Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.76041

Spectral Radius of Extended Adjacency Tensor of Uniform Hypergraphs

Shashwath S. Shetty and K. Arathi Bhat*

ABSTRACT: Hypergraphs are a useful tool in chemistry for representing intricate molecular structures and their interactions. The topological indices of hypergraphs provide valuable insights into the structural characteristics of complex molecular systems. The tensor representations are more preferred than the matrix representations of hypergraphs since they maintain complete information about the hypergraphs. In this article, the notion of adjacency tensor has been generalized to degree-based extended adjacency tensors, and hence the bounds for the spectral radius of uniform hypergraphs. Also, the spectral properties of the extended adjacency tensor of a sunflower hypergraph have been put forward.

Key Words: H-eigenvalues, Spectral Radius, Degree based topological indices, Sunflower.

Contents

1 Introduction **1**

2 Bounds for the Spectral Radius of $\mathcal{F}(\mathcal{H})$ **3**

3 Spectral Radius of Linear Hyperpath **5**

4 Eigenvalues of the Sunflower **10**

1. Introduction

Tensor analysis of hypergraphs offers effective tools for analysis as well as a more methodical and rigorous way to understand their properties and patterns of behavior. The connectivity of hypergraphs and other properties can be studied using spectral analysis of tensors, where the eigenvalues and eigenvectors of tensor representations provide insights into the structural characteristics of the hypergraph [6]. Tensor decomposition techniques such as tensor rank decomposition or tensor singular value decomposition can be applied to hypergraphs for tasks such as clustering, dimensionality reduction, and pattern extraction [4,20,31,36]. Modern quantum chemistry makes use of entanglement optimisation and the tensor product [30].

The study based on the representation of chemical reactions as hypergraphs, in which the participating molecules are represented by nodes, and the reactions between nodes are represented by hyperedges has been carried out in [19]. A new way of depicting saturated hydrocarbon structures as hypergraphs of a particular type is proposed in [27]. A comparative analysis is conducted between the proposed hypergraph model and the standard graph model, taking into account specific quantitative criteria related to specific computer chemistry tasks. Results show that the graphical model is surpassed by the hypergraph model.

A hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}, \mathcal{E})$, where the elements of $\mathcal{V}$ are called the vertices and there exist a bijection between the elements of $\mathcal{E}$ and a collection of subsets of $\mathcal{V}$. A hypergraph is said to be simple, if no hyperedge is completely contained in any other hyperedge. We denote the elements of the vertex set by $1, 2, \ldots, n$, also by e_i a hyperedge that contain the vertex i. For the sake of simplicity, $[n] := \{1, 2, \ldots, n\}$. Two vertices $i, j \in \mathcal{V}$ in a hypergraph $(\mathcal{V}, \mathcal{E})$ are said to be adjacent (resp. non-adjacent) if there exist a (resp. doesn't exist any) hyperedge $e \in \mathcal{E}$ such that $i, j \in e$. The degree of a vertex $i \in \mathcal{V}$ denoted by d_i in a hypergraph $(\mathcal{V}, \mathcal{E})$ is the cardinality of the set, $\{e \in \mathcal{E} : i \in e\}$. A vertex with degree one are called pendant vertices. A hypergraph is said to be d-regular if the degree of every vertex is equal to d. A k-uniform hypergraph is a hypergraph in which every hyperedge contains exactly k vertices.

* Corresponding author

Submitted March 09, 2025. Published August 24, 2025
2010 *Mathematics Subject Classification:* 05C65, 05C07.

If a vertex i is contained in an edge e, we simply say that they are incident with each other. A walk in a hypergraph is an alternating sequence of vertices and hyperedges that starts with a vertex and ends with another vertex such that sequentially adjacent terms (vertices and edges) are incident with each other in the hypergraph. A path between two vertices i and j in a hypergraph is a walk with all vertices and edges distinct. A hypergraph is said to be connected, if every two vertices in a hypergraph are connected by a path. A simple connected hypergraph is a hypertree in which removal of any hyperedge results in a disconnected hypergraph. A hyperpath is a hypertree in which a hyperedge can be adjacent to at most two other hyperedges. A hypergraph is called a linear hypergraph, if the intersection of any two pairs of hyperedges contain at most one vertex.

A sunflower hypergraph $\mathcal{S}(m,c,k)$ on n vertices is a k-uniform hypergraph with m hyperedges such that $0 < c < n-m$ vertices are of degree m and all the remaining vertices are pendant. The sunflower $\mathcal{S}(m,1,k)$ is called a k-uniform hyperstar. A hypergraph $\mathcal{H}$ is said to be a cored hypergraph, if for every hyperedge e of $\mathcal{H}$, there exist a vertex $j_e \in e$ of degree one. The vertices with degree greater than one are called the intersectional vertices.

Let $H=(V,E)$ be a simple graph. Define a k-uniform hypergraph $H^k := (\mathcal{V},\mathcal{E})$, where

$$\mathcal{V} = V \cup \{j_{e'_1}, j_{e'_2}, \dots, j_{e'_{k-2}} \mid e' \in E\}$$

and

$$\mathcal{E} = \{e' \cup \{j_{e'_1}, j_{e'_2}, \dots, j_{e'_{k-2}}\} \mid e' \in E\}.$$

For a hypergraph $\mathcal{H}$, if there exists a graph H and a $k \geq 3$ such that $\mathcal{H} \cong H^k$, then $\mathcal{H}$ is called a power hypergraph.

An adjacency tensor $\mathcal{A}$ of a k-uniform hypergraph on n vertices is an order k, dimension n tensor whose $(j_1, j_2, \dots, j_k)$-th entry is given by

$$a_{j_1 j_2 \dots j_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{j_1, j_2, \dots, j_k\} \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

Two non-adjacent vertices i and j in a hypergraph are said to be duplicate, if $a_{ij_2\dots j_k} = a_{jj_2\dots j_k}$ for every $j_2, j_3, \dots, j_k \in [n]$.

Let $F : \mathbb{R}^n_{++} \longrightarrow R_+$ be a real, positive valued symmetric function defined on strictly positive orthant. By symmetric function on k variables, we mean that $F(x_1, x_2, \dots, x_k) = F(\sigma(x_1), \sigma(x_2), \dots, \sigma(x_k))$ for any permutation $\sigma \in S_k$, where S_k is the symmetric group on a set of size k. Also F is assumed to be bounded if all the components of x are finite.

The concept of adjacency matrix of simple graphs has been generalized to degree based extended adjacency matrix in [7] and for graphs with self-loops in [24]. For uniform hypergraphs, the generalization of adjacency tensor to ABC tensor has been done in [17]. Similarly, the notion of Sombor index of graphs has been generalized to general hypergraphs in [25]. Here we generalize the notion of adjacency tensor of hypergraphs to degree based extended adjacency tensor of hypergraphs with respect to the function F defined above.

