Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–28.
ISSN-0037-8712
doi:10.5269/bspm.76053

Some Symmetric Holomorphic Function Subfamilies and Their Hankel Determinants Involving the $(\mathfrak{p},\mathfrak{q})$-Derivative Operator

Shadea Ahmad Abdullah and Pishtiwan Othman Sabir *

ABSTRACT: Coefficient function estimates play a vital role in building a mathematical framework for nonlocal problems, providing the groundwork necessary to develop innovative methods that enhance the precision and efficiency of engineering applications in applied science fields. In this study, we establish coefficient bounds for normalized holomorphic functions of the form

$$f(z) = z + \sum_{\ell=1}^{\infty} b_{\jmath\ell+1} z^{\jmath\ell+1}, \ \left(z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}, \ \jmath \in \mathbb{N} := \{1, 2, 3, \cdots\}, \ b_{\jmath\ell+1} \in \mathbb{C}\right),$$

that belongs to some subfamilies of $\jmath$-fold symmetric functions defined by the $(\mathfrak{p},\mathfrak{q})$-derivative operator. The derived estimates pertain to the bounds of $|b_{\jmath+1}|$, $|b_{2\jmath+1}|$, and $|b_{3\jmath+1}|$, along with the Fekete–Szegö funtional $\left|b_{2\jmath+1} - \mu b_{\jmath+1}^2\right|$. Additionally, we obtain the upper bound of the second Hankel determinant $\left|b_{\jmath+1}b_{3\jmath+1} - b_{2\jmath+1}^2\right|$, which serves as an important indicator of the relationship between the coefficients. Python 3.12 (2023) was used for graphical illustrations and verification to confirm the accuracy of our comprehensive theoretical results.

Key Words: Holomorphic function, univalent function, symmetric bi-univalent function, fractional calculus, Fekete-Szegö functional, Hankel determinant.

Contents

1. Motivation

A significant and fascinating area in complex analysis involves examining the geometric properties of holomorphic functions within $\mathbb{U}$. Properties such as starlikeness and convexity are closely associated with the coefficients of their Taylor-Maclaurin series expansion. Although the graph of an holomorphic function cannot be directly depicted, its image set exhibits a distinct and well-defined geometric structure.

The study of univalent functions ($\mathbb{UF}$) plays a fundamental role in geometric function theory ($\mathbb{GFT}$). Over the years, this field has expanded significantly, leading to new research directions and numerous important findings and applications. Its foundations are closely linked to Bieberbach's famous conjecture. Bieberbach demonstrated that for any $\mathbb{UF}$ f with a normalized Taylor-Maclaurin series, the bound $|b_2| \leqslant 2$ holds. Durin [1] further conjectured that, in general, the inequality $|b_\ell| \leqslant \ell, \ \ell \in \mathbb{N}\backslash\{1\}$ holds. The conjecture stimulated further research on the coefficient bounds of univalent (and bi-univalent) functions, leading to significant advancements in $\mathbb{GFT}$.

The study of bi-univalent functions ($\mathbb{BUF}$) was initiated by Lewin [2], who established the bound $|b_2| < 1.51$, which was later refined by Brannan and Clunie [3]. Furthermore, Netanyahu [4] demonstrated that $|b_2| \leqslant 4/3$, while Tan [5] further improved this estimate to $|b_2| \leqslant 1.485$. A significant advancement in

* Corresponding author. Email: pishtiwan.sabir@univsul.edu.iq

Submitted March 10, 2025. Published August 10, 2025
2010 *Mathematics Subject Classification:* 26A51, 30C45, 30C50, 30C80.

this area was made by Srivastava et al. [6], who investigated specific subfamilies of $\mathbb{BUF}$. Their research established bounds on the first, $|b_2|$, and second $|b_3|$, initial Taylor-Maclaurin coefficient functions of $f(z) = z + \sum_{\ell=2}^{\infty} b_\ell z^\ell$. The problem of determining coefficient bounds in this setting remains an active area of research. In $\mathbb{GFT}$, finding coefficient bounds for $|b_\ell|$, where $\ell \in \mathbb{N}\backslash\{1, 2, 3\}$, remains an unresolved challenge. More recently, researchers have investigated $\mathbb{BUF}$ concerning Taylor-Maclaurin coefficient estimates (see [7,8,9,10]).

Quantum calculus (Jackson calculus or briefly, $\mathfrak{q}$-calculus) is a mathematical discipline that derives $\mathfrak{q}$-analogous results without relying on limits. Unlike classical calculus, where derivatives are defined through the limit process of the difference quotient as the input change approaches zero, $\mathfrak{q}$-calculus operates without this limiting procedure. Initially introduced by Jackson [11,12], this field has since garnered significant interest from researchers in both pure and applied mathematics. Jackson calculus finds applications in multiple fields. For example, within number theory, it introduces concepts such as $\mathfrak{q}$-(real and complex numbers). In combinatorics, fundamental ideas include $\mathfrak{q}$-(binomial, Taylor, and hypergeometric series). The $\mathfrak{q}$-analogues involve the $\mathfrak{q}$-(gamma, zeta, beta, exponential, and trigonometric function) in the context of special functions. Significant topics in calculus include $\mathfrak{q}$-(fractional, differentiation, and integration). Although substantial progress has been made in this domain, many open problems persist, ensuring that quantum calculus remains an active and evolving field of study.

Sadjang [13] studied various properties of the $(\mathfrak{p}, \mathfrak{q})$-derivative and $(\mathfrak{p}, \mathfrak{q})$-integration, while also introducing two appropriate polynomial bases for the $(\mathfrak{p}, \mathfrak{q})$-derivative and analyzing their key characteristics. George et al. [14] offer essential contributions to the study of $[\mathfrak{p}, \mathfrak{q}]$-difference equations, with a particular emphasis on demonstrating the existence of solutions for initial value problems. Their application of this theory to the model of vibrating eardrums highlights an interdisciplinary approach, bridging complex mathematical concepts with tangible engineering problems.

For each $f \in \mathcal{S}$, the transformation defined by

$$f(z) = \sqrt[\jmath]{f\left(z^\jmath\right)},\ \jmath \in \mathbb{N},$$

produces a function belonging to $\mathbb{UF}$, which maps $\mathbb{U}$ onto a domain characterized by $\jmath$-fold symmetry. In their work [15], Srivastava et al. introduced the family of $\jmath$-fold symmetric $\mathbb{BUF}$ functions, denoted by $\bar{\Sigma}_\jmath$. Their study led to several important results, including the observation that every function in $\bar{\Sigma}$ gives rise to an element of $\bar{\Sigma}_\jmath$ for any $\jmath \in \mathbb{N}$. Recent investigations have highlighted compelling reasons to examine new general subfamilies of $\bar{\Sigma}_\jmath$ by incorporating the $(\mathfrak{p}, \mathfrak{q})$-derivative operator (see [16]).

The Fekete–Szegö problem holds significant importance in $\mathbb{GFT}$, as it concerns the determination of bounds for the functional $\left|b_3 - \mu b_2^2\right|$, where the parameter μ is a real or complex number. Its origins can be traced to the research of Fekete and Szegö [17], who disproved the Littlewood-Paley conjecture. Biagi et al. [18] generalize the Hong-Krahn-Szegö-type inequality, traditionally applicable to purely local or nonlocal contexts, to hybrid operators that integrate local and nonlocal elements. Using variational methods and functional analysis, the authors derive precise bounds for the first eigenvalue of related operators under volume constraints. Several studies have investigated various $\mathbb{BUF}$ subfamilies using diverse mathematical concepts. Amini et al. [19] examine the Fekete-Szegö problem for a specific subfamily of $\mathbb{BUF}$ by extending classical results; their research provides refined coefficient estimates, which contribute to a deeper understanding of $\mathbb{GFT}$ within the framework of nonlinear analysis and incorporates $\mathfrak{q}$-analogs of logarithmic functions, offering a novel perspective on the structural properties of these functions. Alsoboh et al. [20] investigate the interaction between $\mathbb{BUF}$ and special polynomial familyes associated with $\mathfrak{q}$-calculus, leading to new bounds and insights into the behavior of these functions. Jadhav et al. [21] focus on coefficient estimates for $\jmath$-fold symmetric holomorphic $\mathbb{BUF}$ and provide bounds on initial coefficients and the Fekete-Szegö problem that contribute to ongoing developments in complex function theory specifically in $\mathbb{GFT}$. Sabir et al. [22] establish new convexity conditions that present a study on Bazilevič-type close-to-convex functions and analyze functions behavior and Fekete-Szegö inequalities for specific subfamilies of $\mathbb{UF}$ and $\mathbb{BUF}$. Further advancements by Srivastava et al. [23] examined some symmetric $\mathbb{BUF}$ subfamilies and their initial coefficient bounds for functions of the form $f(z) = z + \sum_{\ell=1}^{\infty} b_{\jmath\ell+1} z^{\jmath\ell+1}$.

