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Non-Homogeneous Linearized Impulsive Kd.V And Rivlin-Ericksen Fluid

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ABSTRACT: In the current work, the author used integral transforms to solve time fractional differential equations. We show the advantage of our approach through a few concrete examples like the solution for a variant of the non-homogeneous time fractional impulsive Kd.V equation and integral equations. In this study, the author derives exact solutions for a fractional generalization to Stokes first problem for a Rivlin-Ericksen fluid of second grade in a porous half space.The Laplace transform is employed to obtain the exact solutions of the model. Integral representations of the solutions are presented. The obtained results reveal that, the integral transform method is very effective and reliable.

Key Words: Laplace transform; Fourier transform; Rivlin-Ericksen fluid; Kd.V equation; Bessel function.

Contents

1. Introduction and Notations

In [3], the authors studied influence of chemical response on flash convective Hele-Shaw inflow of Rivlin-Ericksen fluid through a plate in vertical position. In[9], the authors considered Rivlin-Ericksen inflow on MHD fluid with capricious porousness. Rana [11],Sharma and Sunil [13], independently considered an impregnated dust patches Rivlin-Ericksen inflow liquid and its thermal insecurity in previous media. In[8], the author studied magneto-hydrodynamic unstable inflow of energy and species transport influence on Rivlin-Ericksen fluid along previous media with passable plate. The attained result reveals that viscoelastic material term decreases fluid instigation but the fluid heat and species weren't affected by the rises in the parameter.

1.1. Definitions and Notations

Definition 1.1. The left Riemann-Liouville fractional derivative of order α, $(0<\alpha<1)$ of continous function $\phi(t)$ is defined as follows [6,10]

$$D_a^{R.L,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{1}{(t-\xi)^\alpha}\phi(\xi)d\xi. \qquad (1.1)$$

Definition 1.2. The left Caputo fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$ of class $\mathcal{C}^1$ is defined as follows [6,10]

$$D_a^{c,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \qquad (1.2)$$

From the above definitions, it can be seen that fractional derivative represents a global property of the function, in contrast to the integer order derivative, which is a local property. The main reason for the success of applications fractional calculus is that these new fractional order models are more accurate than integer order models, i.e. there are more degree of freedom in the fractional order models.

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Definition 1.3. If $\phi(x)$ is a function of class $\mathcal{S}(R)$ space of rapidly decreasing functions defined on the real line, the Fourier transform of $\phi(x)$ is the function $\Phi(w)$ of the same class $\mathcal{S}(R)$ and defined by

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x) e^{iwx} dx.$$

On the other hand, we have the following inversion formula for the Fourier transform as below

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(w) e^{-ixw} dw.$$

Lemma 1.1. Let $\mathcal{F}\{\phi(x)\} = \Phi(w)$ then the following identities hold true.

1. $\mathcal{F}(\phi(x-\lambda)) = e^{i\lambda w}\Phi(w)$,
2. $\mathcal{F}(D^{C,\alpha}_{-\infty,x}\phi(x)) = (-iw)^{\alpha}\Phi(w)$,
3. $\mathcal{F}(D^{RL,\alpha}_{-\infty,x}\phi(x)) = (-i\omega)^{\alpha}\Phi(w)$.

Proof. See [1,10].
Lemma 1.2. The following identities hold true.

1. $\mathcal{F}(e^{-k|x|}) = \sqrt{\frac{2}{\pi}}\frac{k}{w^2+k^2}$,
2. $\mathcal{F}(e^{-\frac{x^2}{2}}) = e^{-\frac{w^2}{2}}$.

Proof. See[2]
Lemma 1.3. Convolution
Let $\mathcal{F}\{f(x)\} = F(\omega)$,$\mathcal{F}\{g(x)\} = G(\omega)$, then the following identity holds true.

1. $\mathcal{F}(\int_{-\infty}^{+\infty} f(x-\xi)g(\xi)d\xi) = \sqrt{2\pi}F(w)G(w)$.