A degree based extended adjacency tensor $\mathcal{F}$ of a k-uniform of hypergraph on n vertices is an order k, dimension n tensor whose $(j_1, j_2, \dots, j_k)$-th entry is given by

$$f_{j_1 j_2 \dots j_k} = \begin{cases} \frac{F(d_{i_1}, \dots, d_{i_k})}{(k-1)!}, & \text{if } \{j_1, j_2, \dots, j_k\} \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}$ be an order $k \geq 2$, dimension n tensor and $\mathcal{S}$ be a order $r \geq 1$, dimension n tensor. The product [23] $\mathcal{TS}$ is a tensor of order $(k-1)(r-1)+1$ and dimension n whose entries are given by,

$$[\mathcal{TS}]_{j\alpha_1 \dots \alpha_{k-1}} = \sum_{j_2, \dots, j_k \in [n]} t_{jj_2 \dots j_k} s_{j_2 \alpha_1} \dots s_{j_k \alpha_{k-1}},$$

where $j \in [n]$ and $\alpha_1, \dots, \alpha_{m-1} \in [n]^{r-1}$.

Then, $\mathcal{T}y^k$ is a complex number defined as

$$\mathcal{T}y^k = \sum_{j_1,\ldots,j_k\in[n]} t_{j_1\ldots j_n} y_{j_1}\ldots y_{j_k}.$$

Also, $\mathcal{T}y^{k-1}$ is an n-dimensional vector whose j^{th} entry is defined as

$$\left(\mathcal{T}y^{k-1}\right)_j = \sum_{j_2,\ldots j_k} t_{jj_2\ldots j_n} y_{j_2}\ldots y_{j_k}.$$

A complex number λ is called an eigenvalue of $\mathcal{T}$ corresponding to an eigenvector $y\in\mathbb{C}^n$, if the pair (λ, y) satisfies the equation

$$\left(\mathcal{T}y^{k-1}\right)_j = \lambda y_j^{k-1},$$

for all $j\in[n]$. An identity tensor $\mathcal{I}$ of order k and dimension n, whose $(j_1\ldots j_k)-th$ entry is given by

$$\mathcal{I}_{j_1\ldots j_k} = \begin{cases} 1, & \text{if } j_1 = j_2 = \cdots = j_k \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $\mathcal{T}$ is defined as the resultant of the polynomial system $(\mathcal{T}-\lambda\mathcal{I})$. The different types of eigenvalue multiplicities and related problems has been discussed in [13,38]. The multiplicity of λ as a root of the characteristic polynomial is its algebraic multiplicity and is denoted by $Am(\lambda)$. The dimension of set of all eigenvectors of $\mathcal{T}$ corresponding to the eigenvalue λ is called the geometric multiplicity of λ and is denoted by $Gm(\lambda)$. The dimension of the vector space,

$$span\,\{y\in\mathbb{C}^n : y \text{ is an eigenvector } \mathcal{T} \text{ corresponding to } \lambda\}$$

is called the *span* multiplicity of λ and is denoted by $Sm(\lambda)$. Similarly the dimension of the vector space,

$$span\,\{y\in\mathbb{R}^n : y \text{ is an eigenvector } \mathcal{T} \text{ corresponding to } \lambda\in\mathbb{R}\}$$

is called the *Hspan* multiplicity of λ and is denoted by $HSm(\lambda)$.

In spectral graph theory, the Perron-Frobenius theorem for non-negative matrices play a very significant role. Unlike the matrix case, there are multiple versions of Perron-Frobenius theorems for non-negative tensors in the literature [2,3,11,12,14].

Theorem 1.1 *[21] Let $T\in\mathcal{T}_{k,n}$ be a non-negative tensor of order k (≥ 2), dimension n (≥ 2).*

1 Then, $\lambda_1(T)$ is an H-eigenvalue of T with a non-negative H-eigenvector.

2 If T is weakly irreducible, then $\lambda_1 > 0$ is the unique H-eigenvalue of T, with the unique positive eigenvector up to a multiplicative constant. If furthermore T is irreducible, the $\lambda_1(T)$ is also the unique H-eigenvalue with non-negative eigenvector.

In [33], authors have discussed the spectral radius of adjacency, Laplacian and signless Laplacian tensors of uniform hypergraphs. Also, the spectral properties (adjacency, Laplacian and many more) of general hypergraphs is explored in [10]. While not exhaustive, some of the most recent developments on the spectral properties of adjacency tensors [5,16,18,26,28,29,35], Laplacian and signless Laplacian tensors [8,9,15,22,34,37] can be found in the literature. One may refer to [1] and [21] for definitions of all other undefined terminology in hypergraph theory and tensor analysis, respectively.

2. Bounds for the Spectral Radius of $\mathcal{F}(\mathcal{H})$

The preliminary bounds for the spectral radius of the degree based extended adjacency tensor of a k-uniform hypergraph is discussed in this section.

Proposition 2.1 *Let $\mathcal{F} \in \mathcal{T}_{k,n}$ be a degree based extended adjacency tensor of a k-uniform hypergraph $(\mathcal{V}, \mathcal{E})$ on n vertices and $F(d_e) := F(d_{i_1}, d_{i_2}, \ldots, d_{i_k})$, where $e = \{i_1, i_2, \ldots, i_k\} \in \mathcal{E}$, then*

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{e_i \in \mathcal{E}} F(d_{e_i}) \leq \lambda_1(\mathcal{F}(\mathcal{H})) \leq \max_i \sum_{e_i \in \mathcal{E}} F(d_{e_i}),$$

and the equality holds if $\sum_{e_i \in \mathcal{E}} F(d_{e_i}) = \sum_{e_j \in \mathcal{E}} F(d_{e_j})$, for all $i \neq j$.

Proof: From Theorem 1.1, the spectral radius of $\mathcal{F}$ is an H-eigenvalue with a non-negative H-eigenvector y. Let y_j be the largest component of y and hence $y_j > 0$. Now,

$$\lambda_1(\mathcal{F})y_j^{k-1} = (\mathcal{F}y^{k-1})_j = \sum_{i_2,\ldots,i_k \in [n]} f_{ji_2\ldots i_k}\, y_{i_2}\cdots y_{i_k}$$

$$\begin{aligned}\implies \lambda_1(\mathcal{F}) &= \sum_{i_2,\ldots,i_k \in [n]} f_{ji_2\ldots i_k}\, \frac{y_{i_2}}{y_j}\cdots\frac{y_{i_k}}{y_j} \\ &\leq \sum_{i_2,\ldots,i_k \in [n]} f_{ji_2\ldots i_k} \\ &= \sum_{e_j \in \mathcal{E}} F(d_{e_j})\end{aligned}$$

where $F(d_{e_j}) = F(d_j, d_{i_2}, \ldots, d_{i_k})$ and $e_j = \{j, i_2, \ldots, i_k\}$ is a hyperedge in $\mathcal{E}$ containing the vertex j. Since $\mathcal{F}$ is non-negative and symmetric, we have from [21],

$$\lambda_1(\mathcal{F}) = \max\{\mathcal{F}y^k | y \in \mathbb{R}^n, \sum_{i=1}^{n} y_i^k = 1\}.$$

