basar [3,4] over the sequence spaces c_o, c, ;ℓ_1, bv and ℓ_p, bv_p respectively. After this, the operator Δ was generalized to $B(r,s)$, $B(r,s)(y_k) = (ry_k + sy_{k-1})$. The spectrum of this operator was studied by Altay and Basar [9], Kayaduman et al. [18], Bilgic and Furkan [12], and Dutta and Tripathy [16] over the sequence spaces c_0, c; ℓ_1, bv; ℓ_p, bv_p $(1 \leq p < \infty)$ and cs respectively. Yaying et al. [41,7] studied the spectrum of second order $q-$difference operator over the sequence space c_0, ℓ_1 respectively. Spectrum of weighted $q-$difference operator was studied by Yaying et al. [40] over the sequence space c_0. $B(r,s)$ was further generalized to $B(r,s,t)$, $B(r,s,t)(y_k) = (ry_k + sy_{k-1} + ty_{k-2})$. The spectrum of this operator was studied by Bilgic and Furkan [13]; Furkan et al. [19,20] over the sequence spaces ℓ_1, bv; c_0, c and ℓ_p, bv_p respectively. Srivastava and Kumar [33,34]; Akhmedov and El-Shabrawy [5] studied the spectrum of the generalized difference operator Δ_v, where $\Delta_v(y_k) = (v_k y_k - v_{k-1} y_{k-1})$ over the sequence spaces c_0, ℓ_1 and c, ℓ_p respectively.

Akhmedov and El-Shabrawy [17,6]; Dutta and Baliarsingh [15] obtained the spectrum of the operator Δ_{ab}, where $\Delta_{ab}(y_k) = (a_k y_k + b_{k-1} y_{k-1})$ over the sequence spaces c_0, c and ℓ_p, bv_p respectively. Panigrahi and Srivastava [28,29] also found out the spectrum of Δ^2_{uv}, where $\Delta^2_{uv}(y_k) = (u_k y_k - v_{k-1} y_{k-1} + u_{k-2} y_{k-2})$ and Δ^2_{uvw}, where $\Delta^2_{uvw}(y_k) = (u_k y_k + v_{k-1} y_{k-1} + w_{k-2} y_{k-2})$ over the spaces c_0 and ℓ_1 respectively. Tripathy and Paul [36,30] studied on the spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces c_0, c and ℓ_p, bv_p respectively.

In this study, our aim is to discuss the spectrum of the quantum analog of the Cesàro matrix and its boundedness on the sequence space ℓ_p. Using Goldberg's classification of spectrum of an operator, we determine various spectral subdivisions of this operator and provide some examples with graphical representations.

2. Some Definitions and Preliminaries

Consider $M : U \longrightarrow V$ be a bounded linear operator, in which U and V are Banach spaces, the following collections

$$\begin{aligned} R(M) &= \{v \in V : v = Mu,\ u \in U\} \\ \text{and}\ \ B(U,V) &= \{M : U \longrightarrow V : M \text{ is continuous and linear}\} \end{aligned}$$

are termed as the range of the operator M and the set of all bounded linear operators from U to V respectively. The adjoint operator M^* of M is defined from V^* to U^*, where V^* and U^* represent the dual space of V and U respectively. Again, it is defined as $(M^* f)(u) = f(Mu)$, for all $f \in V^*$ and $u \in U$.

Let $M : D(M) \longrightarrow U$, where $D(M)$ denotes the domain of M. From M we can get an operator,

$$M_\mu = M - \mu I,$$

where $\mu \in \mathbb{C}$ and I is the identity operator. A regular value $\mu \in \mathbb{C}$ of M is such that M_μ is invertible, and its inverse (M_μ^{-1}) is bounded and defined on a set A and call it the resolvent operator of M, where A is dense in U. The collections of such μ is called the resolvent set and is denoted by $\rho(M, U)$. In the complex plane $\mathbb{C}$, the compliment of $\rho(M, U)$ is denoted by $\sigma(M, U)$, and is called the spectrum of M.

Further, $\sigma(M, U)$ is classified into three disjoint subsets, namely, the point spectrum $\sigma_p(M, U)$, the continuous spectrum $\sigma_c(M, U)$, and the residual spectrum $\sigma_r(M, U)$. In point spectrum, M_μ^{-1} does not exist for any $\mu \in \sigma_p(M, U)$, while in continuous spectrum, M_μ^{-1} exist but unbounded for every $\mu \in \sigma_c(M, U)$, and also defined on a set that is dense in U. On the other hand, in residual spectrum, M_μ^{-1} exist but may or may not be bounded for $\mu \in \sigma_r(M, U)$ and is not dense in U.