Proof. See [2]

Example 1.1. Let us evaluate the Fourier transform of the function

$$\psi(x) = \frac{e^{-ax}\mathcal{H}(x)}{\sqrt[n]{x}}, a > 0.,$$

where $\mathcal{H}(.)$ stands for the Heaviside unit step function.
Solution.
By definition of the Fourier transform of the function $\psi(x)$, we have

$$\mathcal{F}[\psi(x); x \to w] = \frac{1}{\sqrt{2pi}} \int_{-\infty}^{+\infty} \frac{e^{-ax}\mathcal{H}(x)}{\sqrt[n]{x}} dx = .$$

$$. = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{iwx}\frac{e^{-ax}}{\sqrt[n]{x}} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{e^{-(a-iw)x}}{\sqrt[n]{x}} dx = \frac{\Gamma(1-\frac{1}{n})}{\sqrt{2\pi}(a-iw)^{1-\frac{1}{n}}}.$$

The use of the Fourier transform to obtain a form of solution to fractional partial differential equation is a very general technique. For certain problems, the integral representation obtained as a solution will be amenable to exact analysis, more often the method converts the original problem to the technical matter of evaluating a difficult integral. In a variety of areas of life sciences, such as population dynamics, mathematical modelling with delayed differential equations (DDE) is widely used for analysis and prediction. In these models, time delays can be related to the length of certain processes, such as the time between when a cell is infected and when new viruses are made, and so on [12].

Lemma 1.3. Let us consider the following fractional differential equation with retarded argument (delay)

$$D^{C,\alpha}_{-\infty,x}\phi(x) - \phi(x-\lambda) = f(x),$$

where $f(x)$ is chosen as definition 1.3. and $|D^{C,(\alpha k)} f(x)| \leq \frac{1}{2^{\alpha k}}$

$$B.C. \quad lim_{|x|\to+\infty}\phi(x) = 0, \quad 0 < \alpha < 1.$$

$$\phi(x) = -\sum_{k=0}^{+\infty} D^{C,(\alpha k)} f(x+\lambda(k+1)).$$

Note. The above sum is absolutely convergent.

Note. It should be noted that such a delayed fractional differential equation is not considered in the literature.

Proof. In view of the above Lemma 1.1. taking the Fourier transform of the fractional delayed differential equation above, yields;

$$(-iw)^{\alpha}\Phi(w) - e^{iwx}\Phi(w) = F(w),$$

after solving the transformed equation, we obtain

$$\Phi(w) = -\frac{F(w)}{e^{i\lambda w} - (-iw)^{\alpha}}.$$

or,

$$\Phi(w) = -\sum_{k=0}^{+\infty} e^{-i\lambda w}[(\frac{(-iw)^{\alpha}}{e^{i\lambda w}})^k]F(w) = -\sum_{k=0}^{+\infty} e^{-i\lambda w(k+1)}(-iw)^{\alpha k}F(w),$$

by taking the inverse Fourier transform, we get

$$\phi(x) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}[\sum_{k=0}^{+\infty} e^{-i\lambda w(k+1)}(-iw)^{\alpha k}F(w)]dw,$$

or,

$$\phi(x) = -\sum_{k=0}^{+\infty}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}e^{-i\lambda w(k+1)}(-iw)^{\alpha k}F(w)dw] = .$$

$$= -\sum_{k=0}^{+\infty}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(x+\lambda(k+1))w}(-iw)^{\alpha k}F(w)dw] = -\sum_{k=0}^{+\infty} D^{R.L,(\alpha k)} f(x+\lambda(k+1)).$$

Lemma 1.4. Let us consider the following system of fractional differential equations with delay

$$D^{R.L,\alpha}_{-\infty,x}\phi(x) - \psi(x-\lambda) = f(x),$$

$$D^{R.L,\alpha}_{-\infty,x}\psi(x) + \phi(x-\lambda) = g(x),$$

where $f(x), g(x)$ are chosen as Definition 1.3. and $|D^{C,(\alpha k)} f(x)| \leq \frac{1}{2^{\alpha k}}, \quad |D^{C,(\alpha k)} g(x)| \leq \frac{1}{2^{\alpha k}},$

$$B.C. \quad lim_{|x|\to+\infty}\phi(x) = lim_{|x|\to+\infty}\psi(x) = 0, \quad 0 < \alpha < 1,$$

then, the above system of fractional differential equation with delay, has the following formal solutions

$$\phi(x)=\sum_{k=0}^{+\infty}(-1)^K D^{R.L,(\alpha k)}[\cos(\frac{(k+1)\pi}{2})f(x+\lambda(k+1))+\sin(\frac{(k+1)\pi}{2})g(x+\lambda(k+1))],$$

$$\psi(x)=\sum_{k=0}^{+\infty}(-1)^k D^{R.L,(\alpha k)}[\cos(\frac{(k+1)\pi}{2})g(x+\lambda(k+1))-\sin(\frac{(k+1)\pi}{2})f(x+\lambda(k+1))].$$