By taking $y = \left[\frac{1}{\sqrt[k]{n}}, \frac{1}{\sqrt[k]{n}}, \ldots, \frac{1}{\sqrt[k]{n}}\right]^\mathsf{T}$,

$$\begin{aligned}\lambda_1(\mathcal{F}) \geq \mathcal{F}y^k &= \sum_{i_1,i_2,\ldots,i_k \in [n]} f_{i_1 i_2\ldots i_k}\, y_{i_1}y_{i_2}\cdots y_{i_k} \\ &= \frac{1}{n}\sum_{i_1,i_2,\ldots,i_k \in [n]} f_{i_1 i_2\ldots i_k} \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{e_i \in \mathcal{E}} F(d_{e_i}).\end{aligned}$$

The equality part is clear, as the lower and upper bound coincides when $\mathcal{F}$ has an equal row sum. □

Corollary 2.1 *Let $\mathcal{H}$ be a k-uniform d-regular hypergraph, then all-one vector is an H-eigenvector of $\mathcal{F}$ corresponding to the eigenvalue*

$$\lambda_1(\mathcal{F}) = dF(e),$$

where e is an arbitrary hyperedge of $\mathcal{H}$.

Theorem 2.1 *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices and t_i be the i^{th} row sum of the degree based extended adjacency tensor $\mathcal{F}(\mathcal{H})$. If $t_1 \geq t_2 \geq \cdots \geq t_n$ denotes the sequence of row sums, then*

$$\lambda_1(\mathcal{F}(\mathcal{H})) \leq t_1^{\frac{1}{k}} t_2^{1-\frac{1}{k}},$$

with equality if and only if $t_2 = \cdots = t_n$.

Proof: Suppose $t_1 = t_2$, then by using Proposition 2.1,

$$\lambda_1(\mathcal{F}) \le t_1 = t_1^{\frac{1}{k}} t_2^{1-\frac{1}{k}}.$$

Let us assume that $t_1 > t_2$ and let $P = diag(z, 1, \dots, 1)$ and $z \ge 1$ be a real. Then

$$\begin{aligned} t_1(D^{-(k-1)}\mathcal{F}D) &= \sum_{j_2 \dots j_k} (D^{-(k-1)}\mathcal{F}D)_{1 j_2 \dots j_k} \\ &= \sum_{j_2 \dots j_k} D_{11}^{-(k-1)} \mathcal{F} 1 j_2 \dots j_k D_{j_2 j_2} \dots D_{j_k j_k} \\ &= \frac{1}{z^{k-1}} \sum_{j_2 \dots j_k} \mathcal{F} 1 j_2 \dots j_k = \frac{t_1}{z^{k-1}}. \end{aligned}$$

Let $t_{\{1,j\}} = \sum\limits_{\{1,j\} \subset e \in \mathcal{E}(\mathcal{H})} F(d_e)$ be the sum over all the hyperedges that contains both 1 and j. For $2 \le j \le n$, we have

$$\begin{aligned} t_j(D^{-(k-1)}\mathcal{F}D) &= \sum_{j_2 \dots j_k} (D^{-(k-1)}\mathcal{F}D)_{j j_2 \dots j_k} \\ &= \sum_{j_2 \dots j_k} D_{jj}^{-(k-1)} \mathcal{F} j j_2 \dots j_k D_{j_2 j_2} \dots D_{j_k j_k} \\ &= z \sum_{e_j \in \mathcal{E}, 1 \in e_j} F(d_{e_j}) + \sum_{e_j \in \mathcal{E}, 1 \notin e_j} F(d_{e_j}) \\ &= z\ t_{\{1,j\}} + t_i - t_{\{1,j\}} \\ &\ge z\ t_j \text{ (since } z > 1) \\ &\ge z\ t_2. \end{aligned}$$

Now, by taking $z = \left(\frac{t_1}{t_2}\right)^{\frac{1}{k}} \ge 1$,

$$t_j((D^{-(k-1)}\mathcal{F}D) \le z\ t_2 = t_1^{\frac{1}{k}} t_2^{1-\frac{1}{k}},\ 2 \le j \le n.$$

Also,

$$t_1((D^{-(k-1)}\mathcal{F}D) = \frac{t_1}{z^{k-1}} = t_1^{\frac{1}{k}} t_2^{1-\frac{1}{k}}.$$

When $t_1 = t_2$, the equality holds if and only if $t_i = t_j$ for all $1 \le i, j \le n$, and hence $\mathcal{H}$ is a regular hypergraph.

Also when $t_1 > t_2$, equality holds in the upper bound if $z\ t_2 = z\ t_j$ for all $3 \le j \le n$, and $t_i = t_{1,i}$ for all $2 \le i \le n$. That is, $\mathcal{H}$ is a blow-up of some $k-1$-regular hypergraph on $n-1$ vertices. □

Note 1 *One can get the exact expression for the spectral radius of the sunflower $\mathcal{S}(m, 1, k)$ (a hyperstar on $1 + (k-1)m$ vertices) as a special case of the extremal hypergraph of the above theorem, which will be discussed in detail in Section 4.*

3. Spectral Radius of Linear Hyperpath

In this section, we have put forward an attempt to give the expression for the spectral radius of linear hyperpath with m hyperedges.

Lemma 3.1 *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform cored hypergraph and $y \in \mathbb{R}^n$ be an H-eigenvector of its degree based extended adjacency tensor of $\mathcal{H}$ corresponding to an H-eigenvalue $\lambda \ne 0$. If $e \in \mathcal{E}$ and $i, j \in e$ be two pendant vertices then $|y_i| = |y_j|$. Moreover, $y_i = y_j$ when k is an odd integer.*

Proof: By definition,

$$\begin{aligned}\lambda y_i^{k-1} &= (\mathcal{F}y^{k-1})_i \\ &= \sum_{i_2,\ldots,i_k\in[n]} f_{ii_2\ldots i_k}\, y_{i_2}\ldots y_{i_k} \\ &= F(d_e)\prod_{t\in e\setminus\{i\}} y_t.\end{aligned}$$

Hence,

$$\lambda y_i^k = F(d_e)\prod_{t\in e} y_t.$$

Similarly,

$$\lambda y_j^k = F(d_e)\prod_{s\in e} y_s.$$

Therefore, if $\lambda \neq 0$ then $|y_i| = |y_j|$, and if k is odd then $y_i = y_j$. □

Lemma 3.2 *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform power hypergraph and k be even. Let $y \in \mathbb{R}^n$ be an H-eigenvector of its degree based extended adjacency tensor $\mathcal{F}$ corresponding to an H-eigenvalue λ. Let $e \in \mathcal{E}$ be an arbitrary but fixed edge and $l = |\{i \text{ is a pendant vertex of } e \text{ such that } y_i < 0\}|$.*