The Hankel determinant for f was initiated by Fekete and Szegö [17], play a crucial role in $\mathbb{GFT}$ who specifically investigated the determinant

$$\mathfrak{h}_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix}.$$

Subsequent research on these determinants was conducted by Pommerenke [24], and followed by Noonan and Thomas [25]. They generalized the determinant as follows:

$$\mathfrak{h}_k(n) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+k-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+k} \\ & \vdots & & \\ b_{n+k-1} & b_{n+k} & \cdots & b_{n+2k-2} \end{vmatrix},$$

where $b_1 = 1$ and $n, k \in \mathbb{N}$. Hankel determinants have diverse applications, particularly in random matrix theory and the study of orthogonal polynomials. For an in-depth discussion, see, for instance, the recent contributions by Min and Chen [26]. Additionally, Sabir [27] established sharp bounds for Toeplitz determinants related to bilinear transformations. In contrast, Al-Ameedee et al. [28] investigated $\mathfrak{h}_2(2)$ for specific subfamilies within the function family $\bar{\Sigma}$.

This study introduces new general subfamilies of $\bar{\Sigma}_\jmath$ by utilizing the $(\mathfrak{p}, \mathfrak{q})$-derivative operator. Section 2 provides a comprehensive background on the topic. Sections 3 and 4, establish coefficient bounds for $|b_{\jmath+1}|$, $|b_{2\jmath+1}|$, and $|b_{3\jmath+1}|$ for functions belonging to the subfamilies $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\alpha, \lambda)$ and $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\beta, \lambda)$. Additionally, Fekete–Szegö inequalities $\mathfrak{h}_2(1)$, and the second Hankel estimates $\mathfrak{h}_2(2)$, for these subfamilies are derived. Section 5 presents several consequences of our results. As an open problem, our findings inspire further research into additional properties, such as determining bounds for $|b_{\jmath\ell+1}|$ for $\ell \geqslant 4$ and investigating bounds for the third and fourth Hankel determinants.

2. Background

A widely recognized subfamily of holomorphic functions, denoted by $\mathcal{A}$, consists of functions represented as

$$f(z) = z + \sum_{\ell=2}^{\infty} b_\ell z^\ell, \tag{2.1}$$

which are holomorphic in $\mathbb{U}$ and satisfy $f(0) = 0$ and $f'(0) = 1$. Within this family, the subset $\mathcal{S} \subseteq \mathcal{A}$ includes all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$.

Additionally, the subfamily $\mathcal{H}$ is composed of functions of the form

$$h(z) = 1 + \sum_{\ell=1}^{\infty} h_\ell z^\ell, \tag{2.2}$$

which are holomorphic in $\mathbb{U}$ and satisfy $\mathrm{Re}(h(z)) > 0$ for every $z \in \mathbb{U}$.

For $0 \leqslant \alpha < 1$ and $0 < \beta \leqslant 1$, the familyes of starlike functions $x^*(\alpha)$, convex functions $y^*(\alpha)$, strongly starlike functions $x^*(\beta)$, and strongly convex functions $y^*(\beta)$ are respectively characterized by the following conditions (see [1]) for all $z \in \mathbb{U}$:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\beta}{2}, \quad \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\beta}{2}.$$

Each of these conditions defines a distinct subfamily within $\mathbb{U}$, describing various geometric properties such as starlikeness and convexity.

The Koebe 1-quarter theorem [1] asserts that the image of $\mathbb{U}$ under a $\mathbb{UF}$ contains a disk of radius at least $1/4$, ensuring a significantly lower bound for the range of these functions. This establishes that every $\mathbb{UF}$ $f \in \mathcal{S}$ has an inverse function f^{-1} given by:

$$f^{-1}(w) := \mathfrak{g}(w) = w - b_2 w^2 + \left(2b_2^2 - b_3\right) w^3 - \left(5b_2^3 - 5b_2 b_3 + b_4\right) w^4 + \cdots . \tag{2.3}$$

Brannan and Taha [29] studied specific subfamilies of $\bar{\Sigma}$ that correspond to the well-established function familyes $x^*(\alpha)$ and $y^*(\alpha)$ of order α $(0 \leqslant \alpha < 1)$. A function $f \in \bar{\Sigma}$ is familyified as a member of the subfamily $x^*_{\bar{\Sigma}}(\alpha)$ of bi-starlike functions if both f and its inverse exhibit starlikeness of order α. Similarly, f belongs to the subfamily $y^*_{\bar{\Sigma}}(\alpha)$ of bi-convex functions of order α if both f and its inverse satisfy the convexity condition of the same order. Additionally, the same authors extended this familyification to define strongly bi-starlike functions $x^*_{\bar{\Sigma}}(\beta)$ and strongly bi-convex functions $y^*_{\bar{\Sigma}}(\beta)$ for $0 < \beta \leqslant 1$, maintaining a parallel structure to the previous familyifications.

Over the past decade, research has focused on studying various subfamilies of $\bar{\Sigma}$, particularly in establishing initial coefficient bounds for specialized subfamilies of this function family. Farsin and Aouf [30] introduced and analyzed two new subfamilies of $\bar{\Sigma}$, while Deniz [31] investigated four subfamilies of $\bar{\Sigma}$, investigating coefficient bounds for $|b_2|$ and $|b_3|$ in these subfamilies. Sivasubramanian [32] examined functions for which both f and its inverse are close-to-convex. The concept of a strongly bi-close-to-convex function of order β $(0 < \beta \leqslant 1)$ was introduced. A function belongs to this family if there exist $\varphi, \vartheta \in y^*_{\bar{\Sigma}}(0) \equiv y^*_{\bar{\Sigma}}$ such that:

$$f \in \bar{\Sigma}, \ \left|\arg\left(\frac{f'(z)}{\varphi'(z)}\right)\right| < \frac{\pi\beta}{2}, \ \text{and} \ \left|\arg\left(\frac{\mathfrak{g}'(w)}{\vartheta'(w)}\right)\right| < \frac{\pi\beta}{2}.$$

Furthermore, Deniz et al. [33] investigated the coefficient bounds for $\mathfrak{h}_2(2)$ within the subfamilies $x^*_{\bar{\Sigma}}(\alpha)$ and $y^*_{\bar{\Sigma}}(\alpha)$, with Orhan et al. [34] later extending these results.

A function is familyified as $\jmath$-fold symmetric [35] if it can be represented in the following normalized form:

$$f(z) = z + \sum_{\ell=1}^{\infty} b_{\jmath\ell+1} z^{\jmath\ell+1}, \ \ (z \in \mathbb{U}, \ \jmath \in \mathbb{N}). \tag{2.4}$$

The subfamily of $\mathbb{UF}$ in $\mathbb{U}$ that exhibit $\jmath$-fold symmetry and are expressed through the series expansion (2.4) is denoted as $\mathcal{S}_\jmath$. In particular, when $\jmath = 1$, the subfamily $\mathcal{S}_1$ coincides with $\mathcal{S}$, meaning that functions in this family are one-fold symmetric. Moreover, the inverse function of (2.4), denoted by $\mathfrak{g}(w)$, has the following series representation:

$$\begin{aligned}\mathfrak{g}(w) = w - b_{\jmath+1} w^{\jmath+1} + \left[(\jmath+1) b_{\jmath+1}^2 - b_{2\jmath+1}\right] w^{2\jmath+1} \\ - \left[\frac{1}{2}(\jmath+1)(3\jmath+2) b_{\jmath+1}^3 - (3\jmath+2) b_{\jmath+1} b_{2\jmath+1} + b_{3\jmath+1}\right] w^{3\jmath+1} + \cdots .\end{aligned} \tag{2.5}$$

Several recent studies have concentrated on determining coefficient bounds for various subfamilies of $\bar{\Sigma}_\jmath$. Srivastava et al. [36] study coefficient estimates for specific $\jmath$-fold symmetric subfamilies of $\mathbb{BUF}$ and provide new insights by refining known bounds and developing techniques that enhance the understanding of their holomorphic properties. Abd and Wanas [37] extend some results by establishing new inequalities and constraints on $\mathbb{BUF}$ and investigate bounds for a novel subfamily of $\jmath$-fold symmetric functions associated with Bazilevič convex functions. Sabir et al. [38] introduce $\jmath$-fold symmetric holomorphic function subfamilies and address coefficient estimate problems that improved bounds by employing advanced mathematical techniques. Al-Rawashdeh [39] analyze coefficient estimates and the Fekete-Szegö inequality for some $\mathbb{BUF}$ subfamilies by utilizing a generalized operator with Gegenbauer polynomials. Further developments on Hankel determinants have been studied by various authors (see [40,41,42]).