Proof. Let us assume that $\phi(x)+i\psi(x)=\theta(x)$ and $f(x)+ig(x)=h(x)$, then we have

$$D^{R.L,\alpha}_{-\infty,x}\theta(x)+i\theta(x-\lambda)=h(x).$$

In view of the Lemma 1.1. Taking the Fourier transform of the above system of fractional differential equations with delay term wise, yields;

$$(-iw)^\alpha\Theta(w)+ie^{iw\lambda}\Theta(w)=H(w),$$

after solving the transformed equation, we obtain

$$\Theta(w)=\frac{H(w)}{ie^{i\lambda w}+(-iw)^\alpha},$$

or,

$$\Theta(w)=-i\sum_{k=0}^{+\infty}(-1)^k e^{-i\lambda w}[(\frac{(-iw)^\alpha}{ie^{i\lambda w}})^k]H(w)=-i\sum_{k=0}^{+\infty}(-1)^k e^{-i(\lambda w(k+1)+\frac{k\pi}{2})}(-iw)^{\alpha k}F(w),$$

taking the inverse Fourier transform term-wise, we get

$$\phi(x)+i\psi(x)=-i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ixw}[\sum_{k=0}^{+\infty}(-1)^k e^{-i(\lambda w(k+1)+\frac{k\pi}{2})}(-iw)^{\alpha k}H(w)]dw,$$

or,

$$\phi(x)+i\psi(x)=\sum_{k=0}^{+\infty}(-1)^k e^{-i(\frac{(k+1)\pi}{2})}D^{R.L,(\alpha k)}h(x+\lambda(k+1)),$$

from which we deduce that

$$\phi(x)=\sum_{k=0}^{+\infty}(-1)^k D^{R.L,(\alpha k)}[\cos(\frac{(k+1)\pi}{2})f(x+\lambda(k+1))+\sin(\frac{(k+1)\pi}{2})g(x+\lambda(k+1))],$$

$$\psi(x)=\sum_{k=0}^{+\infty}(-1)^k D^{R.L,(\alpha k)}[\cos(\frac{(k+1)\pi}{2})g(x+\lambda(k+1))-\sin(\frac{(k+1)\pi}{2})f(x+\lambda(k+1))].$$

Lemma 1.5. Let us consider the following fractional singular integral equation

$$\int_{-\infty}^{+\infty}e^{-|x-\xi|}D^{R.L,\alpha}_{-\infty,x}\phi(\xi)d\xi=e^{-x^2},$$

where $\phi(.)$ is an unknown differentiable function. Then the above integral equation has the following formal solution

$$\phi(x)=\mathcal{I}^{R.L,\alpha}_{-\infty,x}[e^{-\frac{x^2}{2}(1-\frac{x^2}{2})}].$$

Proof. In view of the Lemma 1.1., taking the Fourier transform of the singular integral equation above leads to.

$$\sqrt{2\pi}[\sqrt{\frac{2}{\pi}}\frac{1}{w^2+1}](-iw)^{\alpha}\Phi(w) = e^{-\frac{w^2}{2}},$$

after solving the transformed equation, we arrive at

$$(-iw)^{\alpha}\Phi(w) = \pi[e^{-\frac{w^2}{2}} + w^2 e^{-\frac{w^2}{2}}],$$

at this point, by taking the inverse Fourier transform, we obtain,

$$D^{R.L,\alpha}_{-\infty,x}\phi(x) = e^{-\frac{x^2}{2}}[1-\frac{x^2}{2}],$$

or

$$\phi(x) = \mathcal{I}^{R.L,\alpha}_{-\infty,x}[e^{-\frac{x^2}{2}(1-\frac{x^2}{2})}].$$

Definition 1.4. Laplace transform of the function $f(t)$ of exponential order is as follows

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st}f(t)dt := F(s). \qquad (1.3)$$