- *If e has only one intersectional vertex i and $x_t \neq 0$ for some pendant vertex $t \in e$, then $\lambda y_i > 0$ when l is even and $\lambda y_i < 0$ when l is odd. Also, $y_t = \frac{F(d_e)y_i}{\lambda}$ or $\frac{-F(d_e)y_i}{\lambda}$.*
- *If e has exactly two intersectional vertices i, j and $x_t \neq 0$ for some pendant vertex $t \in e$, then $\lambda y_i y_j > 0$ when l is even and $\lambda y_i y_j < 0$ when l is odd. Also, $y_t = \pm\sqrt{\frac{F(d_e)y_i y_j}{\lambda}}$ or $\pm\sqrt{\frac{-F(d_e)y_i y_j}{\lambda}}$.*

Proof: Let i be the only intersectional vertex and t be an arbitrary pendant vertex of e. If l is even, then

$$\begin{aligned}\lambda y_t^k &= F(d_e)\prod_{j\in e_t} y_j \\ &= F(d_e)|y_t|^{k-1} y_i \\ \implies \lambda |y_t| &= F(d_e) y_i.\end{aligned}$$

If l is odd, then

$$\begin{aligned}\lambda y_t^k &= F(d_e)\prod_{j\in e_t} y_j \\ &= F(d_e)\left(-|y_t|^{k-1}\right) y_i \\ \implies \lambda |y_t| &= -F(d_e) y_i.\end{aligned}$$

Now, let us assume that i and j be the two intersectional vertices and t be an arbitrary pendant vertex of e. If l is even, then

$$\begin{aligned}\lambda y_t^k &= F(d_e)\prod_{j\in e_t} y_j \\ &= F(d_e)|y_t|^{k-2} y_i y_j \\ \implies \lambda |y_t^2| &= F(d_e) y_i y_j.\end{aligned}$$

If l is odd, then

$$\begin{aligned}\lambda y_t^k &= F(d_e)\prod_{j\in e_t} y_j\\ &= F(d_e)\left(-|y_t|^{k-1}\right) y_i y_j\\ \implies \lambda|y_t| &= -F(d_e)y_iy_j.\end{aligned}$$

□

Lemma 3.3 *Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a k-uniform power hypergraph and k be odd. Let $y\in\mathbb{R}^n$ be an H-eigenvector of its degree based extended adjacency tensor $\mathcal{F}$, corresponding to an H-eigenvalue λ. Let $e\in\mathcal{E}$ be an arbitrary but fixed edge.*

- *If e has only one intersectional vertex i and $x_t\neq 0$ for some pendant vertex $t\in e$, then $y_t=\frac{F(d_e)y_i}{\lambda}$.*
- *If e has exactly two intersectional vertices i,j and $x_t\neq 0$ for some pendant vertex $t\in e$, then $y_t=\pm\sqrt{\frac{F(d_e)y_iy_j}{\lambda}}$.*

Proof: The proof is similar to the proof of Lemma 3.2. □

Theorem 3.1 *Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a k-uniform linear hyperpath with $m\geq 3$ hyperedges and let $\mathcal{F}$ be its degree based extended adjacency tensor. Then we have*

- *When $m=3$, $\lambda_1(\mathcal{F})=\left(\frac{C_1^{\frac{k}{2}}+\sqrt{C_1^k+4C_2^k}}{2}\right)^{\frac{2}{k}}$;*
- *When $m=4$, $\lambda_1(\mathcal{F})=\left(2C_1^k+C_2^k\right)^{\frac{1}{k}}$;*
- *When $m=5$, $\lambda_1(\mathcal{F})$ is the largest (real) positive root of*

$$-C_1^{\frac{k}{2}}\lambda^{\frac{3k}{2}}+C_1^k\lambda^k+\left(C_1^{\frac{3k}{2}}+C_2^kC_1^{\frac{k}{2}}\right)\lambda^{\frac{k}{2}}-C_1^kC_2^k.$$

Here, $C_1=F(2,2,1,\ldots,1)$ and $C_2=F(2,1,1,\ldots,1)$.

Proof: With no loss of generality, let us assume that $\mathcal{E}=\{\{1,\ldots,k\},\{k,\ldots,2k-1\},\ldots,\{(m-1)k-m+2,\ldots,mk-m+1\}\}$. Since $\mathcal{H}$ is connected, by using Theorem 1.1 a real eigenvalue λ of $\mathcal{F}$ is a spectral radius of $\mathcal{F}$, if there exists a positive eigenvector $y\in\mathbb{R}^n$ corresponding to it. Suppose that (λ,y) is the pair satisfying the above. Then by using Lemmas 3.1, 3.2 and 3.3, if $y_t\neq 0$ for some $t\in\{1,2,\ldots,k-1\}$, we have that

$$y_1=\cdots=y_{k-1}=\frac{C_2y_k}{\lambda},$$

and if $y_t\neq 0$ for some $t\in\{(i-1)k-i+3,\ldots,ik-i\}$, we have that

$$y_{(i-1)k-i+3}=\cdots=y_{ik-i}=\sqrt{\frac{C_1\ y_{(i-1)k-i+2}\ y_{ik-i+1}}{\lambda}}.$$

Also, if $y_t\neq 0$ for some $t\in\{(m-1)k-m+3,\ldots,mk-m+1\}$, we have

$$y_{(m-1)k-m+3}=\cdots=y_{mk-m+1}=\frac{C_2y_{(m-1)k-m+2}}{\lambda},$$

where $C_1=F(2,2,1,\ldots,1)$ and $C_2=F(2,1,1,\ldots,1)$.

By definition, we have

$$\begin{aligned}\lambda y_k^{k-1} = \left(\mathcal{F} y^{k-1}\right)_k &= \sum_{j_2,\ldots,j_k} f_{kj_2\ldots j_k} y_{j_2}\cdots y_{j_k} \\ &= C_1 y_t^{k-2} y_{2k-1} + C_2 y_t^{k-1} \\ \implies \lambda y_k^{k-1} &= C_1 \left(\sqrt{\frac{C_1 y_k y_{2k-1}}{\lambda}}\right)^{k-2} y_{2k-1} + C_2\left(\frac{C_2 y_k}{\lambda}\right)^{k-1}\end{aligned}$$

Multiplying the above equation by $\lambda^{k-1} y_k^{-\frac{k-2}{2}}$ and by rearranging the terms we get,

$$\left(\lambda^k - C_2^k\right) y_k^{\frac{k}{2}} = \left(\lambda C_1\right)^{\frac{k}{2}} y_{2k-1}^{\frac{k}{2}} \tag{1}$$