For $0 < \mathfrak{q} < 1$, the $\mathfrak{q}$-derivative operator [43] for a function f of the from (2.1) is defined as follows:

$$D_{\mathfrak{q}} f(z) = \begin{cases} \frac{f(qz)-f(z)}{(\mathfrak{q}-1)z}, & z \in \mathbb{U}^* := \mathbb{U}\backslash\{0\}, \\ f'(0), & z = 0. \end{cases}$$

Or, equivalently,

$$D_{\mathfrak{q}} f(z) = 1 + \sum_{\ell=2}^{\infty} [\ell]_{\mathfrak{q}} b_\ell z^{\ell-1},$$

where

$$[\ell]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^\ell}{1-\mathfrak{q}} = 1 + \mathfrak{q} + \cdots + \mathfrak{q}^{\ell-2} + \mathfrak{q}^{\ell-1}.$$

Noted that

$$\lim_{\mathfrak{q}\to 1^-} [\ell]_{\mathfrak{q}} = \ell.$$

Utilizing $D_{\mathfrak{q}} f(z)$, the subfamily of $\mathfrak{q}$-starlike function of order α, $x^*_{\mathfrak{q}}(\alpha)$, is defined by meeting:

$$\operatorname{Re}\left(\frac{zD_{\mathfrak{q}} f(z)}{f(z)}\right) > \alpha,$$

and $\mathfrak{q}$-convex function of order α, $y^*_{\mathfrak{q}}(\alpha)$, is defined by the criteria:

$$\operatorname{Re}\left(\frac{D_{\mathfrak{q}}(zD_{\mathfrak{q}} f(z))}{D_{\mathfrak{q}} f(z)}\right) > \alpha.$$

It is observed that

$$f \in y^*_{\mathfrak{q}}(\alpha) \iff zD_{\mathfrak{q}} f \in x^*_{\mathfrak{q}}(\alpha), \; \lim_{\mathfrak{q}\to 1^-} x^*_{\mathfrak{q}}(\alpha) = x^*(\alpha), \text{ and } \lim_{\mathfrak{q}\to 1^-} y^*_{\mathfrak{q}}(\alpha) = y^*(\alpha).$$

For $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$, the $(\mathfrak{p},\mathfrak{q})$-derivative operator associated with a function f of the form (2.1) is defined as follows:

$$D_{\mathfrak{p},\mathfrak{q}} f(z) = \begin{cases} \frac{f(pz)-f(qz)}{(\mathfrak{p}-\mathfrak{q})z}, & z \in \mathbb{U}^* := \mathbb{U}\backslash\{0\}, \\ f'(0), & z = 0. \end{cases}$$

Or, equivalently,

$$D_{\mathfrak{p},\mathfrak{q}} f(z) = 1 + \sum_{\ell=2}^{\infty} [\ell]_{\mathfrak{p},\mathfrak{q}} b_\ell z^{\ell-1},$$

where

$$[\ell]_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}^\ell - \mathfrak{q}^\ell}{\mathfrak{p}-\mathfrak{q}} = \mathfrak{p}^{\ell-1} + \mathfrak{p}^{\ell-2}\mathfrak{q} + \cdots + \mathfrak{p}\mathfrak{q}^{\ell-2} + \mathfrak{q}^{\ell-1}, \quad \mathfrak{p} \neq \mathfrak{q},$$

serves as a natural generalization of the $\mathfrak{q}$-number. It follows that

$$\lim_{\mathfrak{p}\to 1} [\ell]_{\mathfrak{p},\mathfrak{q}} = [\ell]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^\ell}{1-\mathfrak{q}}.$$

The possibility of extending $\mathfrak{q}$-calculus to post-quantum calculus, referred to as $(\mathfrak{p},\mathfrak{q})$-calculus, has been considered. However, this extension cannot be achieved by replacing $\mathfrak{q}$ with $\mathfrak{q}/\mathfrak{p}$ in the traditional $\mathfrak{q}$-calculus framework. Milovanović et al. [44] studied fundamental aspects of $(\mathfrak{p},\mathfrak{q})$-calculus, including $(\mathfrak{p},\mathfrak{q})$-exponentials, $(\mathfrak{p},\mathfrak{q})$-integrals, and $(\mathfrak{p},\mathfrak{q})$-differentiation. Frasin et al. [45] study a distinct family of $\mathbb{BUF}$ by employing the $(\mathfrak{p},\mathfrak{q})$-derivative operator and establish a framework in which these functions are subordinate to generalized bivariate Fibonacci polynomials, extending previous findings in geometric function theory.

It is important to observe that for $\mathfrak{p}=1$, the subfamilies $x^*_{\mathfrak{p},\mathfrak{q}}(\alpha)$ and $y^*_{\mathfrak{p},\mathfrak{q}}(\alpha)$ reduce to the corresponding subfamilies $x^*_{\mathfrak{q}}(\alpha)$ and $y^*_{\mathfrak{q}}(\alpha)$. Furthermore, in the case where $\mathfrak{p}=1$ and $\mathfrak{q}\to 1^-$, these subfamilies further simplify to $x^*(\alpha)$ and $y^*(\alpha)$, respectively.

Considering an holomorphic function $\mho(z)$ in $\mathbb{U}$ with the properties $\operatorname{Re}(\mho(z))>0$, $\mho(0)=1$, and $\mho'(0)>0$, Srivastava et al. [46] introduced generalized subfamilies of $(\mathfrak{p},\mathfrak{q})$-starlike functions, denoted as $x^*_{\mathfrak{p},\mathfrak{q}}(\mho)$, and $(\mathfrak{p},\mathfrak{q})$-convex functions, denoted as $y^*_{\mathfrak{p},\mathfrak{q}}(\mho)$. These function familyes generalize the well-known starlike and convex functions, and the authors derived Fekete-Szegö inequalities for holomorphic functions belonging to these subfamilies. In particular, for $\mathfrak{p}=1$ and

$$\mho(z)=\frac{1+(1-2\alpha)z}{1-z}, \tag{2.6}$$

the subfamilies $x^*_{\mathfrak{p},\mathfrak{q}}(\mho)$ and $y^*_{\mathfrak{p},\mathfrak{q}}(\mho)$ simplify to $x^*_{\mathfrak{q}}(\alpha)$ and $y^*_{\mathfrak{q}}(\alpha)$, respectively. Similarly, when $\mathfrak{p}=1$, $\mathfrak{q}\to 1^-$, and $\mho(z)$ given by (2.6), these subfamilies further reduce to $x^*(\alpha)$ and $y^*(\alpha)$, respectively.

3. The function subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\alpha,\lambda)$

Definition 3.1 *A function $f(z)$ of the form given in (2.4) is said to be a member of the subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\alpha,\lambda)$ if both the function f and its inverse satisfy the following condition:*

$$\operatorname{Re}\left((1-\lambda)\frac{f(z)}{z}+\lambda D_{\mathfrak{p},\mathfrak{q}}f(z)\right)>\alpha,$$

where the parameters are constrained by $0<\mathfrak{q}<\mathfrak{p}\leqslant 1$, $0\leqslant\alpha<1$, and $\lambda\geqslant 1$.

Lemma 3.1 *[1] If $h\in\mathcal{H}$ is given in the form of (2.2), then the following bound holds:*

$$|h_\ell|\leqslant 2,$$

for all $\ell\in\mathbb{N}$.

Theorem 3.1 *If f is an element of the subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\alpha,\lambda)$ and is expressed in the form given by (2.4), then*

$$|b_{\jmath+1}|\leqslant\min\left\{\frac{2(1-\alpha)}{1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}},\sqrt{\frac{1-\alpha}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})\,(\jmath+1)}}\right\}, \tag{3.1}$$

$$|b_{2\jmath+1}|\leqslant\frac{2(1-\alpha)}{1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}}, \tag{3.2}$$

$$|b_{3\jmath+1}|\leqslant\frac{2(1-\alpha)}{1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}}+\frac{2(3\jmath+2)(1-\alpha)^2}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})\,(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}. \tag{3.3}$$

Proof: From Definition 3.1, it follows that:

$$(1-\lambda)\frac{f(z)}{z}+\lambda D_{\mathfrak{p},\mathfrak{q}}f(z)=\alpha+(1-\alpha)h(z) \tag{3.4}$$

and

$$(1-\lambda)\frac{\mathfrak{g}(w)}{w}+\lambda D_{\mathfrak{p},\mathfrak{q}}\mathfrak{g}(w)=\alpha+(1-\alpha)s(w), \tag{3.5}$$

where the obtained approximations for h and s, which belong to the family $\mathcal{H}$, are given by:

$$h(z)=1+h_{\jmath}z^{\jmath}+h_{2\jmath}z^{2\jmath}+h_{3\jmath}z^{3\jmath}+\cdots \tag{3.6}$$

and

$$s(w)=1+s_{\jmath}w^{\jmath}+s_{2\jmath}w^{2\jmath}+s_{3\jmath}w^{3\jmath}+\cdots. \tag{3.7}$$

Clearly, we obtain

$$\alpha+(1-\alpha)h(z)=1+(1-\alpha)h_{\jmath}z^{\jmath}+(1-\alpha)h_{2\jmath}z^{2\jmath}+(1-\alpha)h_{3\jmath}z^{3\jmath}+\cdots \tag{3.8}$$

and

$$\alpha+(1-\alpha)s(w)=1+(1-\alpha)s_{\jmath}w^{\jmath}+(1-\alpha)s_{2\jmath}w^{2\jmath}+(1-\alpha)s_{3\jmath}w^{3\jmath}+\cdots. \tag{3.9}$$

Additionally, we find

$$\begin{aligned}(1-\lambda)\frac{f(z)}{z}+\lambda D_{\mathfrak{p},\mathfrak{q}}f(z)&=1+b_{\jmath+1}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)z^{\jmath}\\&+b_{2\jmath+1}\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)z^{2}\jmath+b_{3\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)z^{3\jmath}+\cdots\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(1-\lambda)\frac{g(w)}{w}+\lambda D_{\mathfrak{p},\mathfrak{q}}g(w)&=1-b_{\jmath+1}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)w^{\jmath}\\&+\left((\jmath+1)b_{\jmath+1}^{2}-b_{2\jmath+1}\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)w^{2}\jmath\\&-\frac{1}{2}(\jmath+1)(3\jmath+2)b_{\jmath+1}^{3}-(3\jmath+2)b_{\jmath+1}b_{2\jmath+1}+b_{3\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)w^{3\jmath}+\cdots.\end{aligned} \tag{3.11}$$