There are more subdivisions of the spectrum of a bounded operator such as approximate point spectrum $\sigma_{ap}(M, U)$, defect spectrum $\sigma_\delta(M, U)$ and compression spectrum $\sigma_{co}(M, U)$, which are defined as follows:

- $\sigma_{ap}(M, U) = \{\mu \in \mathbb{C} : (M - \mu I)\ \text{ is not bounded below}\}$.
- $\sigma_\delta(M, U) = \{\mu \in \mathbb{C} : (M - \mu I)\ \text{ is not surjective}\}$.
- $\sigma_{co}(M, U) = \{\mu \in \mathbb{C} : \overline{R(M - \mu I)} \neq U\}$.

3. Goldberg's Classification of Spectrum

A detailed classification of the spectrum of an operator was given by Goldberg [21]. This classification is based on the nature of the set $R(M_\mu)$ and the operator M_μ^{-1}.

If $M \in B(U, U)$, then there are three possibilities for $R(M_\mu)$:

$$\text{(P)}\ R(M_\mu) = U,\ \text{(Q)}\ \overline{R(M_\mu)} = U,\ \text{but}\ R(M_\mu) \neq U,\ \text{and (R)}\ \overline{R(M_\mu)} \neq U$$

and three possibilities for M_μ^{-1}:

$$\text{(1) Exist and continuous},\ \text{(2) Exist but discontinuous},\ \text{and (3) Does not exist}.$$

Combination of the possibilities $P,\ Q,\ R$ and $1,\ 2,\ 3$ lead to nine different states. They are identified as $P_1,\ P_2,\ P_3,\ Q_1,\ Q_2,\ Q_3,\ R_1,\ R_2,$ and R_3.

If $M_\mu \in P_1$ or $M_\mu \in Q_1$ then $\mu \in \rho(M,X)$. If $M_\mu \in R_2$ then M_μ^{-1} exist and unbounded and $\overline{R(M_\mu)} \neq X$ and we can write $\mu \in R_2\sigma(M,X)$. We can summarize this classification in the following Table-[1].

		1	2	3
		M_μ^{-1} exist and bounded	M_μ^{-1} exist and unbounded	M_μ^{-1} does not exist
P	$R(M_\mu) = U$	$\mu \in \rho(M,U)$	-	$\mu \in \sigma_p(M,U)$ $\mu \in \sigma_{ap}(M,U)$
Q	$\overline{R(M_\mu)} = U$	$\mu \in \rho(M,U)$	$\mu \in \sigma_c(M,U)$ $\mu \in \sigma_\delta(M,U)$ $\mu \in \sigma_{ap}(M,U)$	$\mu \in \sigma_p(M,U)$ $\mu \in \sigma_\delta(M,U)$ $\mu \in \sigma_{ap}(M,U)$
R	$\overline{R(M_\mu)} \neq U$	$\mu \in \sigma_r(M,U)$ $\mu \in \sigma_\delta(M,U)$ $\mu \in \sigma_{co}(M,U)$	$\mu \in \sigma_r(M,U)$ $\mu \in \sigma_\delta(M,U)$ $\mu \in \sigma_{co}(M,U)$ $\mu \in \sigma_{ap}(M,U)$	$\mu \in \sigma_p(M,U)$ $\mu \in \sigma_\delta(M,U)$ $\mu \in \sigma_{co}(M,U)$ $\mu \in \sigma_{ap}(M,U)$

Table 1: Goldberg's classification of spectrum of linear operator

Proposition 3.1 *([10], p.28) Spectral and sub-spectral relationships of an operator M and its adjoint operator M^* are provided below.*

- **(a)** $\sigma(M^*,U^*) = \sigma(M,U)$,
- **(b)** $\sigma_{ap}(M^*,U^*) = \sigma_\delta(M,U)$,
- **(c)** $\sigma_\delta(M^*,U^*) = \sigma_{ap}(M,U)$,
- **(d)** $\sigma_p(M^*,U^*) = \sigma_{co}(M,U)$,
- **(e)** $\sigma(M,U) = \sigma_{ap}(M,U) \bigcup \sigma_p(M^*,U^*) = \sigma_p(M,U) \bigcup \sigma_{ap}(M^*,U^*)$.

Lemma 3.1 *([21], p.60) The adjoint operator M^* of M is onto if and only if M has a bounded inverse.*

Lemma 3.2 *([21], p. 59) The bounded linear operator $M : U \longrightarrow U$ has dense range if and only if M^* is one to one.*

Throughout this work, the aforementioned spaces c_0, c, ℓ_1, ℓ_p, bv, bv_p, cs, and ℓ_∞ represent the class of all null, convergent, absolutely summable, $p-$absolutely summable, bounded variation, $p-$bounded variation, convergent series, and bounded sequences respectively.