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by Bromwich integral as follows,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \qquad (1.4)$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Definition 1.5. The Laplace transform of Caputo fractional derivatives of order non-integer α, is as follows

$$\mathcal{L}[D^{c,\alpha}_{a}f(t)] = s^{\alpha}F(s) - s^{\alpha-1}f(0+) = s^{\alpha-1}[sF(s) - f(0+)] =$$
$$= s^{\alpha-1}\mathcal{L}[f^{'}(t)], 0 < \alpha < 1, \qquad (1.5).$$

and generally [10]

$$\mathcal{L}\{D^{C,\alpha}_{a}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{k=m-1} s^{\alpha-1-k}f^{k}(0+), m-1 < \alpha < m. \qquad (1.6)$$

The Laplace transform provides a useful technique for the solution of such fractional singular integro-differential equations.

Lemma 1.6. We have the following integral representation for the Bessel function of order zero $J_0(\alpha x)$[2],

$$J_0(\alpha x) = \frac{2}{\pi}\int_{\alpha x}^{+\infty}\frac{\sin\xi}{\xi}J_0(\xi-\alpha x)d\xi. \qquad (1.1)$$

Proof. Let us start with the trivial identity $\frac{\sin\xi}{\xi} = \int_0^1 \cos\xi y dy$, and by replacing it on the right-hand side, we have the following

$$R.H.S = \frac{2}{\pi}\int_{\alpha x}^{+\infty} J_0(\xi-\alpha x)[\int_0^1 \cos\xi y dy]d\xi.$$

The inner integral is absolutely integrable, so we can change the order of integration to get

$$R.H.S = \frac{2}{\pi}\int_0^1[\int_{\alpha x}^{+\infty}\cos\xi y J_0(\xi-\alpha x)d\xi]dy.$$

At this point, we make a change of variable in the inner integral as $\xi - \alpha x = u$, to obtain

$$R.H.S = \frac{2}{\pi}\int_0^1 [\int_0^{+\infty} \cos(\alpha x + u) y J_0(u) du] dy = \frac{2}{\pi}\int_0^1 \mathcal{R}e[\int_0^{+\infty} e^{-iy(\alpha x+u)} J_0(u) du] d\xi,$$

or,

$$R.H.S = \frac{2}{\pi}\int_0^1 \mathcal{R}e[e^{-i\alpha xy}\int_0^{+\infty} e^{-iyu} J_0(u) du] dy.$$

By contrast, using the Laplace transform of the zero-order Bessel function, we have

$$\int_0^{+\infty} e^{-su} J_0(u) du = \frac{1}{\sqrt{s^2+1}}.$$

By setting $s = iy$ in the above relation, we obtain

$$\int_0^{+\infty} e^{-iyu} J_0(u) du = \frac{1}{\sqrt{1-y^2}},$$

therefore,

$$R.H.S = \frac{2}{\pi}\int_0^1 Re[\frac{e^{-i\alpha xy}}{\sqrt{1-y^2}}] dy = \frac{2}{\pi}\int_0^1 \frac{\cos(\alpha x) y}{\sqrt{1-y^2}} dy = \frac{2}{\pi}\int_0^{\alpha} \frac{\cos xw}{\sqrt{\alpha^2 - w^2}} dw = J_0(\alpha x).$$

Note. In the last integral, we made a change of variable $\alpha y = w$. Finally, in (1.1) if we make a change of variable $\eta = \xi - \alpha x$, after simplifying we arrive at

$$J_0(\alpha x) = \frac{2}{\pi}\int_0^{+\infty} \frac{\sin(\eta + \alpha x)}{\eta + \alpha x} J_0(\eta) d\eta. \tag{1.2}$$

Lemma 1.7. Let $L\{f(t)\} = F(s)$ then, the following identity holds.

$$\mathcal{L}^{-1}(F(s^{\alpha})) = \frac{1}{\pi}\int_0^{\infty} f(u)[\int_0^{\infty} e^{-tr - ur^{\alpha}\cos\alpha\pi} \sin(ur^{\alpha}\sin\alpha\pi) dr] du.$$

Proof. See [2].