By equating the corresponding coefficients in (3.4) and (3.5), we obtain the following expressions:

$$\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{\jmath+1}=(1-\alpha)h_{\jmath}, \tag{3.12}$$

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1}=(1-\alpha)h_{2\jmath}, \tag{3.13}$$

$$\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{3\jmath+1}=(1-\alpha)h_{3\jmath}, \tag{3.14}$$

$$-\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{\jmath+1}=(1-\alpha)s_{\jmath}, \tag{3.15}$$

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^{2}-\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1}=(1-\alpha)s_{2\jmath}, \tag{3.16}$$

and

$$\begin{aligned}&-\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(\frac{1}{2}(\jmath+1)(3\jmath+2)b_{\jmath+1}^{3}\right)\\&\quad+(3\jmath+2)b_{\jmath+1}b_{2\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\\&\quad\quad-b_{3\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)=(1-\alpha)s_{3\jmath}.\end{aligned} \tag{3.17}$$

Taking (3.12) and (3.15) into account, we obtain the following qualities:

$$h_\jmath = -s_\jmath \tag{3.18}$$

and

$$2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2 b_{\jmath+1}^2 = (1-\alpha)^2\left(h_\jmath^2+s_\jmath^2\right). \tag{3.19}$$

By adding (3.13) to (3.16), we derive the following result:

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^2 = (1-\alpha)\left(h_{2\jmath}+s_{2\jmath}\right). \tag{3.20}$$

As a consequence of (3.19) and (3.20), we find that

$$b_{\jmath+1}^2 = \frac{(1-\alpha)\left(h_{2\jmath}+s_{2\jmath}\right)}{\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)} \tag{3.21}$$

and

$$b_{\jmath+1}^2 = \frac{(1-\alpha)^2\left(h_\jmath^2+s_\jmath^2\right)}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}. \tag{3.22}$$

By considering the absolute values of (3.21) and (3.22) and utilizing Lemma 3.1 for $h_\jmath$, $h_{2\jmath}$, $s_\jmath$, and $s_{2\jmath}$, we obtain the first required inequality (3.1).

To establish an upper bound for $|b_{2\jmath+1}|$, we subtract (3.13) from (3.16), yielding

$$2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1} - \left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^2 = (1-\alpha)\left(h_{2\jmath}-s_{2\jmath}\right). \tag{3.23}$$

Now, by replacing the value of $b_{\jmath+1}^2$ from (3.19) into (3.23), we arrive at the following conclusion:

$$b_{2\jmath+1} = \frac{(1-\alpha)\left(h_{2\jmath}-s_{2\jmath}\right)}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)} + \frac{(1-\alpha)^2\left(h_\jmath^2+s_\jmath^2\right)(\jmath+1)}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}. \tag{3.24}$$

On the other hand, by substituting (3.23) into (3.16), we obtain

$$b_{2\jmath+1} = \frac{(1-\alpha)h_{2\jmath}}{\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}. \tag{3.25}$$

By taking the absolute values of (3.24) and (3.25) and applying Lemma 3.1 to the coefficients $h_\jmath$, $h_{2\jmath}$, $s_\jmath$, and $s_{2\jmath}$, we obtain the second required inequality (3.2).

Next, to establish the bound for $|b_{3\jmath+1}|$, we subtract (3.14) from (3.17), yielding

$$\begin{aligned}2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{3\jmath+1} &= (1-\alpha)\left(h_{3\jmath}-s_{3\jmath}\right)\\ &\quad-\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(\frac{1}{2}(\jmath+1)(3\jmath+2)b_{\jmath+1}^3\right)\\ &\quad+(3\jmath+2)b_{\jmath+1}b_{2\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right).\end{aligned} \tag{3.26}$$

By applying (3.19) and (3.23) in (3.26), we derive

$$b_{3\jmath+1}=\frac{(1-\alpha)\left(h_{3\jmath}-s_{3\jmath}\right)}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}-\frac{(\jmath+1)(3\jmath+2)\left(1-\alpha^3\right)\left(h_\jmath^2+s_\jmath^2\right)^{\frac{3}{2}}}{4\sqrt{2^3}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^3}$$
$$+\frac{(3\jmath+2)(1-\alpha)^2\left(h_\jmath^2+s_\jmath^2\right)^{\frac{1}{2}}\left(h_{2\jmath}-s_{2\jmath}\right)}{4\sqrt{2}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}$$
$$+\frac{(3\jmath+2)(1-\alpha)^3\left(h_\jmath^2+s_\jmath^2\right)^{\frac{1}{2}}\left(h_\jmath^2+s_\jmath^2\right)(\jmath+1)}{8\sqrt{2}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^3}. \tag{3.27}$$

Finally, by taking the absolute value of (3.27) and utilizing Lemma 3.1 for the coefficients $h_\jmath$, $h_{2\jmath}$, $h_{3\jmath}$, $s_\jmath$, $s_{2\jmath}$, and $s_{3\jmath}$, we obtain the last required inequality (3.3). □

Lemma 3.2 *[47] If $h\in\mathcal{H}$ is given in the form of (2.2) and μ is a complex number, then*

$$\left|h_2-\mu h_1^2\right|\leqslant 2\max\{1,|1-2\mu|\}.$$

Theorem 3.2 *If f is a member of the subfamily $\mathbb{H}_{\mathfrak{p},\mathfrak{q}}^{\bar{\Sigma}_\jmath}(\alpha,\lambda)$ and is expressed in the form given by (2.4), with $\mu\in\mathbb{C}$, then the following inequality holds:*

$$\left|b_{2\jmath+1}-\mu b_{\jmath+1}^2\right|\leqslant\frac{1-\alpha}{1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}}\Bigg(\max\left\{1,\left|1-\tau_1\right|\right\}+\max\left\{1,\left|1-\tau_2\right|\right\}\Bigg), \tag{3.28}$$

where

$$\tau_1:=\frac{(1-\alpha)\left(2\mu\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)-(\jmath+1)\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2} \tag{3.29}$$

$$\tau_2:=\frac{(1-\alpha)\left((\jmath+1)-2\mu\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}. \tag{3.30}$$

Proof: By substituting equations (3.19) and (3.24) into the functional $b_{2\jmath+1}-\mu b_{\jmath+1}^2$ and rearranging the terms, we obtain

$$b_{2\jmath+1}-\mu b_{\jmath+1}^2=\frac{(1-\alpha)}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\Bigg(\left(h_{2\jmath}-\tau_1 h_\jmath^2\right)-\left(s_{2\jmath}-\tau_1 s_\jmath^2\right)\Bigg), \tag{3.31}$$

where τ_1 and τ_2 are defined by (3.29) and (3.30), respectively.

Finally, by applying Lemma 3.2 to (3.31), we directly obtain (3.28). Thus, the proof is complete. □

Lemma 3.3 *[48] If $h\in\mathcal{H}$ is expressed in the form given by (2.2), then*

$$2h_2=h_1^2+(4-h_1^2)\xi,$$
$$4h_3=h_1^3+2(4-h_1^2)h_1\xi-(4-h_1^2)h_1\xi^2+2(4-h_1^2)(1-|\xi|^2)z,$$

for some $|\xi|\leqslant 1$ and $|z|\leqslant 1$.

Theorem 3.3 *If f is an element of the subfamily $\mathbb{H}_{\mathfrak{p},\mathfrak{q}}^{\bar{\Sigma}_\jmath}(\alpha,\lambda)$ and is represented in the form given by (2.4), then*

$$\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right|\leqslant\begin{cases}k(2^-), & \varpi(h)\geqslant 0 \text{ and } v(h)\geqslant 0,\\ \frac{4(1-\alpha)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}, & \varpi(h)\leqslant 0 \text{ and } v(h)\leqslant 0,\\ \max\left\{\frac{4(1-\alpha)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2},k(2^-)\right\}, & \varpi(h)>0 \text{ and } v(h)<0,\\ \max\left\{k(h),k(2^-)\right\}, & \varpi(h)<0 \text{ and } v(h)>0,\end{cases}$$

where

$$\varpi(h) := \frac{(1-\alpha)}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}\left(\frac{(1-\alpha)(\jmath+1)^2}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} - \frac{\jmath}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\right) + \frac{1}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2} - \frac{1}{(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})\,(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})}, \tag{3.32}$$

$$v(h) := \frac{1}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\left(\frac{2m(1-\alpha)}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})\,(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})} + \frac{6}{(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\right) - \frac{4}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}, \tag{3.33}$$

$$k(h_0) := \frac{4(1-\alpha)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} - \frac{(1-\alpha)^2 v^2(h)}{8\varpi(h)};\ h_0 := \sqrt{\frac{-v(h)}{2\varpi(h)}}, \tag{3.34}$$

and

$$k(2^-) := \frac{4(1-\alpha)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} + (1-\alpha)^2\left(8\varpi(h)+2v(h)\right). \tag{3.35}$$

Proof: Taking (3.12) and (3.15) into account, and applying (3.18), we obtain

$$b_{\jmath+1} = \frac{(1-\alpha)}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath. \tag{3.36}$$

Additionally, combining (3.13) with (3.16), as well as (3.14) with (3.17), respectively, yields

$$b_{2\jmath+1} = \frac{(1-\alpha)^2(\jmath+1)}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}h_\jmath^2 + \frac{1-\alpha}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\left(h_{2\jmath}-s_{2\jmath}\right) \tag{3.37}$$

and

$$b_{3\jmath+1} = \frac{1-\alpha}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\left(h_{3\jmath}-q_{3\jmath}\right) + \frac{(3\jmath+2)(1-\alpha)^2}{4\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath\left(h_{2\jmath}-s_{2\jmath}\right). \tag{3.38}$$