4. Spectrum of $C_1(q)$ on ℓ_p $(1 < p < \infty)$

Theorem 4.1 *$C_1(q) \in B(\ell_p)$ with $||C_1(q)||_{B(\ell_p)} \leq \dfrac{1}{1-q}$, for $0 < q < 1$, where $1 < p < \infty$.*

Proof: To prove this, we first show that $C_1(q) \in B(\ell_1)$. Now,

$$\begin{aligned}
||C_1(q)||_{\ell_1} &= \sup_k \sum_{n=k}^{\infty} \left| \frac{q^{n-k}}{1+q+\ldots+q^n} \right| \\
&= \sup_k \frac{1}{q^k} \sum_{n=k}^{\infty} \left| \frac{q^n}{1+q+\ldots+q^n} \right| \\
&= \sup_k \frac{1}{q^k} \sum_{n=k}^{\infty} \left| \frac{q^n(1-q)}{1-q^{n+1}} \right| \\
&= \sup_k \frac{(1-q)}{q^k} \sum_{n=k}^{\infty} q^n \left(\frac{1}{1-q^{n+1}} \right).
\end{aligned}$$

Here, the sequence $\frac{1}{1-q^{n+1}}$ is a decreasing and convergent sequence. So, we can write

$$\begin{aligned}
||C_1(q)||_{\ell_1} &\leq \sup_k \frac{(1-q)}{q^k} \frac{1}{(1-q)} \sum_{n=k}^{\infty} q^n \\
&= \sup_k \frac{1}{q^k} \sum_{n=k}^{\infty} q^n \\
&= \sup_k \sum_{j=0}^{\infty} q^j \\
&= \frac{1}{1-q}.
\end{aligned}$$

This implies $C_1(q) \in B(\ell_1)$. Again since each entry of $C_1(q)$ is positive and each of the row sum is 1 therefore, $||C_1(q)||_{\ell_\infty} = 1$ and $C_1(q) \in B(\ell_\infty)$. As a result, we get $C_1(q) \in B(\ell_p)$ with the norm less than or equal to $\frac{1}{1-q}$ (see lemma-[4.4]). □

Theorem 4.2 *The point spectrum* $\sigma_p(C_1(q), \ell_p) = \phi$*, the empty set.*

Proof: We prove this theorem by the method of contradiction. Consider, $\sigma_p(C_1(q), \ell_p) \neq \phi$.

Then for any $0 \neq u \in \ell_p$ with $C_1(q)u = \lambda u$, we get the following equalities;

$$\begin{aligned}
u_0 &= \mu u_0 \\
\frac{1}{1+q}(qu_0 + u_1) &= \mu u_1 \\
\frac{1}{1+q+q^2}(q^2u_0 + qu_1 + u_2) &= \mu u_2 \\
&\vdots \\
\frac{1}{1+q+q^2+\ldots+q^n}(q^nu_0 + q^{n-1}u_1 + \ldots + qu_{n-1} + u_n) &= \mu u_n \\
&\vdots
\end{aligned}$$

Considering u_m to be the first non zero element of $u = (u_m)$, $m \geq 1$, from the above equations we get

$$\mu = \frac{1}{1+q+q^2+\ldots+q^m}.$$

Furthermore,

$$\frac{1}{1+q+q^2+\ldots+q^m+q^{m+1}}(qu_m+u_{m+1})=\mu u_{m+1} \tag{4.1}$$

Putting the values of μ in equation(4.1) we get,

$$u_{m+1}=\frac{1+q+q^2+\ldots+q^m}{q^m}u_m.$$

Again

$$\frac{1}{1+q+q^2+\ldots+q^{m+1}+q^{m+2}}(q^2u_m+qu_{m+1+u_{m+2}})=\mu u_{m+2}. \tag{4.2}$$

Putting the values of u_{m+1} and μ in equation (4.2) we get,

$$u_{m+2}=\frac{(1+q+\ldots+q^m)(1+q+\ldots+q^{m+1})}{q^{2m}(1+q)}u_m.$$

By the same process we get,

$$\begin{aligned}
u_{m+3}&=\frac{(1+q+\ldots+q^m)(1+q+\ldots+q^{m+1})(1+q+\ldots+q^{m+2})}{q^{3m}(1+q)(1+q+q^2)}u_m,\\
u_{m+4}&=\frac{(1+q+\ldots+q^m)(1+q+\ldots+q^{m+1})(1+q+\ldots+q^{m+2})(1+q+\ldots+q^{m+3})}{q^{4m}(1+q)(1+q+q^2)(1+q+q^2+q^3)}u_m,\\
&\vdots\\
u_n&=\frac{(1+q+\ldots+q^m)(1+q+\ldots+q^{m+1})\ldots(1+q+\ldots+q^{n-1})}{q^{(n-m)m}(1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-m-1})}, \quad \text{and so on.}
\end{aligned}$$

Rewriting u_n we get,

$$\begin{aligned}
u_n&=\frac{\frac{1-q^{m+1}}{1-q}\frac{1-q^{m+2}}{1-q}\cdots\frac{1-q^n}{1-q}}{q^{(n-m)m}\frac{1-q^2}{1-q}\frac{1-q^3}{1-q}\cdots\frac{1-q^{n-m}}{1-q}}\\
&=\frac{1-q^{m+1}}{1-q}\frac{1-q^{m+2}}{1-q^2}\cdots\frac{1-q^n}{1-q^{(n-m)}}\frac{1}{q^{(n-m)m}}.
\end{aligned} \tag{4.3}$$