Lemma 1.8. (Gross-Levi) Let us assume that $\mathcal{L}[\phi(t); t \to s] = \Phi(s)$ and $s = 0$ is a branch point of $\Phi(s)$, then we have the following inversion formula

$$\phi(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} \mathcal{I}m[lim_{\theta \to -\pi} \Phi(\xi e^{i\theta})] d\xi.$$

Proof. See[2]

Example 1.5. Let us solve the following singular integral equation with trigonometric kernel

$$\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi})}{\sqrt{\pi\xi}} \psi(\xi) d\xi = Erfc(\frac{a}{2\sqrt{t}}).$$

Solution. Taking the Laplace transform of both sides of the above singular integral equation leads to

$$\frac{1}{s\sqrt{s}}\Psi(\frac{1}{s}) = \frac{e^{-a\sqrt{s}}}{s},$$

from which we deduce that

$$\Psi(s) = \frac{1}{\sqrt{s}} e^{-\frac{a}{\sqrt{s}}},$$

since $s=0$ is a branch point of the $\Psi(s)$, we may use Gross-Levi inversion formula for the inverse transform of $\Psi(s)$, that is

$$\mathcal{L}^{-1}[\Psi(s); s\to t] = \psi(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m[\frac{1}{\sqrt{re^{-i\pi}}}e^{-\frac{a}{\sqrt{re^{-i\pi}}}}]dr = \frac{a^2}{\pi}\int_0^{+\infty} e^{-\frac{a^2 t}{\xi^2}}\cos(\xi)\frac{d\xi}{\xi^2}.$$

Lemma 1.8. Let us consider the following fractional differential equation with non-constant coefficient

$$D_{0,x}^{C,2\alpha}\phi(t) - tD_{0,x}^{C,\alpha}\phi(t) = 0. \quad \phi(0) = \phi^{'}(0) = 0, \quad 0.5 < \alpha < 1, \quad t > 0.$$

The above fractional differential equation has the following solution

$$\phi(t) = \frac{c_0}{\pi}\int_0^{+\infty} r^{-\alpha}e^{-(tr+\frac{r^{\alpha+1}}{\alpha+1}\cos(\pi(\alpha+1))}\sin(\pi\alpha + \frac{r^{\alpha+1}}{\alpha+1}\sin(\pi(\alpha+1))dr.$$

Proof. Let us assume that $\mathcal{L}[\phi(t)] = \Phi(s)$, by taking the Laplace transform of the fractional differential equation term-wise and in view of the relations (1.4) and (1.5), we have

$$s^{2\alpha}\Phi(s) + s^{\alpha}\Phi^{'}(s) + \alpha s^{\alpha-1}\Phi(s) = 0,$$

from which, we deduce that

$$\frac{\Phi^{'}(s)}{\Phi(s)} = -s^{\alpha} - \frac{\alpha}{s},$$

after integration and simplifying, we obtain

$$\Phi(s) = \frac{c_0}{s^{\alpha}}e^{-\frac{s^{\alpha+1}}{\alpha+1}},$$

at this point, using the Gross-Levi inversion formula and the fact that $s=0$ is a branch point, we arrive at

$$\phi(t) = \frac{c_0}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m[lim_{\alpha\to\pi}\Phi(re^{i\alpha})]dr,$$

which leads to

$$\phi(t) = \frac{c_0}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m[\frac{e^{-(re^{-i\pi})^{\alpha+1}}}{(re^{-i\pi})^{\alpha+1}}]dr,$$

finally, we have

$$\phi(t) = \frac{c_0}{\pi}\int_0^{+\infty} r^{-\alpha}e^{-(tr+\frac{r^{\alpha+1}}{\alpha+1}\cos(\pi(\alpha+1))}\sin(\pi\alpha + \frac{r^{\alpha+1}}{\alpha+1}\sin(\pi(\alpha+1))dr.$$

1.2. Solution to non-homogenous impulsive Kd.V via the Joint Laplace-Fourier transforms

In this section, the author implemented the joint Laplace-Fourier transform to construct exact solution for a variant of the non-homogeneous time fractional Kd.V equation[14]. The time fractional linearized non-homogeneous Kd.V equation, for the free surface elevation $u(x,t)$ is an invicide water at constant depth h, where $\lambda = \sqrt{gh}$, shalow water speed and $\nu = \frac{\lambda h^2}{6}$, are attracting many researchers around the world and a great deal of work has already been done in some of these equation.