For $2 \le i \le m-2$, $(i)^{th}$ equation is obtained as

$$\begin{aligned}\lambda y_{ik-i+1}^{k-1} = C_1 &\left(\sqrt{\frac{C_1 y_{(i-1)k-i+2} y_{ik-i+1}}{\lambda}}\right)^{k-2} y_{(i-1)k-i+2} \\ &+ C_1\left(\sqrt{\frac{C_1 y_{ik-i+1} y_{(i+1)k-i}}{\lambda}}\right)^{k-2} y_{(i+1)k-i}\end{aligned}$$

$$\lambda^{\frac{k}{2}} y_{ik-i+1}^{\frac{k}{2}} = C_1^{\frac{k}{2}} \left(y_{(i-1)k-i+2} + y_{(i+1)k-i}\right) \tag{i}$$

Similarly,

$$\begin{aligned}\lambda y_{(m-1)k-m+2}^{k-1} = C_1 &\left(\sqrt{\frac{C_1 y_{(m-2)k-m+3} y_{(m-1)k-m+2}}{\lambda}}\right)^{k-2} y_{(m-1)k-m+2} \\ &+ C_2\left(\frac{C_2 y_{(m-1)k-m+2}^{k-1}}{\lambda}\right)^{k-1}.\end{aligned}$$

Now, by simplifying and rearranging the terms we get

$$\left(\lambda^k - C_2^k\right) y_{(m-1)k-m+2}^{\frac{k}{2}} = \left(\lambda C_1\right)^{\frac{k}{2}} y_{(m-2)k-m+3}^{\frac{k}{2}}.$$

Define $a := \lambda^k - C_2^k$, $b := (\lambda C_1)^{\frac{k}{2}}$ and $c := C_1^{\frac{k}{2}}$. Let

$$FX = \begin{bmatrix} a & -b & 0 & \cdots & 0 & 0 & 0 \\ -c & \frac{b}{c} & -c & 0 & \cdots & 0 & 0 \\ 0 & -c & \frac{b}{c} & -c & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -c & \frac{b}{c} & -c & 0 \\ 0 & 0 & \cdots & 0 & -c & \frac{b}{c} & -c \\ 0 & 0 & 0 & \cdots & 0 & -b & a \end{bmatrix} \begin{bmatrix} y_k^{\frac{k}{2}} \\ y_{2k-1}^{\frac{k}{2}} \\ y_{3k-2}^{\frac{k}{2}} \\ \vdots \\ y_{(m-2)k-m+3}^{\frac{k}{2}} \\ y_{(m-1)k-m+2}^{\frac{k}{2}} \end{bmatrix}$$

where F is square matrix of order $m-1$. In order to compute $\lambda_1(\mathcal{F})$, consider the non-zero solutions of $FX = 0$.

Case 1: When $m = 3$. We know that, if $|F| = 0$ implies that $FX = 0$ has a non-zero solution. Therefore

$$|F| = \begin{vmatrix} a & -b \\ -b & a \end{vmatrix} = a^2 - b^2 = 0.$$

From Equation 1, $\lambda^k - C_2^k > 0$ and since F is a positive valued function, we have that $a+b \neq 0$ and hence $|F| = 0 \implies a - b = \lambda^k - C_1^{\frac{k}{2}}\lambda^{\frac{k}{2}} - C_2^k = 0$. Let $g(\lambda) := \lambda^k - C_1^{\frac{k}{2}}\lambda^{\frac{k}{2}} - C_2^k$.
Since $g(\lambda)$ is monotonically increasing after $\lambda = \frac{C_1}{\sqrt[k]{4}}$ and $g\left(\frac{C_1}{\sqrt[k]{4}}\right) < 0$, we have by direct computation

$$\lambda_1(\mathcal{F}) = \left(\frac{C_1^{\frac{k}{2}} + \sqrt{C_1^k + 4C_2^k}}{2}\right)^{\frac{2}{k}}.$$

Case 2: When $m = 4$. Now,

$$|F| = \begin{vmatrix} a & -b & 0 \\ -c & \frac{b}{c} & -c \\ 0 & -b & a \end{vmatrix} = ab(a - 2c^2) = 0.$$

By similar argument, we have that

$$(a - 2c^2) = 0 \implies a = 2c^2 \implies \lambda^k = C_2^k + 2C_1^k.$$

Case 3: When $m = 5$. Now,

$$|F| = \begin{vmatrix} a & -b & 0 & 0 \\ -c & \frac{b}{c} & -c & 0 \\ 0 & -c & \frac{b}{c} & -c \\ 0 & 0 & -b & a \end{vmatrix} = (bc^2 - a(b + c^2))(bc^2 - a(b - c^2) = 0.$$

Let $g_1(\lambda) = bc^2 - a(b + c^2)$ and $g_2(\lambda) = bc^2 - a(b - c^2)$. Since $a,\ b$ and c^2 are greater than zero, we have $g_2(\lambda) > g_1(\lambda)$. Hence, the largest real solution of $g_2(\lambda)$ will be equal to $\lambda_1(\mathcal{F})$. We have,

$$g_2'(\lambda) = \frac{kC_1^{\frac{k}{2}}\lambda^{\frac{k}{2}-1}}{2}\left[-3\lambda^k + 2C_1^{\frac{k}{2}}\lambda^{\frac{k}{2}} + C_1^k + C_2^k\right].$$

As there is no critical point between 0 and $\frac{C_1^{\frac{k}{2}} + \sqrt{C_1^k + 3(C_1 + C_2)}}{3}$, also $g_2(C_1) > 0$ and $g_2(\lambda)$ is monotonically decreasing after $\lambda = \frac{C_1^{\frac{k}{2}} + \sqrt{C_1^k + 3(C_1 + C_2)}}{3}$, we have λ_1 is the positive real root of $g_2(\lambda)$. □

Remark 3.1 However the solution for the spectral radius of linear hyperpaths containing more than 6 hyperedges couldn't be concluded completely, the direction for which is discussed.

Let

$$T_n = \begin{vmatrix} \frac{b}{c} & -c & 0 & 0 & 0 \\ -c & \frac{b}{c} & -c & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & -c & \frac{b}{c} & -c \\ 0 & \cdots & 0 & -c & \frac{b}{c} \end{vmatrix}$$

When $m \geq 6$, if $\lambda^k \neq 4c^2$, then

$$\begin{aligned} |F| = 0 &\Leftrightarrow a^2 T_{m-3} - 2abcT_{m-4} + b^2c^2 T_{m-5} = 0 \\ &\Leftrightarrow a^2\left(\frac{p^{m-2} - q^{m-2}}{2^{m-2}t}\right) - 2abc\left(\frac{p^{m-3} - q^{m-3}}{2^{m-3}t}\right) + b^2c^2\left(\frac{p^{m-4} - q^{m-4}}{2^{m-4}t}\right) = 0 \\ &\Leftrightarrow \frac{1}{t}\left[a^2\left(p^{m-2} - q^{m-2}\right) - 4abc\left(p^{m-3} - q^{m-3}\right) + 4b^2c^2\left(p^{m-4} - q^{m-4}\right)\right] = 0 \\ &\Leftrightarrow \frac{1}{t}\left[a\left(p^{\frac{m-2}{2}} + q^{\frac{m-2}{2}}\right) - 2bc\left(p^{\frac{m-4}{2}} - q^{\frac{m-4}{2}}\right)\right] \\ &\qquad \left[a\left(p^{\frac{m-2}{2}} - q^{\frac{m-2}{2}}\right) - 2bc\left(p^{\frac{m-4}{2}} + q^{\frac{m-4}{2}}\right)\right] = 0 \end{aligned}$$

where $p := \lambda^{\frac{k}{2}} + \sqrt{\lambda^k - 4c^2}$, $q := \lambda^{\frac{k}{2}} - \sqrt{\lambda^k - 4c^2}$ and $t := \sqrt{\lambda^k - 4c^2}$.