We have $0<q<1$, for $n>m$ we get, $q^m>q^n \implies 1-q^m<1-q^n \implies \frac{1-q^n}{1-q^m}>1$. Applying this condition in equation (4.3), we get, $u_n\geq\frac{1}{q^{(n-m)m}}$

$$\implies \sum|u_n|\geq\sum|\frac{1}{q^{(n-m)m}}|$$

$$\implies \sum|u_n|^p\geq\sum|\frac{1}{q^{(n-m)m}}|^p.$$

Now,

$$\frac{1}{q^{(n-m)mp}}\longrightarrow\infty,\quad \text{as}\quad n\longrightarrow\infty \implies \sum\left|\frac{1}{q^{(n-m)m}}\right|^p$$

is divergent. So, $\sum|u_n|^p$ is divergent. From this, we conclude that $(u_n)\notin\ell_p$, which is a contradiction and we get the required result. □

Lemma 4.1 *[[35], p.215] For any $A\in B(\ell_p)$ $(1<p<\infty)$, the adjoint operator $A^*\in B(\ell_q)$, where $\frac{1}{p}+\frac{1}{q}=1$ and can be represented by the transpose of A matrix.*

Theorem 4.3 $\sigma_p(C_1^*(q),\ell_p^*\cong\ell_q)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}\cup\{1\}$, *for* $0<q<1$.

Proof: Let, $u\neq 0$ and $C_1^*(q)u=\mu u$. From this the following equalities are obtained;

$$\begin{aligned}
&u_0+\frac{q}{1+q}u_1+\frac{q^2}{1+q+q^2}u_2+\frac{q^3}{1+q+q^2+q^3}u_3+\ldots=\mu u_0\\
&\frac{1}{1+q}u_1+\frac{q}{1+q+q^2}u_2+\frac{q^2}{1+q+q^2+q^3}u_3+\ldots=\mu u_1\\
&\frac{1}{1+q+q^2}u_2+\frac{q}{1+q+q^2+q^3}u_3+\ldots=\mu u_2\\
&\frac{1}{1+q+q^2+q^3}u_3+\ldots=\mu u_3\\
&\vdots
\end{aligned}$$

Solving the above equations we get,

$$\begin{aligned}
u_n&=\frac{1}{q^n}\left(1-\frac{1}{\mu}\right)\left(1-\frac{1}{(1+q)\mu}\right)\left(1-\frac{1}{(1+q+q^2)\mu}\right)\cdots\left(1-\frac{1}{(1+q^2+\ldots+q^{n-1})\mu}\right)u_0\\
&=\frac{1}{q^n}\left(1-\frac{1}{\mu}\right)\left(1-\frac{(1-q)}{(1-q^2)\mu}\right)\left(1-\frac{(1-q)}{(1-q^3)\mu}\right)\cdots\left(1-\frac{(1-q)}{(1-q^n)\mu}\right)u_0.
\end{aligned}$$

The above expression can also be written as

$$u_n=\frac{u_0}{q^n}\prod_{k=1}^{n}\left(1-\frac{1-q}{1-q^k}\frac{1}{\mu}\right).$$

Here u_0 can not be zero otherwise all terms will become zero.

Now, let us consider Z to be set $\left\{1,\frac{1}{1+q},\frac{1}{1+q+q^2},\ldots\right\}$ and assume that $\mu\in Z$.

If $\mu=1$, then $C_1^*(q)u=u$, for $u=(u_0,0,0,...)\neq\theta$. This implies that $1\in\sigma_p(C_1^*(q),\ell_q)$.

If $\mu=\frac{1}{1+q}$, then $C_1^*(q)u=\frac{1}{1+q}u$, for $u=(u_0,-u_0,0,0,...)\neq\theta$. This implies that $\frac{1}{1+q}\in\sigma_p(C_1^*(q),\ell_q)$.

Again, if $\mu=\frac{1}{1+q+q^2}$, then $C_1^*(q)u=\frac{1}{1+q+q^2}u$, for $u=(u_0,-(1+q)u_0,qu_0,0,0,....)\neq\theta$, this implies that $\frac{1}{1+q+q^2}\in\sigma_p(C_1^*(q),\ell_q)$. Proceeding in a similar way it is seen that $Z\subset\sigma_p(C_1^*(q),\ell_q)$. Now, we have to check whether there any values of μ exist other than $\mu\in Z$.

Consider, $\mu\notin Z$. For this case we apply the ratio test for the sequence (u_n).