Then, we obtain

$$b_{\jmath+1}b_{3\jmath+1} - b_{2\jmath+1}^2 = \frac{-(1-\alpha)^4(\jmath+1)^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^4}h_\jmath^4 + \frac{\jmath(1-\alpha)^3}{4\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}h_\jmath^2\left(h_{2\jmath}-s_{2\jmath}\right) + \frac{(1-\alpha)^2}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath\left(h_{3\jmath}-s_{3\jmath}\right) - \frac{(1-\alpha)^2}{4\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}\left(h_{2\jmath}-s_{2\jmath}\right)^2. \tag{3.39}$$

Based on Lemma 3.3 and considering (3.39), we obtain

$$h_{2\jmath} - s_{2\jmath} = \frac{4-h_\jmath^2}{2}(\xi-\psi) \tag{3.40}$$

and

$$h_{3\jmath}-s_{3\jmath}=\frac{h_\jmath^3}{2}+\frac{h_\jmath\left(4-h_\jmath^2\right)}{2}(\xi+\psi)-\frac{\left(4-h_\jmath^2\right)}{4}h_\jmath\left(\xi^2+\psi^2\right)$$
$$+\frac{4-h_\jmath^2}{2}\left(\left(1-|\xi|^2\right)z-\left(1-|\psi|^2\right)w\right). \quad (3.41)$$

Then, by substituting (3.40) and (3.41) into (3.39), we obtain

$$\begin{aligned}\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right| \leqslant{}& \frac{(1-\alpha)^4(\jmath+1)^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^4}h_\jmath^4+\frac{(1-\alpha)^2}{4\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^4\\
&+\frac{(1-\alpha)^2}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath\left(4-h_\jmath^2\right)\\
&+\left[\frac{\jmath(1-\alpha)^3}{8\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}h_\jmath^2\left(4-h_\jmath^2\right)\right.\\
&\left.+\frac{(1-\alpha)^2}{4\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^2\left(4-h_\jmath^2\right)\right](|\xi|+|\psi|)\\
&+\left[\frac{(1-\alpha)^2}{8\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^2\left(4-h_\jmath^2\right)\right.\\
&\left.-\frac{(1-\alpha)^2}{4\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath\left(4-h_\jmath^2\right)\right]\left(|\xi|^2+|\psi|^2\right)\\
&+\frac{(1-\alpha)^2}{16\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}\left(4-h_\jmath^2\right)^2(|\xi|+|\psi|)^2.\end{aligned}$$

By defining $|h_\jmath|=h$, we can assume, without loss of generality, that h belongs to the interval $[0,2]$. Given that $\gamma_1=|\xi|\leqslant 1$ and $\gamma_2=|\psi|\leqslant 1$, we derive

$$\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right|\leqslant \upsilon_1+(\gamma_1+\gamma_2)\upsilon_2+\left(\gamma_1^2+\gamma_2^2\right)\upsilon_3+(\gamma_1+\gamma_2)^2\upsilon_4=\Omega(\gamma_1,\gamma_2),$$

where

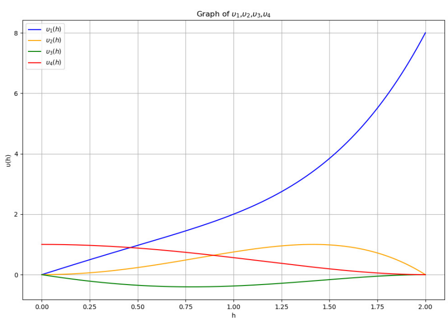
$$\upsilon_1:=\upsilon_1(h)=\frac{(1-\alpha)^2}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\left(\frac{(1-\alpha)^2(\jmath+1)^2}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^3}h^4\right.$$
$$\left.+\frac{1}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h^4-\frac{h}{\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\left(h^2-4\right)\right), \quad (3.42)$$

$$\upsilon_2:=\upsilon_2(h)=\frac{(1-\alpha)^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h^2\left(4-h^2\right)$$
$$\times\left(\frac{\jmath(1-\alpha)}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}+\frac{1}{\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\right), \quad (3.43)$$

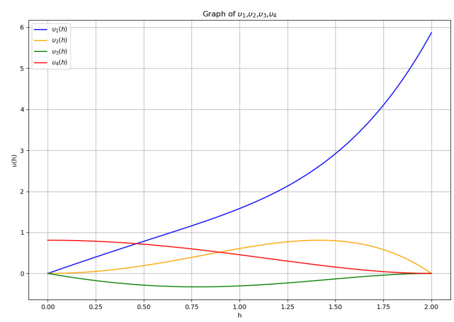
$$\upsilon_3:=\upsilon_3(h)=\frac{(1-\alpha)^2}{8\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h\left(4-h^2\right)(h-2), \quad (3.44)$$

and

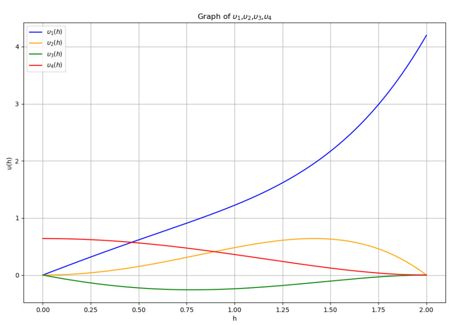
$$\upsilon_4:=\upsilon_4(h)=\frac{(1-\alpha)^2}{16\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}\left(4-h^2\right)^2. \quad (3.45)$$



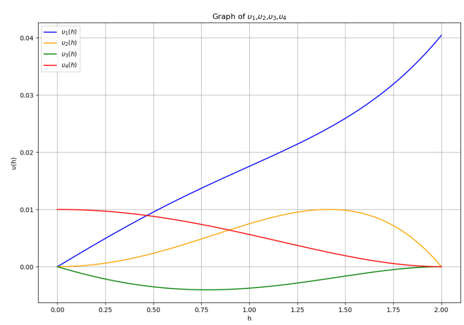
(a) $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for $\alpha = 0$



(b) $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for $\alpha = 0.1$



(c) $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for $\alpha = 0.2$



(d) $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for $\alpha = 0.9$

Figure 1: Graph of $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

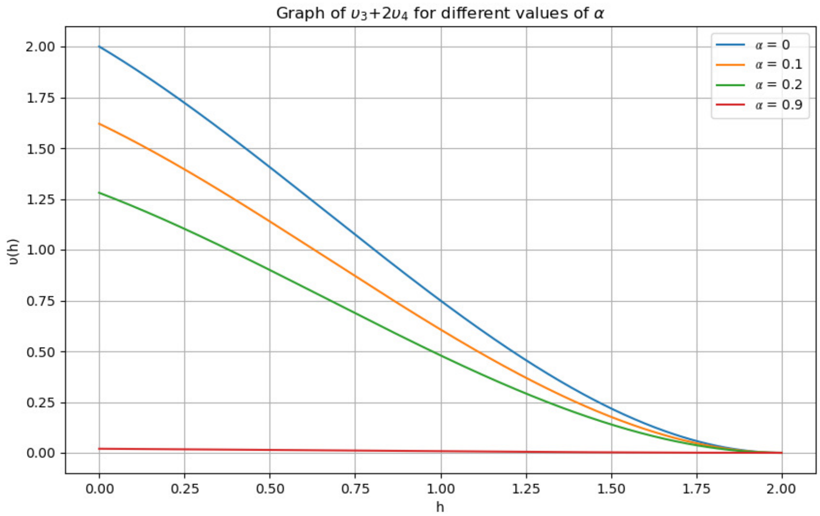


Figure 2: Graph of $\upsilon_3 + 2\upsilon_4$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

Figure 1 verifies that υ_1, υ_2, and υ_4 are non-negative, whereas υ_3 is non-positive.

Our current objective is to maximize $\Omega(\gamma_1, \gamma_2)$ within the square region $[0,1] \times [0,1]$. To achieve this, we need to determine the maximum value of $\Omega(\gamma_1, \gamma_2)$ for three cases:

Case 1. When $h \in (0,2)$, it follows that $\upsilon_3 < 0$ and $\upsilon_3 + 2\upsilon_4 > 0$ (see Fig. 2). Consequently, we deduce that $\Omega_{\gamma_1\gamma_1}\Omega_{\gamma_2\gamma_2} - (\Omega_{\gamma_1\gamma_2})^2$ is negative. This result implies that Ω cannot attain a local maximum within the interior of the square. As a result, we shift our focus to examining the boundary of the square. For $\gamma_1 = 0$ with $0 \leqslant \gamma_2 \leqslant 1$ (and similarly, for $\gamma_2 = 0$ with $0 \leqslant \gamma_1 \leqslant 1$), we obtain the following result:

$$\Omega(0, \gamma_2) = A(\gamma_2) = (\upsilon_3 + \upsilon_4)\gamma_2^2 + \upsilon_2\gamma_2 + \upsilon_1.$$

Subcase 1. If $\upsilon_3 + \upsilon_4 \geqslant 0$, then

$$A'(\gamma_2) = 2\left(\upsilon_3 + \upsilon_4\right)\gamma_2 + \upsilon_2 > 0,$$

which implies that $A(\gamma_2)$ is an increasing function. Consequently, the maximum value of $A(\gamma_2)$ is attained at $\gamma_2 = 1$, leading to

$$\max A(\gamma_2) = A(1) = \upsilon_1 + \upsilon_2 + \upsilon_3 + \upsilon_4.$$

Subcase 2. If $\upsilon_3 + \upsilon_4 < 0$, it follows that $\upsilon_2 + 2\left(\upsilon_3 + \upsilon_4\right) \geqslant 0$ (see Fig. 3). This leads to the following inequality:

$$\upsilon_2 + 2\left(\upsilon_3 + \upsilon_4\right) < 2\left(\upsilon_3 + \upsilon_4\right)\gamma_2 + \upsilon_2 < \upsilon_2.$$

As a result, we obtain $A'(\gamma_2) > 0$, indicating that $A(\gamma_2)$ is also maximized at $\gamma_2 = 1$.