$$\begin{array}{l} D_{0,t}^{C,\alpha}u_t + bu + \lambda u_x + \nu u_{xxx} = \sqrt{2\pi}\beta\delta(t-\tau_0)g(x) \\ u(x,0) = \phi(x), \end{array}$$

then the above equation has the following formal solution,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}[\Phi(w)E_{\alpha,1}(-(ikw - i\gamma w^3 + \alpha)t^{\alpha}) + .$$

$$G(w)t^{\alpha-1}E_{\alpha,\alpha}(-(ikw-i\gamma w^3+\alpha)(t-\tau_0)^\alpha)]dw.$$

Solution. By taking the joint Laplace-Fourier transform of equation and using boundary condition, we get the following transformed equation

$$\hat{\bar{U}}(w,s)=\frac{s^{\alpha-1}\Phi(w)}{s^\alpha+(ikw-i\gamma w^3+b)}+\frac{\beta e^{-\tau_0 s}G(w)}{s^\alpha+(ikw-i\gamma w^3+b)},$$

for the sake of simplicity, let us assume that $\theta=ikw-i\gamma w^3+\alpha$, and using inverse Laplace transform of transformed equation to obtain

$$\hat{U}(w,t)=\mathcal{L}^{-1}\{\frac{s^{\alpha-1}\Phi(w)}{s^\alpha+\theta}+\frac{\beta e^{-\tau_0 s}G(w)}{s^\alpha+\theta};s\to t\},$$

at this point, inverting Fourier transform to get

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)[\mathcal{L}^{-1}[\frac{s^{\alpha-1}\Phi(w)}{s^\alpha+\theta}+\frac{\beta e^{-\tau_0 s}G(w)}{s^\alpha+\theta}]dw,$$

or

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)\Phi(w)[\mathcal{L}^{-1}[\frac{s^{\alpha-1}}{s^\alpha+\theta}]]dw+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ixw)G(w)[\mathcal{L}^{-1}[\frac{\beta e^{-\tau_0 s}}{s^\alpha+\theta}]]dw.$$

At this stage, in view of the following Laplace transform identity [10]

$$\mathcal{L}^{-1}[\frac{k!s^{\alpha-\beta}}{s^\alpha+\theta}]=t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(-\theta t^\alpha),$$

where$E^{(k)}_{\alpha,\beta}(-\theta t^\alpha)$stands for the Mittag-Leffler function with 3-parameters [6,10], from which we have

$$\mathcal{L}^{-1}[\frac{s^{\alpha-1}}{s^\alpha+\theta}]=E_{\alpha,1}(-\theta t^\alpha),$$

$$\mathcal{L}^{-1}[\frac{\beta e^{-\tau_0 s}}{s^\alpha+\theta}]=t^{\beta-1}E_{\alpha,\alpha}(-\theta(t-\tau_0)^\alpha),$$

finally by setting back $\theta=ikw-i\gamma w^3+\alpha$, after simplifying, we arrive at

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ixw}[\Phi(w)E_{\alpha,1}(-(ikw-i\gamma w^3+\alpha)t^\alpha)+G(w)t^{\beta-1}E_{\alpha,\alpha}(-(ikw-i\gamma w^3+\alpha)(t-\tau_0)^\alpha)]dw.$$

Note: This kind of time fractional impulsive Kd.V is not considered in the literature.

2. Main Result : Rivlin-Ericksen Fluid

Since the beginning of Calculus, fractional integrals have been considered, but have not been truly worked on until recently. This field has gained traction as some physical systems have been found to be indescribable using traditional methods. By defining and utilizing Riemann-Liouville fractional integrals and derivatives, Caputo fractional derivatives, and others, the process of solving these physical systems is made possible. Extending classical mechanics to a fractional case is an example of fractional calculus. In the present work, we will focus on non-Newtonian fluid using partial fractional derivatives and utilizing this new definition to solve the problem of the generalized fractional Rivlin-Ericksen fluid.