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\frac{1}{q}\left|1-\frac{1-q}{1-q^{n+1}}\frac{1}{\mu}\right|=\frac{1}{q}\left|1-\frac{1-q}{\mu}\right|.$$

Therefore, $(u_n)\in\ell_p^*\cong\ell_q$ if and only if $\frac{1}{q}\left|1-\frac{1-q}{\mu}\right|<1$, and by taking, $\mu=a+ib$ in the inequality yields,

$$\begin{aligned}
\frac{1}{q}\left|1-\frac{1-q}{a+ib}\right|<1 &\iff \left|1-\frac{1-q}{a+ib}\right|<q\\
&\iff \left|1-\frac{(a-ib)(1-q)}{a^2+b^2}\right|<q\\
&\iff \left|1-\frac{1-q}{a^2+b^2}a+\frac{1-q}{a^2+b^2}bi\right|<q\\
&\iff 1-2\frac{1-q}{a^2+b^2}a+\frac{(1-q)^2}{a^2+b^2}<q^2\\
&\iff 1-q^2<2\frac{(1-q)}{a^2+b^2}a-\frac{(1-q)^2}{a^2+b^2}\\
&\iff (1+q)<\frac{2a}{a^2+b^2}-\frac{1-q}{a^2+b^2}\\
&\iff a^2+b^2<\frac{2a}{1+q}-\frac{1-q}{1+q}\\
&\iff \left(a-\frac{1}{1+q}\right)^2+b^2<\frac{1}{(1+q)^2}-\frac{1-q}{1+q}=\frac{q^2}{(1+q)^2}\\
&\iff \left|\mu-\frac{1}{1+q}\right|<\frac{q}{1+q}
\end{aligned}$$

So, finally we get $(u_n)\in\ell_q$ iff $\left|\mu-\frac{1}{1+q}\right|<\frac{q}{1+q}$, then

$$\sigma_p(C_1^*(q),l_q)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}\cup\{1\}.$$

□

Lemma 4.2 *[[39], p.126]: A matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 4.3 *[[39], p.126]: A matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

Lemma 4.4 *[[25], p.174, Theorem 9]: Let $1<p<\infty$ and suppose $A\in(\ell_1,\ell_1)\cap(\ell_\infty,l_\infty)$. Then $A\in(\ell_p,\ell_p)$.*

Theorem 4.4 $\sigma(C_1(q),\ell_p)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|\le\frac{q}{1+q}\right\}$.

Proof: Consider S be the set $\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|\le\frac{q}{1+q}\right\}$. Here we will show that the compliment of the set S will represent the resolvent set, i.e. $\rho(C_1(q),\ell_p)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|>\frac{q}{1+q}\right\}$.

In Theorem 4.2, we have proved that $\sigma_p(C_1(q),\ell_p)=\phi$. It directly implies that $\mu I-C_1(q)$ is one-to-one. Now our next step will be to show that $(\mu I-C_1(q))^{-1}\in B(\ell_p)$, for $\mu\in\mathbb{C}$, for $\left|\mu-\frac{q}{1+q}\right|>\frac{q}{1+q}$.

Solving the system of equation $(\mu I - C_1(q))u = v$, we get

$$
\begin{aligned}
u_0 &= \frac{1}{\mu-1}v_0\\
u_1 &= \frac{1}{\mu-1}\frac{1}{\mu-\frac{1}{1+q}}\frac{q}{1+q}v_0 + \frac{1}{\mu-\frac{1}{1+q}}v_1\\
u_2 &= \frac{1}{\mu-1}\frac{1}{\mu-\frac{1}{1+q}}\frac{1}{\mu-\frac{1}{1+q+q^2}}\frac{\mu q^2}{1+q+q^2}v_0 + \frac{1}{\mu-\frac{1}{1+q}}\frac{1}{\mu-\frac{1}{1+q+q^2}}\frac{q}{1+q+q^2}v_1 + \frac{1}{\mu-\frac{1}{1+q+q^2}}v_2\\
&\vdots\\
u_n &= \frac{1}{1+q+\ldots+q^n}\sum_{k=0}^{k=n-1}\frac{v_k}{q}\frac{1}{\mu^2}\prod_{i=k}^{n}\frac{\mu q}{\mu-\frac{1}{1+q+\ldots+q^i}} + \frac{1}{\mu-\frac{1}{1+q+\ldots+q^n}}v_n
\end{aligned}
$$

Therefore, we have $(\mu I - C_1(q))^{-1} = (a_{nk})$ defined by

$$
a_{nk} = \begin{cases} \frac{1}{1+q+\ldots+q^n}\frac{1}{q}\left(\prod_{i=k}^{n}\frac{q}{\mu-\frac{1}{1+q+\ldots+q^i}}\right)\mu^{n-k-1} & , \quad 0\le k\le n-1\\ \frac{1}{\mu-\frac{1}{1+q+\ldots+q^n}} & , \quad k=n\\ 0 & , \quad k>n. \end{cases}
$$