If $\lambda^k = 4c^2$, then

$$\begin{aligned}|F| = 0 &\Leftrightarrow a^2 T_{m-3} - 2abcT_{m-4} + b^2c^2T_{m-5} = 0\\ &\Leftrightarrow a^2\left(\frac{(m-2)b^{m-3}}{(2c)^{m-3}}\right) - 2abc\left(\frac{(m-3)b^{m-4}}{(2c)^{m-4}}\right) + b^2c^2\left(\frac{(m-4)b^{m-5}}{(2c)^{m-5}}\right) = 0\\ &\Leftrightarrow \frac{1}{c^{m-3}}\left[(m-2)a^2b^{m-3} - 4(m-3)ab^{m-3}c^2 - 4(m-4)b^{m-3}c^4\right] = 0.\end{aligned}$$

By solving the above equations separately one may get the solution for $m \geq 6$ in two cases.

4. Eigenvalues of the Sunflower

The spectral radius of the sunflower hypergraph and few additional eigenvalues of a particular class of sunflower hypergraph are the focus of this section.

Proposition 4.1 *Let $\mathcal{H}$ be a k (≥ 3)-uniform hypergraph on n vertices. Then 0 is an eigenvalue of $\mathcal{F}$ with* Hspan *multiplicity equal to n.*

Proof: By the definition of extended adjacency tensor, it is clear that $f_{j_1 \ldots j_k}$ is a non-zero entry of $\mathcal{F}$ if and only if j_i's are all distinct. Let us assume that $y = e_i$, the standard basis of $\mathbb{R}^n$. Then for given $j \in [n]$, each term in the left hand side of the equation

$$\left(\mathcal{F}y^{k-1}\right)_j = \lambda y_j^{k-1}$$

vanishes. Hence 0 is an eigenvalue of $\mathcal{F}$ with *Hspan* multiplicity n. □

Without loss of generality, we assume that $1, 2, \ldots, c$ are the central vertices of the sunflower $\mathcal{S}(m, c, k)$.

Proposition 4.2 *Let $\mathcal{S}(m, 1, k)$ (hyperstar) be a k-uniform sunflower hypergraph with m hyperedges. If $F^* = F(m, 1, \ldots, 1)$ then the spectral radius*

$$\lambda_1(\mathcal{F}(\mathcal{S})) = \sqrt[k]{m}F^*$$

with corresponding H-eigenvector $y := (\sqrt[k]{m}, 1, \ldots, 1)$.

Proof: Let 1 be the central vertex and $i \neq 1$ be a pendant vertex of the sunflower $\mathcal{S}(m, 1, k)$. If λ is assumed to be the spectral radius of $\mathcal{F}$ corresponding to the eigenvector y, then by definition and Lemma 3.2 and 3.3,

$$\begin{aligned}\lambda y_1^{k-1} = \left(\mathcal{F}y^{k-1}\right)_1 &= \sum_{e\in\mathcal{E}} F^* y_i^{k-1}\\ &= mF^*\left(\frac{F^*y_1}{\lambda}\right)^{k-1}\\ \implies \lambda &= \sqrt[k]{m}F^*.\end{aligned}$$

Now, for $1 \neq i \in [n]$,

$$\begin{aligned}\lambda y_i^{k-1} = \left(\mathcal{F}y^{k-1}\right)_i &= \sum_{e\in\mathcal{E}} F^* y_i^{k-2} y_1\\ &= F^*\left(\frac{F^*y_1}{\lambda}\right)^{k-2} y_1\\ \implies \sqrt[k]{m}\, y_i^{k-1} &= y_1,\end{aligned}$$

hence the result follows.

Alternative proof: By using Theorem 2.1, we can observe that hyperstar is an extremal hypergraph that attains the upper bound. Therefore,

$$\lambda_1(\mathcal{F}) = t_1^{\frac{1}{k}} t_i^{1-\frac{1}{k}} = (mF^*)^{\frac{1}{k}} (F^*)^{1-\frac{1}{k}} = \sqrt[k]{m} F^*,$$

where $1 \neq i \in [n]$. □

Theorem 4.1 *Let $\mathcal{S}(m, c, k)$ be a k-uniform sunflower hypergraph with m hyperedges. If $F^* = F(\underbrace{m, \ldots, m}_{c \text{ times}}, 1, \ldots, 1)$, then*

$$\lambda_1(\mathcal{F}(\mathcal{S})) = \sqrt[k]{m^c}\ F^*$$

with corresponding H-eigenvector $y := (\underbrace{\sqrt[k]{m}, \ldots, \sqrt[k]{m}}_{c \text{ times}}, 1, \ldots, 1)$.

Proof: Assume that y is an eigenvector of $\mathcal{F}$ corresponding to the eigenvalue $\lambda = \lambda_1(\mathcal{F})$ and hence from Theorem 1.1, we have $y > 0$. Since $y > 0$ and by Lemma 3.1, the components of y corresponding to the pendant vertices of a fixed hyperedge are constant. Let $y_i = 1$, for all components of y corresponding pendant vertex i. Then for $t \in [c]$,

$$\begin{aligned} \lambda y_t^{k-1} = (\mathcal{F} y^{k-1})_t &= \sum_{e_t \in \mathcal{E}} F(d_{e_t}) \frac{y_1 \ldots y_c}{y_t} y_i^{k-c} \\ &= mF^* \frac{y_1 \ldots y_c}{y_t} \\ \implies \lambda y_t^k &= mF^* y_1 \ldots y_c. \end{aligned}$$

For any pendant vertex i, we have

$$\begin{aligned} \lambda y_i^{k-1} = (\mathcal{F} y^{k-1})_i &= \sum_{e_i \in \mathcal{E}} F(d_{e_i}) y_1 \ldots y_c\ y_i^{k-c-1} \\ &= F^* y_1 \ldots y_c \\ \implies \lambda y_t^k &= F^* y_1 \ldots y_c. \\ \implies \lambda y_t^k &= F^* \frac{\lambda y_t^k}{mF^*} \\ \implies y_t^k &= m \end{aligned}$$