Next, for $\gamma_1 = 1$ and $\gamma_2 \in [0,1]$ (similarly, for $\gamma_2 = 1$ and $\gamma_1 \in [0,1]$), we derive the following expression:

$$\Omega(1, \gamma_2) = L(\gamma_2) = \left(\upsilon_3 + \upsilon_4\right)\gamma_2^2 + \left(\upsilon_2 + 2\upsilon_4\right)\gamma_2 + \upsilon_1 + \upsilon_2 + \upsilon_3 + \upsilon_4.$$

Using a similar argument as in **Subcases 1** and **2**, we obtain:

$$\max L(\gamma_2) = L(1) = \upsilon_1 + 2\upsilon_2 + 2\upsilon_3 + 4\upsilon_4 \geqslant 0.$$

Case 2. If $h = 2$, then

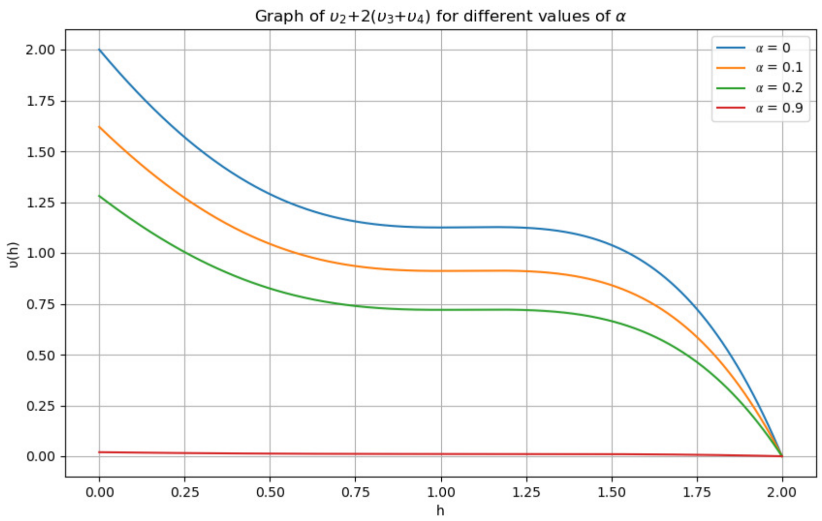


Figure 3: Graph of $\upsilon_2 + 2(\upsilon_3 + \upsilon_4)$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

$$\Omega(\gamma_1, \gamma_2) = \frac{(1-\alpha)^2}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})} \left(\frac{4(1-\alpha)^2(\jmath+1)^2}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^3} + \frac{4}{(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})} \right). \tag{3.46}$$

Considering the constant value given in (3.46), we obtain

$$\max\{\Omega(\gamma_1, \gamma_2) : \gamma_1 \in [0,1], \gamma_2 \in [0,1]\} = \Omega(1,1) = \upsilon_1 + 2(\upsilon_2 + \upsilon_3) + 4\upsilon_4.$$

Case 3. If $h = 0$, then

$$\Omega(\gamma_1, \gamma_2) = \frac{(1-\alpha)^2(\gamma_1+\gamma_2)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}.$$

It is observed that the maximum value of $\Omega(\gamma_1, \gamma_2)$ is attained at $\gamma_1 = \gamma_2 = 1$, leading to

$$\max\{\Omega(\gamma_1, \gamma_2) : \gamma_1 \in [0,1], \gamma_2 \in [0,1]\} = \Omega(1,1) = \upsilon_1 + 2(\upsilon_2 + \upsilon_3) + 4\upsilon_4.$$

By considering all cases together, and noting that $\upsilon_1 + 2(\upsilon_2 + \upsilon_3) + 4\upsilon_4 \geqslant 0$ (see Fig. (4)) for $h \in [0,2]$, we conclude that

$$\max\{\Omega(\gamma_1, \gamma_2) : \gamma_1 \in [0,1], \gamma_2 \in [0,1]\} = \Omega(1,1).$$

Consider the function $k : [0,2] \to \mathbb{R}$, defined as

$$k(h) = \max \Omega(\gamma_1, \gamma_2) = \Omega(1,1) = \upsilon_1 + 2\upsilon_2 + 2\upsilon_3 + 4\upsilon_4. \tag{3.47}$$

Substituting the values of $\upsilon_1, \upsilon_2, \upsilon_3$, and υ_4 from (3.42), (3.43), (3.44), and (3.45), respectively, into (3.47), we obtain

$$k(h) = \frac{(1-\alpha)^2}{2}\left(\varpi(h)h^4 + v(h)h^2\right) + \frac{4(1-\alpha)^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2},$$

where $\varpi(h)$ and $v(h)$ are given by (3.32) and (3.33), respectively.

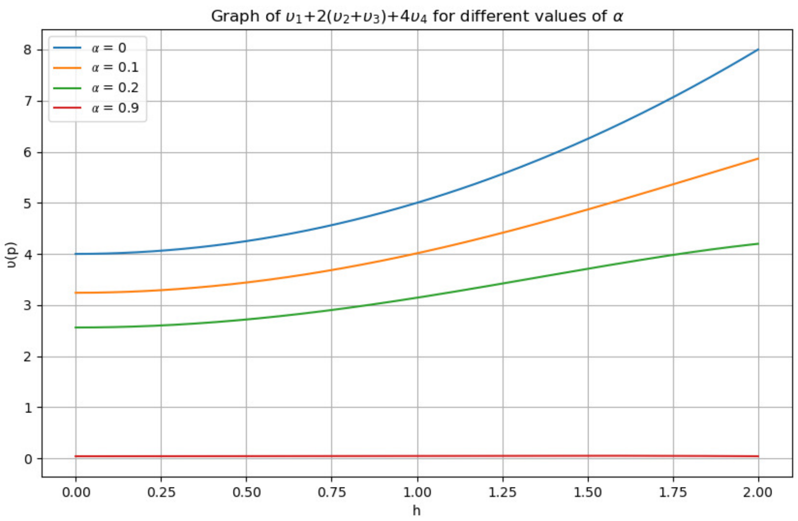


Figure 4: Graph of $\upsilon_1 + 2(\upsilon_2 + \upsilon_3) + 4\upsilon_4$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

Additionally, we note that

$$k'(h) = (1-\alpha)^2 \left(2\varpi(h)h^3 + v(h)h\right).$$

To analyze the behavior of $k'(h)$, we must consider different cases for $\varpi(h)$ and $v(h)$:

Result 1. If $\varpi(h) \geqslant 0$ and $v(h) \geqslant 0$, then $k'(h) \geqslant 0$, implying that $k(h)$ is an increasing function. Consequently,

$$\max\{k(h) : h \in (0,2)\} = k(2^-).$$

Thus,

$$\max\{\max\{\Omega(\gamma_1,\gamma_2) = \gamma_1,\gamma_2 \in [0,1]\} : h \in (0,2)\} = k(2^-),$$

where $k(2^-)$ is given by (3.35).

Result 2. If $\varpi(h) \leqslant 0$ and $v(h) \leqslant 0$, then $k'(h) \leqslant 0$ indicating that $k(h)$ is a decreasing function. Therefore,

$$\max\{k(h) : h \in (0,2)\} = k(0^+) = \frac{4(1-\alpha)^2}{\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}.$$

Result 3. If $\varpi(h) > 0$ and $v(h) < 0$, then $h_0 = \sqrt{\frac{-v(h)}{2\varpi(h)}} \in (0,2)$ is a critical point. Since $k''(h) > 0$, h_0 serves as a local minimum point of $k(h)$. Consequently, $k(h)$ does not attain a local maximum in this case.

Result 4. If $\varpi(h) < 0$ and $v(h) > 0$, then h_0 is critical point of $k(h)$. Given that $k''(h) < 0$, $h_0 \in (0,2)$ represents a local maximum point. Thus, the maximum value occurs at $h = h_0$, and we have

$$\max\{k(h) = h \in (0,2)\} = k(h_0),$$

where $k(h_0)$ is given by (3.34). This concludes the proof. □

4. The function subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\beta,\lambda)$

Definition 4.1 *A function $f(z)$, expressed in the form of (2.4), is familyified as a member of the subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\beta,\lambda)$ if both f and its inverse satisfy the following condition:*

$$\left|\arg\left((1-\lambda)\frac{f(z)}{z}+\lambda D_{\mathfrak{p},\mathfrak{q}}f(z)\right)\right|<\frac{\beta\pi}{2},$$

where $0<\mathfrak{q}<\mathfrak{p}\leqslant 1$, $0<\beta\leqslant 1$, and $\lambda\geqslant 1$.