Now we analyze the matrix (a_{nk}). Taking k fixed we consider,

$$
a_n = \frac{1}{1+q+\ldots+q^n}\frac{1}{q}\left(\prod_{i=k}^{n}\frac{q}{\mu-\frac{1}{1+q+\ldots+q^i}}\right)\mu^{n-k-1}.
$$

Here, we apply the ratio test to check whether the sequence $(a_n)\in\ell_1$ or not.

$$
\frac{a_{n+1}}{a_n} = \frac{1+q+\ldots+q^n}{1+q+\ldots+q^{n+1}}\frac{\left(\prod_{i=k}^{n+1}\frac{q}{\mu-\frac{1}{1+q+\ldots+q^i}}\right)\mu^{n-k}}{\left(\prod_{i=k}^{n}\frac{q}{\mu-\frac{1}{1+q+\ldots+q^i}}\right)\mu^{n-k-1}} = \frac{1-q^{n+1}}{1-q^{n+2}}\frac{q}{\mu-\frac{1-q}{1-q^{n+2}}}\mu
$$

$$
\implies \lim_{n\longrightarrow\infty}\frac{a_{n+1}}{a_n} = \lim_{n\longrightarrow\infty}\left(\frac{1-q^{n+1}}{1-q^{n+2}}\frac{q}{\mu-\frac{1-q}{1-q^{n+2}}}\mu\right) = \frac{q}{\mu-(1-q)}\mu.
$$

From here, we can say that $(a_n)\in\ell_1$ if $\left|\frac{q}{\mu-(1-q)}\mu\right|<1$. This implies $(\mu I - C_1(q))^{-1}\in(\ell_1,\ell_1)$ if $\left|\frac{q}{\mu-(1-q)}\mu\right|<1$. Simplifying this inequality, we get $\left|1-\frac{1-q}{\mu}\right|>q$. From the previous theorem it can be verified that this inequality satisfies the condition $\left|\mu-\frac{1}{1+q}\right|>\frac{q}{1+q}$. Also, since $(a_n)\in\ell_1$, the supremum of the ℓ_1 norms of the rows of $(\mu I - C_1(q))^{-1}$ is finite, it implies $(\mu I - C_1(q))^{-1}\in(\ell_\infty,\ell_\infty)$. As a result, we obtain $(\mu I - C_1(q))^{-1}\in(\ell_p,\ell_p)$ or $B(\ell_p)$, if $\left|\mu-\frac{1}{1+q}\right|>\frac{q}{1+q}$ (see Lemma-[4.4]). Consequently, we get $\sigma(C_1(q),\ell_p)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|\le\frac{q}{1+q}\right\}$. □

Theorem 4.5 *The residual spectrum,* $\sigma_r(C_1(q),\ell_p)=\left\{\mu\in\mathbb{C}:\left|\mu-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}\cup\{1\}$.

Proof: Using Lemma-[3.2], it can be proved that $\sigma_r(C_1(q),\ell_p)=\sigma_p(C_1^*(q),\ell_q)\backslash\sigma_p(C_1(q),\ell_p)$. Now the result follows from Theorems 4.2 and 4.3. □

Theorem 4.6 *The continuous spectrum,* $\sigma_c(C_1(q),\ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \frac{1}{1+q}\right| < \frac{q}{1+q}\right\} \setminus \{1\}$.

Proof: We have $\sigma(C_1(q),\ell_p) = \sigma_p(C_1(q),\ell_p) \cup \sigma_r(C_1(q),\ell_p) \cup \sigma_c(C_1(q),\ell_p)$ and the corresponding sets are pairwise disjoint. Now the result follows from Theorems 4.2, 4.4, and 4.5. □

Theorem 4.7 $P_3\sigma(C_1(q),\ell_p) = Q_3\sigma(C_1(q),\ell_p) = R_3\sigma(C_1(q),\ell_p) = \phi$.

Proof: From Table-[1] we get,
$\sigma_p(C_1(q),\ell_p) = P_3\sigma(C_1(q),\ell_p) \cup Q_3\sigma(C_1(q),\ell_p) \cup R_3\sigma(C_1(q),\ell_p)$. Now, the result follows from the Theorem 4.2. □

Theorem 4.8 $R_1\sigma(C_1(q),\ell_p) = \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \dots\right\}$.

Proof: Already we have considered that Z represents the set $\left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \dots\right\}$. From Theorem 4.5, we get $Z \subseteq \sigma_r(C_1(q),\ell_p)$. In addition, from Table-[1], it is found that Z is a subset of R. Now we have to show whether Z belongs to (1) or (2).