In particular for $t \in [c]$, let $y_t = \sqrt[k]{m}$ and hence we have

$$\lambda = F^* y_1 \ldots y_c = F^* \sqrt[k]{m^c}.$$

Now, from Theorem 1.1, the result follows. □

Lemma 4.1 *Let i and j be two duplicate vertices in a k-uniform hypergraph $\mathcal{H}$. If $\lambda \neq 0$ is an eigenvalue of $\mathcal{F}$ with corresponding eigenvector y. Then $y_i^{k-1} = y_j^{k-1}$.*

Proof: By definition, we have

$$\left(\mathcal{F} y^{k-1}\right)_i = \lambda y_i^{k-1} \text{ and } \left(\mathcal{F} y^{k-1}\right)_j = \lambda y_j^{k-1}.$$

Since i and j are duplicate and by the definition of $\mathcal{F}$, we have $\left(\mathcal{F} y^{k-1}\right)_i = \left(\mathcal{F} y^{k-1}\right)_j$ and since $\lambda \neq 0$ the result follows. □

Theorem 4.2 *Let $\mathcal{S}(m, k-1, k)$ be a k uniform sunflower hypergraph with m hyperedges and on $n = m+k-1$ vertices. Then a non-zero λ is an eigenvalue of $\mathcal{F}$ corresponding to an eigenvector y (not all components zero) if and only if all of the following hold (up to a multiplicative constant).*

- $y_j^{k-1} = 1$ *for each* $j \in \{k, \ldots, n\}$;
- $y_i^k = \sum_{j=k}^{n} y_j$ *for each* $i \in [k-1]$;
- $\lambda = F(1, m, \ldots, m) y_1 y_2 \cdots y_{k-1}$.

Proof: Let $\lambda \neq 0$ be an eigenvalue of $\mathcal{F}$ corresponding to an eigenvector y. By using Lemma 4.1, note that for $j \in \{k, \ldots, n\}$, y_j^{k-1} is a constant. Suppose that $y_j = 0$ for all $j \in \{k, \ldots, n\}$, then $\left(\mathcal{F}y^{k-1}\right)_i = 0$ for all $i \in \{1, \ldots, k-1\}$, implies that $\lambda y_i^{k-1} = 0$ for all $i \in \{1, \ldots, k-1\}$. Since $\lambda \neq 0$, $y_i = 0$ for all $i \in \{1, \ldots, k-1\}$ which is a contradiction, since $y \neq 0$. Therefore, y_j^{k-1} is a non-zero constant for $j \in \{k, \ldots, n\}$. Hence, without loss of generality let us assume that $y_j^{k-1} = 1$ for all $j \in \{k, \ldots, n\}$. By definition, λ is an eigenvalue of $\mathcal{F}$ corresponding to an eigenvector y if and only if

$$\sum_{\{l, j_2, \ldots, j_k\} \in \mathcal{E}} F(d_l, d_{j_2}, \ldots, d_{j_k}) y_{j_2} \ldots y_{j_k} = \lambda y_l^{k-1}$$

for all $l \in [n]$. If j is a pendant vertex, then

$$F(d_j, d_{j_2}, \ldots, d_{j_k}) y_{j_2} \ldots y_{j_k} = F(1, m, \ldots, m) y_1 \ldots y_{k-1} = \lambda y_j^{k-1} = \lambda.$$

Since, $\lambda \neq 0$ and by the definition of F, we have that $y_i \neq 0$ for all $i \in [k-1]$. If i is a central vertex, then

$$\begin{aligned}
&\sum_{\{l, j_2, \ldots, j_k\} \in \mathcal{E}} F(d_l, d_{j_2}, \ldots, d_{j_k}) y_{j_2} \ldots y_{j_k} = \lambda y_l^{k-1} \\
\implies &\sum_{j=k}^{n} \left(y_j \frac{y_1 \cdots y_{k-1}}{y_i} \right) F(1, m, \ldots, m) = \lambda y_l^{k-1} \\
\implies &\sum_{j=k}^{n} y_j = y_i^k \text{ for all } i \in [k-1].
\end{aligned}$$

□

Theorem 4.3 *Let $\mathcal{S}(m, k-1, k)$ be a k uniform sunflower hypergraph with m hyperedges and on $n = m+k-1$ vertices. If z is a k^{th} root of unity and $F' = F(1, m, \ldots, m)$, then*

$$F' \sqrt[k]{m^{k-1}}, zF' \sqrt[k]{m^{k-1}} \ldots, z^{k-1} F' \sqrt[k]{m^{k-1}}.$$

Furthermore, if $m = t(k-1)+1$ for some positive integer t, then
$F', zF', \ldots, z^{k-1}F'$ are also the eigenvalues of $\mathcal{F}$.

Proof: Let z be a k^{th} root of unity and α denote the $(k-1)^{th}$ root of z. For a fixed $r \in \{0, 1, \ldots, k-1\}$, if $y_j = \alpha^{rk}$ for $j \in \{k, \ldots, n\}$; and $y_i = \alpha^r \sqrt[k]{mF'}$ for $i \in [k-1]$, it is easy to observe that, $y_j^{k-1} = 1$ for $j \in \{k, \ldots, n\}$ and $y_i^k = \sum_{j=k}^{n} y_j$ for each $i \in [k-1]$.

By Theorem 4.2, we have $\sum_{j=k}^{n} |y_j| = m$ and $\lambda = F' z^r \sqrt[k]{m^{k-1}}$ is an eigenvalue of $\mathcal{F}$ corresponding to the eigenvector y by which we have the first part of the theorem.

Now assume that $m = t(k-1)+1$ for some positive integer t. Let z be a k^{th} root of 1 and β be a $k-1^{th}$ root of z and let $\eta := \beta^k$. Suppose the P elements $y_k, \dots, y_n$ are given by,

$$\eta^r \ (t+1 \text{ times }) \text{ and each } \eta^i (t \text{ times }) \text{ where } i \neq r, i \in \{0, 1, \dots, k-1\}.$$

If y is an eigenvector of $\mathcal{F}$,

$$\sum_{j=k}^{n} y_j = t \sum_{l=1}^{k-1} \eta^l + \eta^r = \eta^r = y_i^k.$$

If $y_i = \beta^r$ for each $i \in [k-1]$, then the condition in Theorem 4.2 is satisfied and hence y is an eigenvector corresponding to the eigenvalue

$$\lambda = F' y_i^{k-1} = \beta^{r(k-1)} F' = F' z^r.$$

Hence the proof. □

Acknowledgment

Authors acknowledge the organization for overall support.