Theorem 4.1 *Let f be an element of the subfamily $\mathbb{H}^{\bar{\Sigma}_\jmath}_{\mathfrak{p},\mathfrak{q}}(\beta,\lambda)$, where f is expressed in the form given by (2.4). Then,*

$$|b_{\jmath+1}|\leqslant\frac{2\beta}{\sqrt{\beta\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)-(\beta-1)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}},\tag{4.1}$$

$$|b_{2\jmath+1}|\leqslant\frac{2\beta}{1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}}+\frac{\beta^2(\jmath+1)}{\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2},\tag{4.2}$$

$$|b_{3\jmath+1}|\leqslant\frac{4\beta(\beta-1)(3+(\beta-2))+6\beta}{3\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}+\frac{2\beta^2(3\jmath+2)}{\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}.\tag{4.3}$$

Proof: From Definition 4.1, we have

$$(1-\lambda)\frac{f(z)}{z}+\lambda D_{\mathfrak{p},\mathfrak{q}}f(z)=[h(z)]^\beta\tag{4.4}$$

and

$$(1-\lambda)\frac{\mathfrak{g}(w)}{w}+\lambda D_{\mathfrak{p},\mathfrak{q}}\mathfrak{g}(w)=[s(w)]^\beta,\tag{4.5}$$

where $h(z)$ and $s(w)$ are given in the form of (3.6) and (3.7), respectively. It is clear that

$$\begin{aligned}[h(z)]^\beta&=1+\beta h_\jmath z^\jmath+(\frac{1}{2}\beta(\beta-1)h_\jmath^2+\beta h_{2\jmath})z^{2\jmath}\\&\quad+(\frac{1}{6}\beta(\beta-1)(\beta-2)h_\jmath^3-\beta(1-\beta)h_\jmath h_{2\jmath}+\beta h_{3\jmath})z^{3\jmath}+\cdots\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}[s(w)]^\beta&=1+\beta s_\jmath w^\jmath+(\frac{1}{2}\beta(\beta-1)s_\jmath^2+\beta s_{2\jmath})w^{2\jmath}\\&\quad+(\frac{1}{6}\beta(\beta-1)(\beta-2)s_\jmath^3-\beta(1-\beta)s_\jmath s_{2\jmath}+\beta s_{3\jmath})w^{3\jmath}+\cdots.\end{aligned}\tag{4.7}$$

By equating the corresponding coefficients in (4.4) and (4.5), we derive the following expressions:

$$\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{\jmath+1}=\beta h_\jmath,\tag{4.8}$$

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1}=\beta h_{2\jmath}+\frac{\beta(\beta-1)}{2}h_\jmath^2,\tag{4.9}$$

$$\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{3\jmath+1}=\beta(\beta-1)(h_\jmath h_{2\jmath}+\frac{1}{6}(\beta-2)h_\jmath^3)+\beta h_{3\jmath},\tag{4.10}$$

$$-\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{\jmath+1}=\beta s_\jmath,\tag{4.11}$$

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^{2}-\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1}=\beta s_{2\jmath}+\frac{\beta(\beta-1)}{2}s_{\jmath}^{2},\tag{4.12}$$

and

$$-\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(\frac{1}{2}(\jmath+1)(3\jmath+2)b_{\jmath+1}^{3}\right)+(3\jmath+2)b_{\jmath+1}b_{2\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)$$
$$-b_{3\jmath+1}\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)=\beta(\beta-1)\left(s_{\jmath}s_{2\jmath}+\frac{1}{6}(\beta-2)s_{\jmath}^{3}\right)+\beta s_{3\jmath}.\tag{4.13}$$

Taking (4.8) and (4.11) into account, we obtain that

$$h_{\jmath}=-s_{\jmath}\tag{4.14}$$

and

$$2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^{2}b_{\jmath+1}^{2}=\beta^{2}\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right).\tag{4.15}$$

By adding (4.9) to (4.12), we derive

$$\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^{2}=\beta\left(h_{2\jmath}+s_{2\jmath}\right)+\frac{\beta(\beta-1)}{2}\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right).\tag{4.16}$$

As a consequence of (4.15) and (4.16), we find that

$$b_{\jmath+1}^{2}=\frac{\beta^{2}\left(h_{2\jmath}+s_{2\jmath}\right)}{\beta\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)-(\beta-1)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^{2}}.\tag{4.17}$$

By considering the absolute values of (4.17) and utilizing Lemma 3.1 for $h_{2\jmath}$ and $s_{2\jmath}$, we obtain the first required inequality (4.1).

Next, to establish the bound on $|b_{2\jmath+1}|$, we subtract (4.9) from (4.12), yielding

$$2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)b_{2\jmath+1}-\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)(\jmath+1)b_{\jmath+1}^{2}$$
$$=\beta\left(h_{2\jmath}-s_{2\jmath}\right)+\frac{\beta(\beta-1)}{2}\left(h_{\jmath}^{2}-s_{\jmath}^{2}\right).\tag{4.18}$$

Now, by replacing the value of $b_{\jmath+1}^{2}$ from (4.15) and (4.14) into (4.18), we arrive

$$b_{2\jmath+1}=\frac{\beta\left(h_{2\jmath}-s_{2\jmath}\right)}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}+\frac{\beta^{2}\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right)(\jmath+1)}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^{2}}.\tag{4.19}$$

By considering the absolute values of (4.19) and utilizing Lemma 3.1 for the coefficients $h_{2\jmath}$ and $s_{2\jmath}$, we obtain the second required inequality (4.2).

Finally, to obtain the bound for $|b_{3\jmath+1}|$, we subtract (4.10) from (4.13), yielding

$$b_{3\jmath+1}=\frac{\beta(\beta-1)\left(h_{\jmath}h_{2\jmath}-s_{\jmath}s_{2\jmath}\right)}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}+\frac{\beta(\beta-1)(\beta-2)\left(h_{\jmath}^{3}-s_{\jmath}^{3}\right)}{12\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}-\frac{(\jmath+1)(3\jmath+2)}{4}b_{\jmath+1}^{3}$$
$$+\frac{1}{2}(3\jmath+2)b_{\jmath+1}b_{2\jmath+1}+\frac{\beta\left(h_{3\jmath}-s_{3\jmath}\right)}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}.\tag{4.20}$$

By incorporating (4.14), (4.15), and (4.18) into (4.20), we derive the following conclusion:

$$b_{3\jmath+1}=\frac{\beta(\beta-1)\left(h_{\jmath}h_{2\jmath}-s_{\jmath}s_{2\jmath}\right)}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}+\frac{\beta(\beta-1)(\beta-2)\left(h_{\jmath}^{3}-s_{\jmath}^{3}\right)}{12\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}$$
$$-\frac{(\jmath+1)(3\jmath+2)\beta^{3}\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right)^{\frac{3}{2}}}{2^{\frac{7}{2}}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^{3}}+\frac{\beta^{2}(3\jmath+2)\left(h_{2\jmath}-s_{2\jmath}\right)\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right)^{\frac{1}{2}}}{4\sqrt{2}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}$$
$$+\frac{(3\jmath+2)\beta^{3}\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right)(\jmath+1)\left(h_{\jmath}^{2}+s_{\jmath}^{2}\right)^{\frac{1}{2}}}{8\sqrt{2}\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^{3}}+\frac{\beta\left(h_{3\jmath}-s_{3\jmath}\right)}{2\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}.\tag{4.21}$$

Finally, by considering the absolute value of (4.21) and utilizing Lemma 3.1 for the coefficients $h_{\jmath}$, $h_{2\jmath}$, $h_{3\jmath}$, $s_{\jmath}$, $s_{2\jmath}$, and $s_{3\jmath}$, we obtain the last required inequality (4.3). □

Theorem 4.2 *Let f be an element of the subfamily $\mathbb{H}^{\bar{\Sigma}_{\jmath}}_{\mathfrak{p},\mathfrak{q}}(\beta,\lambda)$ and be expressed in the form given by (2.4). Then,*

$$\left|b_{2\jmath+1}-\mu b_{\jmath+1}^2\right| \leqslant \frac{\beta\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)-2\mu\beta^2\varepsilon_2}{\varepsilon_2\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)}\max\left\{1,|1-\varepsilon_3|\right\} + \frac{\beta\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)+2\mu\beta^2\varepsilon_2}{\varepsilon_2\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)}\max\left\{1,|1-\varepsilon_4|\right\}, \quad \mu\in\mathbb{C}, \tag{4.22}$$

where

$$\varepsilon_1 := 1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}, \tag{4.23}$$

$$\varepsilon_2 := 1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}, \tag{4.24}$$

$$\varepsilon_3 := \frac{(\jmath+1)\beta^2\Big(2\varepsilon_2\big((\beta-1)\varepsilon_1^2-\beta\varepsilon_2(\jmath+1)\big)\Big)}{4\varepsilon_1^2\Big(\beta\big(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\big)-2\mu\beta^2\varepsilon_2\Big)}, \tag{4.25}$$

$$\varepsilon_4 := \frac{(\jmath+1)\beta^2\Big(2\varepsilon_2\big(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\big)\Big)}{4\varepsilon_1^2\Big(\beta\big(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\big)+2\mu\beta^2\varepsilon_2\Big)}. \tag{4.26}$$

Proof: For $\mu\in\mathbb{C}$, by utilizing equations (4.17) and (4.19), we obtain

$$b_{2\jmath+1}-\mu b_{\jmath+1}^2 = \frac{\beta\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)-2\mu\beta^2\varepsilon_2}{2\varepsilon_2\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)}\left(h_{2\jmath}-\varepsilon_3 h_{\jmath}^2\right) - \frac{\beta\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)+2\mu\beta^2\varepsilon_2}{2\varepsilon_2\left(\beta\varepsilon_2(\jmath+1)-(\beta-1)\varepsilon_1^2\right)}\left(s_{2\jmath}-s_{\jmath}^2\right), \tag{4.27}$$

where ε_1, ε_2, ε_3, and ε_4 are given by (4.23), (4.24), (4.25), and (4.26), respectively.