Let $v = (v_n) \in \ell_q$ be such that $(C_1(q) - \mu I)^* u = v$ for some $u = (u_n)$. Solving this we get the following equations:

$$\begin{aligned}
(1-\mu)u_0 + \frac{q}{1+q}u_1 + \frac{q^2}{1+q+q^2}u_2 + \frac{q^3}{1+q+q^2+q^3}u_3 + \dots &= v_0 \\
(\frac{1}{1+q} - \mu)u_1 + \frac{q}{1+q+q^2}u_2 + \frac{q^2}{1+q+q^2+q^3}u_3 + \dots &= v_1 \\
(\frac{1}{1+q+q^2} - \mu)u_2 + \frac{q}{1+q+q^2+q^3}u_3 + \dots &= v_2 \\
(\frac{1}{1+q+q^2+q^3} - \mu)u_3 + \dots &= v_3 \\
&\vdots
\end{aligned}$$

from the above equations, we can write u_n in terms of u_0 and v_n in the following way.

$$\begin{aligned}
u_1 &= \frac{1}{q\mu}\left[(\mu-1)u_0 + v_0 - qv_1\right] \\
u_2 &= \frac{1}{q\mu}\left[\frac{1}{q\mu}\left(\mu - 1)(\mu - \frac{1}{1+q}\right)u_0 + \frac{1}{q\mu}\left(\mu - \frac{1}{1+q}\right)v_0 + \frac{1}{\mu(1+q)}v_1 - qv_2\right] \\
u_3 &= \frac{1}{q\mu}\left[\frac{1}{(q\mu)^2}(\mu-1)\left(\mu - \frac{1}{1+q}\right)\left(\mu - \frac{1}{1+q+q^2}\right)u_0 + \frac{1}{(q\mu)^2}\left(\mu - \frac{1}{1+q}\right)\right. \\
&\qquad \left(\mu - \frac{1}{1+q+q^2}\right)v_0 + \frac{1}{q\mu}\left(\mu - \frac{1}{1+q+q^2}\right)\frac{1}{\mu(1+q)}v_1 + \\
&\qquad\qquad \left.\left(1 - \frac{1}{\mu}\left(\mu - \frac{1}{1+q+q^2}\right)\right)v_2 - qv_3\right] \\
&\vdots \\
u_{n+1} &= \frac{u_0}{(q\mu)^n}\prod_{j=0}^{n}\left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right) + \frac{v_0}{(q\mu)^n}\prod_{j=1}^{n}\left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right) \\
&\qquad + \sum_{i=1}^{n}\frac{1}{\mu\sum_{k=0}^{i} q^k}\frac{v_i}{(q\mu)^{n-1}}\prod_{j=i+1}^{n}\left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right) - \frac{1}{\mu}v_{n+1}.
\end{aligned}$$

Now, we have to analyze the conditions under which the sequence $(u_n) \in \ell_q$. If $\mu \in Z$, then we get $u_{n+1} = -\frac{1}{\mu} v_{n+1}$. Since $v_{n+1} \in \ell_q$ this implies $u_{n+1} \in \ell_q$. From this we directly conclude that $(C_1(q) - \mu I)^*$ is onto. Now using the Lemma 3.1, we get for $\mu \in Z$, the inverse of $(C_1(q) - \mu I)$ exist and it is bounded. So, $Z \subseteq (1)\sigma(C_1(q), \ell_p)$ see Table-[1].

If $\mu \in Z^{\complement}$, then the belonging of u_n in ℓ_q can be said from the convergence of the infinite product $\prod_{j=0}^{n} \left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right)$. Now, $\lim_{j \longrightarrow \infty} \left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right) = \mu - \frac{1}{\frac{1}{1-q}} = \mu - 1 + q$. Here the infinite product $\prod_{j=0}^{n} \left(\mu - \frac{1}{\sum_{k=0}^{j} q^k}\right)$ is divergent, if $\mu \neq 2 - q$. Therefore, for $\mu \in Z^{\complement}$ and $\mu \neq 2 - q$ the sequence $(u_n) \notin \ell_q$. From this, we directly conclude that $(C_1(q) - \mu I)^*$ is not onto. So, using the Lemma 3.1 we get for $\mu \notin Z \cup \{2 - q\}$, $(C_1(q) - \mu I)$ doesn't have bounded inverse. For the point $2 - q$, it is also obtained that $u_n \notin \ell_q$ and consequently bounded inverse of $(C_1(q) - \mu I)$ doesn't exist. As a result, it can be concluded that $R_1\sigma(C_1(q), \ell_p) = \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$. $\square$

Corollary 4.1 $R_2\sigma(C_1(q), \ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \frac{1}{1+q}\right| < \frac{q}{1+q}\right\} \setminus \left\{\frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$.