References

1. C. Berge, *Hypergraphs: combinatorics of finite sets*, 45, Elsevier, 1984.
2. K. C. Chang, K. Pearson, T. Zhang, *Perron-Frobenius theorem for nonnegative tensors.* Commun. Math. Sci. 6(2), 507-520, (2008).
3. K. C. Chang, L. Qi, T. Zhang, *A survey on the spectral theory of nonnegative tensors,* Numer. Linear Algebra Appl. 20(6), 891-912, (2013).
4. P. Chodrow, N. Eikmeier, J. Haddock, *Non-backtracking spectral clustering of nonuniform hypergraphs* SIAM J. Math. Data Sci. 5(2), 251-279, (2023).
5. G. J. Clark, F. Thomaz, A. T. Stephen, *Comparing the principal eigenvector of a hypergraph and its shadows*, Linear Algebra Appl. 673, 46-68, (2023).
6. C. Cui, Z. Luo, L. Qi, H. Yan, *The analytic connectivity in uniform hypergraphs: Properties and computation*, Numer. Linear Algebra Appl. 30(2), e2468, (2023).
7. K. C. Das, I. Gutman, I. Milovanović, , E. Milovanović, B. Furtula, *Degree-based energies of graphs*, Linear Algebra Appl. 554, 185-204, (2018).
8. C. Duan, L. Wang, X. Li, *Some properties of the signless Laplacian and normalized laplacian tensors of general hypergraphs*, Taiwanese J. Math. 24(2), 265-281, (2020).
9. Y. Z. Fan, Y. Wang, Y. H. Bao, J. C. Wan, M. Li, Z. Zhu, *Eigenvectors of Laplacian or signless Laplacian of hypergraphs associated with zero eigenvalue.* Linear Algebra and its Applications 579, 244-261, (2019).
10. F. Galuppi, R. Mulas, L. Venturello, *Spectral theory of weighted hypergraphs via tensors*, Linear Multilinear Algebra 71(3), 317-347, (2023).
11. A. Gautier, F. Tudisco, M. Hein, *Nonlinear Perron-Frobenius theorems for nonnegative Tensors*, SIAM Review 65(2), 495-536, (2023).
12. A. Gautier, F. Tudisco, M. Hein, *A unifying Perron–Frobenius theorem for nonnegative tensors via multihomogeneous maps*, SIAM J. Matrix Anal. Appl. 40(3), 1206-1231, (2019).
13. S. Hu, K. Ye, *Multiplicities of tensor eigenvalues*, Commun. Math. Sci. 14(4), 1049-1071, (2016).
14. S. Hu, L. Qi, *A necessary and sufficient condition for existence of a positive Perron vector*, SIAM J. Matrix Anal. Appl. 37(4), 1747-1770, (2016).
15. S. Hu, L. Qi, *The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph*, Discrete Appl. Math., 169, 140-151, (2014).
16. L. J. Kurian, A. V. Chitra, *On the Adjacency and Seidel Spectra of Hypergraphs*, (2023) arXiv:2309.09751.
17. H. Lin, B. Zhou, *On ABC spectral radius of uniform hypergraphs* (2023) arXiv:2303.14929.
18. C. Lü, L. You, Y. Huang, *Sharp bounds for the spectral radii of nonnegative tensors* Front. Math. 18(4), 883-901, (2023).
19. V. Mann, V. Venkatasubramanian, *AI-driven hypergraph network of organic chemistry: network statistics and applications in reaction classification*, Reaction Chem. Eng. 8(3), 619-635, (2023).

20. D. Maurya, Ravindran, B. *Hypergraph partitioning using tensor eigenvalue decomposition*, Plos one 18(7), e0288457, (2023).

21. L. Qi, Z. Luo, *Tensor Analysis*, Philadelphia (PA), Society for Industrial and Applied Mathematics; 2017.

22. L. Qi, *H^+-eigenvalues of Laplacian and signless Laplacian tensors*. Commun. Math. Sci. 12(6), 1045-1064, (2014).

23. J. Y. Shao, *A general product of tensors with applications*, Linear Algebra Appl. 439, 2350-2366, (2013).

24. S. S. Shetty, K. A. Bhat, *Degree based energy and spectral radius of a graph with self-loops*, AKCE Int. J. Graphs Comb. 20(3), 326-331, (2024).

25. S. S. Shetty, K. A. Bhat, *Sombor index of hypergraphs*. MATCH Commun. Math. Comput. Chem. 91(1), 235-254, (2024).

26. G. H. Shirdel, A. Mortezaee, E. Golpar-Raboky, *On spectral theory of a k-uniform directed hypergraph*, Appl. Anal. Discrete Math. 17(2), 296-320, (2023).

27. M.I. Skvortsova, I.I. Fashutdinova, N.A. Mikhailova, *Hypergraph models of hydrocarbon molecules and their applications in computer chemistry*, Fine Chemical Tech. 9(5), 86-93, (2014).

28. Y. M. Song, Y. Z. Fan, Y. Wang, M. Y. Tian, J. C. Wan, *The spectral property of hypergraph coverings*, Discrete Math. 347(3), 113830, (2024) .

29. R. Sun, W. H. Wang, *On the spectral radius of uniform weighted hypergraph*, Discrete Math. Algorithms Appl. 15(1), 2250067, (2023).

30. S. Szalay, M. Pfeffer, V. Murg, et. al., *Tensor product methods and entanglement optimization for ab initio quantum chemistry*, Int J Quant Chem. 115, 1342–1391, (2015).

31. W. Yin, Y. Qu, Z. Ma, Q. Liu, *Hyperntf: A hypergraph regularized nonnegative tensor factorization for dimensionality reduction*, Neurocomputing 512, 190-202, (2022).

32. X. Yuan, M. Zhang, M. Lu, *Some upper bounds on the eigenvalues of uniform hypergraphs*, Linear Algebra Appl. 484, 540-549, (2015).

33. J. Yue, L. Zhang, M. Lu, *Largest adjacency, signless Laplacian, and Laplacian H-eigenvalues of loose paths*, Front. Math. China 11, 623-645, (2016).

34. X. Yuan, L. Qi, J. Shao, *The proof of a conjecture on largest Laplacian and signless Laplacian H-eigenvalues of uniform hypergraphs*, Linear Algebra Appl. 490, 18-30, (2016).

35. L. Zheng, B. Zhou, *Pareto H-eigenvalues of nonnegative tensors and uniform hypergraphs*, Electron. J. Linear Algebra. 39, 423-433, (2023).

36. P. Zheng, H. Su, H. Lu, Q. Du, *Adaptive hypergraph regularized multilayer sparse Tensor factorization for hyperspectral unmixing*, IEEE Trans. Geosci. Remote Sensing 61, 1-18, (2023).

37. Y. N. Zheng, *The zero eigenvalue of the Laplacian tensor of a uniform hypergraph*, Linear Multilinear Algebra 72, 1094-1111, (2023).

38. Y. N. Zheng, *Algebraic and geometric multiplicities for some hypergraphs*, Discrete Math. 347(2), 113780, (2024).

Shashwath S Shetty,
Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal, Karnataka,
India-576104
E-mail address: shashwathsshetty01334@gmail.com

and

K Arathi Bhat,
Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal, Karnataka,
India-576104
E-mail address: arathi.bhat@manipal.edu