Definition 2.1. NON-NEWTONIAN FLUID
Non-Newtonian fluid does not obey Newton's viscosity law, which states that viscosity should remain constant regardless of stress.The viscosity and flow of non-Neowtonian fluids can change from liquid to

solid, if the force is applied. For instance, ketchup is a non-Newtonian fluid because it becomes more fluid when shaken.They do not contain constant viscosity. For example, butter and mayonnaise are not Newtonian fluids. Recently, interest in non-Newtonian flows through porous media has increased considerably, largely because of demands from fields as diverse as the chemical and oil industries [4,5,7,8]. Noting that, the dilatant fluid is composed of a cornstarch-water mixture with 40 percent water and 60 percent cornstrach of total weight, it is commonly known as oobleck. Oobleck is commonly used to display the properties of non-Newtonian fluids. You could easily run/walk over a big container of oobleck because of the forces applied by your weight causes the oobleck under your feet to become solid. If we remove the force, the oobleck will return to its original state of liquid.

In recent years, a growing number of research works done by many researchers from various fields of engineering and science deal with dynamical systems described by equations of fractional order which means equations involving derivatives and integrals of fractional order. In the sequel, we provide an exact solution to a fractional generalization to stokes first problem for Rivlin-Ericksen fluid of second grade in a porous half space which is not considered in the literature yet.

Problem 2.1. Suppose we wish to solve a fractional generalization for Rivlin-Ericksen fluid of second grade in a porous half space formulated as follows. To the best of author's knowledge, this kind of fractional model with non-homogeneous boundary condition is not considered in the literature yet.

$$D_t^{C,\alpha}\phi(x,(t)+\lambda\phi(x,t)=\phi_{xx}(x,t)+\beta D_t^{C,\alpha}\phi_{xx}(x,(t),(2.1)$$

$$I.C: \qquad \phi(x,0)=\beta\phi_{xx}(x,0), \quad (2.2).$$

$$B.C. \qquad \phi(0,t)=\psi(t), \quad (2.3)$$

$$lim_{x\to+\infty}|\phi(x,t)|<+\infty, \qquad (2.4)$$

$$0<x<+\infty, 0<t<+\infty, \lambda,\beta>0,<\alpha<1.$$

Solution. Let us take the Laplace transform of the fractional partial differential equation (2.1) and using (2.2), we get the following second order differential equation

$$(1+\beta s^{\alpha})\frac{d^2U}{dx^2}-(s^{\alpha}+\lambda)U(x,s)=0, \quad (2.5).$$

$$U(x,s)=\mathcal{L}[u(x,t);x\to s],$$

solving equation (2.5) leads to

$$U(x,s)=c_1e^{-x\sqrt{\frac{s^{\alpha}+\lambda}{1+\beta s^{\alpha}}}}+c_2e^{x\sqrt{\frac{s^{\alpha}+\lambda}{1+\beta s^{\alpha}}}}, \qquad (2.6).$$

Notice that, taking account of the condition (2.4) gives $c_2=0$, it follows that

$$U(x,s)=c_1e^{-x\sqrt{\frac{s^{\alpha}+\lambda}{1+\beta s^{\alpha}}}} \qquad (2.7),$$

at this stage, using boundary condition (2.3), leads to $c_1=\mathcal{L}[\psi(t)]=\Psi(s)$, from which we get

$$U(x,s)=\Psi(s)e^{-x\sqrt{\frac{s^{\alpha}+\lambda}{1+\beta s^{\alpha}}}}, \qquad (2.8),$$

taking the inverse Laplace transform and in view of the convolution theorem for the Laplace transform, we have

$$u(x,t)=[\mathcal{L}^{-1}\Psi(s)]*[\mathcal{L}^{-1}e^{-x\sqrt{\frac{s^{\alpha}+\lambda}{1+\beta s^{\alpha}}}}],$$

in order to evaluate $[\mathcal{L}^{-1}e^{-x\sqrt{\frac{s^\alpha+\lambda}{1+\beta s^\alpha}}}]$, let us assume that

$$G(s^\alpha)=e^{-x\sqrt{\frac{s^\alpha+\lambda}{1+\beta s^\alpha}}},\quad \to G(s)=e^{-x\sqrt{\frac{s+\lambda}{1+\beta s}}},$$

we need to evalulate the residues of $G(s)$ at two branch points $s=-\lambda$ and $s=-\frac{1}{\beta}$ by Gros-Levi theorem as below