Proof: We have, the residual spectrum $\sigma_r(C_1(q), \ell_p) = R_1\sigma(C_1(q), \ell_p) \cup R_2\sigma(C_1(q), \ell_p)$. Now applying Theorems 4.5 and 4.8 we get the required result. $\square$

Theorem 4.9 *The operator $C_1(q)$ satisfies the following statements.*

(a) $\sigma_{ap}(C_1(q), \ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \dfrac{1}{1+q}\right| \leq \dfrac{q}{1+q}\right\} \setminus \left\{1, \dfrac{1}{1+q}, \dfrac{1}{1+q+q^2}, \ldots\right\}$,

(b) $\sigma_{ap}((C_1(q))^*, \ell_p^*) = \left\{\mu \in \mathbb{C} : \left|\mu - \dfrac{1}{1+q}\right| \leq \dfrac{q}{1+q}\right\}$,

(c) $\sigma_\delta(C_1(q), \ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \dfrac{1}{1+q}\right| \leq \dfrac{q}{1+q}\right\}$,

(d) $\sigma_{co}(C_1(q), \ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \dfrac{1}{1+q}\right| < \dfrac{q}{1+q}\right\} \cup \{1\}$,

(e) $\sigma_\delta((C_1(q))^*, \ell_p^*) = \left\{\mu \in \mathbb{C} : \left|\mu - \dfrac{1}{1+q}\right| \leq \dfrac{q}{1+q}\right\} \setminus \left\{1, \dfrac{1}{1+q}, \dfrac{1}{1+q+q^2}, \ldots\right\}$.

Proof: (a) From table-[1], we get

$$\sigma_{ap}(C_1(q), \ell_p) = \sigma(C_1(q), \ell_p) \setminus C_1\sigma(C_1(q), \ell_p).$$

Now applying the Theorems 4.4 and 4.8, we get the required result.

(b) The result in (b) is obtained from the relation (e) in proposition 3.1.

(c) The result in (c) is obtained from the relation (b) in proposition 3.1.

(d) The result in (d) is obtained from the relation (d) in proposition 3.1.

(e) The result in (e) is obtained from the relation (c) in proposition 3.1. $\square$

5. Example

Taking particular values for $q \in (0, 1)$, we get some examples of spectrum of $C_1(q)$ and that are given below.

(i) If $q = \frac{1}{2}$, then the spectrum of $C_1(\frac{1}{2})$ is given by

$$\sigma(C_1(q), \ell_p) = \left\{\mu \in \mathbb{C} : \left|\mu - \frac{2}{3}\right| \leq \frac{1}{3}\right\}.$$

(ii) If $q = \frac{1}{3}$, then the spectrum of $C_1(\frac{1}{3})$ is given by

$$\sigma(C_1(q), \ell_p) = \left\{ \mu \in \mathbb{C} : \left| \mu - \frac{3}{4} \right| \leq \frac{1}{4} \right\}.$$

The graphical representation of the spectra for these examples are provided in Figure-[1] and Figure-[2]

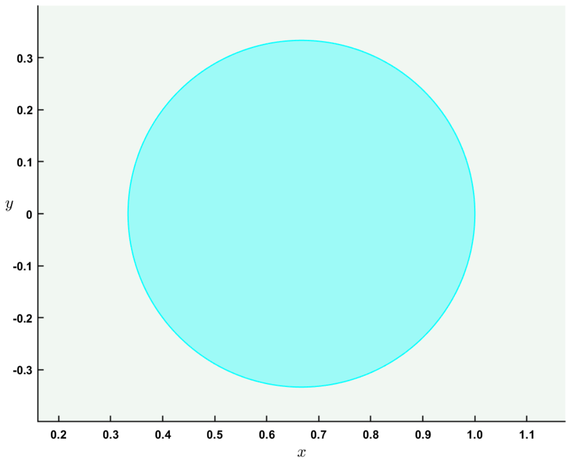

Figure 1: Spectrum of $C_1(\frac{1}{2})$.

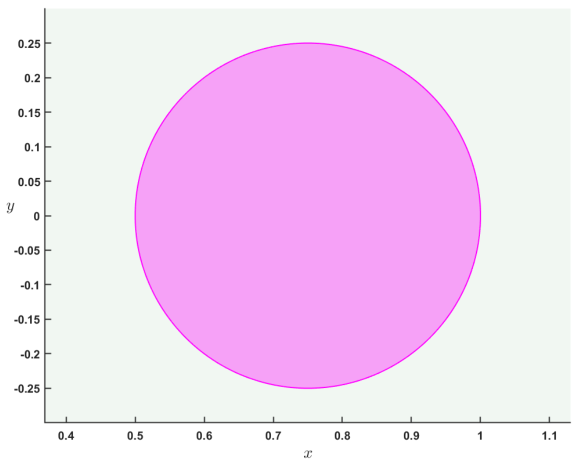

Figure 2: Spectrum of $C_1(\frac{1}{3})$.

6. Conclusion

In our study, we have discussed some important aspects of quantum calculus and analyzed the spectrum of the Quantum Cesàro matrix. Using Goldberg's classification we have also determined various spectral decompositions of this matrix. There are many spaces left unstudied where we can explore the spectrum of this $q-$Cesàro matrix.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable suggestions.

Conflict of interest: The authors declare that there is no conflict of interest.

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