1. At branch point $s=-\lambda$, we have

$$s+\lambda=re^{i\phi},\quad b_1=\frac{1}{\pi}\int_0^{+\infty}e^{-tr}Im[lim_{\phi\to-\pi}e^{-x\sqrt{\frac{re^{i\phi}+\lambda}{1+\beta re^{i\phi}}}}]dr.$$

$$b_1=\frac{e^{t\lambda}}{\pi}\int_0^{+\infty}e^{-t\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})d\xi.$$

2. At branch point $s=-\frac{1}{\beta}$, we have

$$s+\frac{1}{\beta}=re^{i\phi},\quad b_2=\frac{1}{\pi}\int_0^{+\infty}e^{-tr}Im[lim_{\phi\to-\pi}e^{-\frac{x}{\sqrt{\beta}}\sqrt{\frac{re^{i\phi}-\beta+\lambda}{\beta re^{i\phi}}}}]dr,$$

or,

$$b_2=\frac{1}{\pi}\int_0^{+\infty}e^{-tr}\mathcal{I}m[e^{-\frac{x}{\beta}\sqrt{\frac{\beta r+1-\beta\lambda}{r}}}]dr=0,\qquad 0<\beta\lambda\leq 1,$$

in view of the Lemma 1.7. we conclude that

$$\mathcal{L}^{-1}(G(s^\alpha))=\frac{1}{\pi}\int_0^{\infty}\frac{e^{u\lambda}}{\pi}[\int_0^{+\infty}e^{-u\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})[\int_0^{\infty}e^{-tr-ur^\alpha\cos\alpha\pi}\sin(ur^\alpha\sin\alpha\pi)dr]d\xi]du.$$

We have the solution to probelm is as follows

$$u(x,t)=\frac{1}{\pi^2}\int_0^t\psi(t-\eta)[\int_0^{+\infty}e^{u\lambda}[\int_0^{+\infty}e^{-u\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})[\int_0^{\infty}e^{-\eta r-ur^\alpha\cos\alpha\pi}\sin(ur^\alpha\sin\alpha\pi)dr]d\xi]du]d\eta.$$

Let us consider the special case $\beta=\lambda=1,\alpha=0.5$ and $\psi(t)=\mathcal{H}(t)$, then we obtain the following boundary value problem

$$D_t^{C,0.5}\phi(x,(t)+\phi(x,t)=\phi_{xx}(x,t)+D_t^{C,0.5}\phi_{xx}(x,(t),(2.9)$$

$$I.C:\qquad \phi(x,0)=\phi_{xx}(x,0),\quad (2.10).$$

$$B.C.\qquad \phi(0,t)=\mathcal{H}(t),\quad (2.11)$$

$$lim_{x\to+\infty}|\phi(x,t)|<+\infty,\qquad (2.12)$$

$$0<x<+\infty,0<t<+\infty,\lambda=\beta=1,\alpha=0.5.$$

with the following solution

$$u(x,t)=\frac{1}{\pi^2}\int_0^t[\int_0^{+\infty}e^{u\lambda}[\int_0^{+\infty}e^{-u\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})[\int_0^{\infty}e^{-\eta r}\sin(u\sqrt{r})dr]d\xi]du]d\eta.$$

After evaluation of the inner integral and simplifying, we arrive at

$$u(x,t)=\frac{1}{2\pi\sqrt{\pi}}\int_0^t \eta^{-\frac{3}{2}}[\int_0^{+\infty} u^3 e^{u\lambda}[\int_0^{+\infty} e^{-u\xi-\frac{u^2}{4\eta}}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})d\xi]du]d\eta,$$

changing the order of integration and evaluation of the inner integral leads to

$$u(x,t)=\frac{1}{2\pi\sqrt{\pi}}\int_0^{+\infty} u^3 e^{u\lambda}[\int_0^{+\infty} e^{-u\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})[\int_0^t \eta^{-\frac{3}{2}}e^{-\frac{u^2}{4\eta}}d\eta]d\xi]du,$$

after simplifying we have

$$u(x,t)=\frac{1}{\pi}\int_0^{+\infty} u^2 e^{\lambda u}Erfc(\frac{u}{2\sqrt{t}})[\int_0^{+\infty} e^{-u\xi}\sin(x\sqrt{\frac{\xi-\lambda}{1+\beta\xi}})d\xi]du,$$

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