Finally, by applying Lemma 3.2 to (4.27), we derive (4.22). This completes the proof. □

Theorem 4.3 *et f be an element of the subfamily $\mathbb{H}^{\bar{\Sigma}_{\jmath}}_{\mathfrak{p},\mathfrak{q}}(\beta,\lambda)$ and be expressed in the form given by (2.4). Then,*

$$\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right| \leqslant \begin{cases} \delta(2^-), & \chi_1(h)\geqslant 0 \text{ and } \chi_2(h)\geqslant 0,\\ \frac{4\beta^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}, & \chi_1(h)\leqslant 0 \text{ and } \chi_2(h)\leqslant 0,\\ \max\left\{\frac{4\beta^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2},\delta(2^-)\right\}, & \chi_1(h)>0 \text{ and } \chi_2(h)<0,\\ \max\{\delta(h),\delta(2^-)\}, & \chi_1(h)<0 \text{ and } \chi_2(h)>0. \end{cases}$$

where

$$\chi_1(h) := \frac{\beta}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}\left(\frac{\beta(\jmath+1)^2}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}-\frac{(5\jmath+4)}{2(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\right) + \frac{(1-\beta)(2-\beta)-3}{3(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})} + \frac{1}{2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}, \tag{4.28}$$

$$\chi_2(h) := \frac{2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\left(\frac{\beta(5\jmath+4)}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} - \frac{2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\right) + \frac{6}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}, \quad (4.29)$$

$$\delta(h_0) := \frac{4\beta^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} - \frac{\beta^2\chi_2^2(\beta,h)}{8\chi_1(\beta,h)};\ h_0 := \sqrt{\frac{-\chi_2(\beta,h)}{2\chi_1(\beta,h)}}, \quad (4.30)$$

and

$$\delta(2^-) := \frac{4\beta^2}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2} + \beta^2\left(8\chi_1(h) + 2\chi_2(h)\right). \quad (4.31)$$

Proof: By taking into account (4.8) and (4.11) and making use of (4.14), we obtain

$$b_{\jmath+1} = \frac{\beta}{(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath. \quad (4.32)$$

Additionally, applying (4.9) and (4.12), along with (4.10) and (4.13), respectively, yields:

$$b_{2\jmath+1} = \frac{\beta^2(\jmath+1)}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}h_\jmath^2 + \frac{\beta}{2(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}(h_{2\jmath} - s_{2\jmath}) \quad (4.33)$$

and

$$\begin{aligned} b_{3\jmath+1} = {} & \frac{\beta(\beta-1)}{2(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}(h_\jmath h_{2\jmath} - s_\jmath s_{2\jmath}) \\ & + \frac{\beta^2(3\jmath+2)}{4(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath(h_{2\jmath} - s_{2\jmath}) \\ & + \frac{\beta(\beta-1)(\beta-2)}{12(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}(h_\jmath^3 - s_\jmath^3) + \frac{\beta}{2(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}(h_{3\jmath} - s_{3\jmath}). \quad (4.34) \end{aligned}$$

Next, we obtain confirmation demonstrating that:

$$\begin{aligned} b_{\jmath+1}b_{3\jmath+1} - b_{2\jmath+1}^2 = {} & \frac{-\beta^4(\jmath+1)^2}{4(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^4}h_\jmath^4 + (\frac{\beta^3(3\jmath+2)}{4(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})} \\ & - \frac{\beta^3(\jmath+1)}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})})h_\jmath^2(h_{2\jmath} - s_{2\jmath}) \\ & + \frac{\beta^2(\beta-1)}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath(h_\jmath h_{2\jmath} - s_\jmath s_{2\jmath}) \\ & + \frac{\beta^2(\beta-1)(\beta-2)}{12(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath(h_\jmath^3 - s_\jmath^3) \\ & + \frac{\beta^2}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath(h_{3\jmath} - s_{3\jmath}) \\ & - \frac{\beta^2}{4(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}(h_{2\jmath} - s_{2\jmath})^2. \end{aligned} \quad (4.35)$$

According to Lemma 3.3 and taking (4.35) into account, we obtain

$$h_{2\jmath}-s_{2\jmath}=\frac{4-h_\jmath^2}{2}(\xi-\psi), \tag{4.36}$$

$$h_{3\jmath}-s_{3\jmath}=\frac{h_\jmath^3}{2}+\frac{h_\jmath\left(4-h_\jmath^2\right)}{2}(\xi+\psi)-\frac{\left(4-h_\jmath^2\right)}{4}h_\jmath\left(\xi^2+\psi^2\right)+\frac{4-h_\jmath^2}{2}\left(\left(1-|\xi|^2\right)z-\left(1-|\psi|^2\right)w\right), \tag{4.37}$$

$$h_\jmath h_{2\jmath}-s_\jmath s_{2\jmath}=h_\jmath^2+\frac{1}{2}\left(4-h_\jmath^2\right)(\xi+\psi). \tag{4.38}$$

Next, by applying (4.36), (4.37), and (4.38) in (4.35), we derive

$$\begin{aligned}\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right| \leqslant{}& \frac{\beta^4(\jmath+1)^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^4}h_\jmath^4+\frac{\beta^2(1-\beta)(2-\beta)}{6\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^4\\
&+\frac{\beta^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^4+\frac{\beta^2(1-\beta)}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath^3\\
&+\frac{\beta^2}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h_\jmath(4-h_\jmath^2)\\
&+\left[\frac{\beta^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath\left(4-h_\jmath^2\right)\left(h_\jmath+(1-\beta)\right)\right.\\
&\left.+\frac{\beta^3(5\jmath+4)}{8\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath^2(4-h_\jmath^2)\right](|\xi|+|\psi|)\\
&+\left[\frac{\beta^2}{8\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}\ h_\jmath^2(4-h_\jmath^2)\right.\\
&\left.-\frac{\beta^2}{4\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h_\jmath(4-h_\jmath^2)\right](|\xi|^2+|\psi|^2)\\
&+\frac{\beta^2}{16\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)^2}\left(4-h_\jmath^2\right)^2(|\xi|+|\psi|)^2.\end{aligned}$$

Let $|h_\jmath|=h$. Without loss of generality, we can assume that $h\in[0,2]$. Given that $\zeta_1=|\xi|\leqslant 1$ and $\zeta_2=|\psi|\leqslant 1$, we obtain

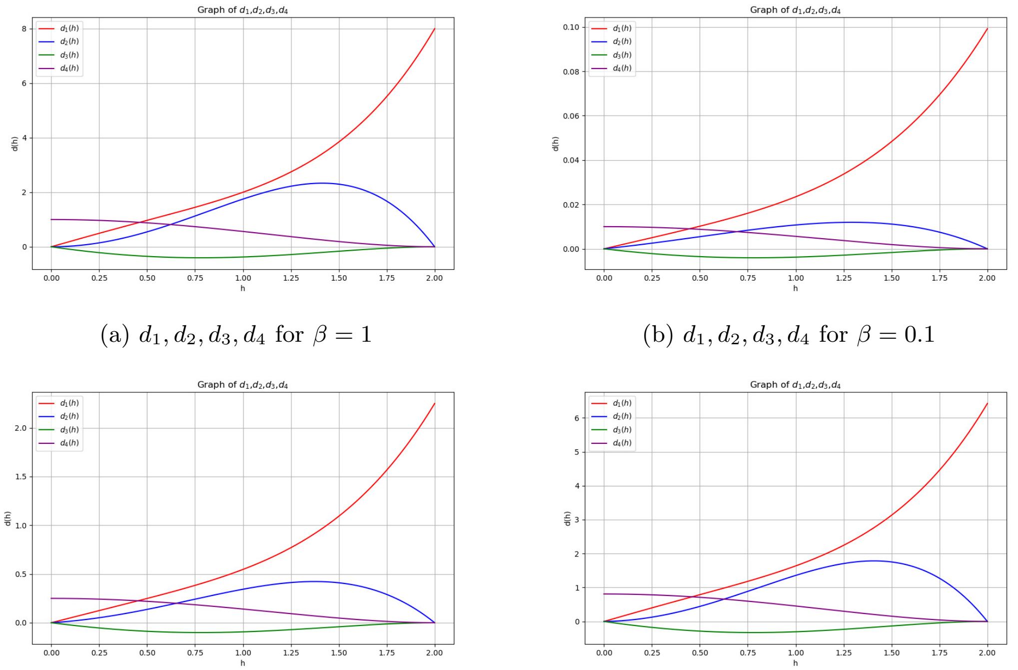
$$\left|b_{\jmath+1}b_{3\jmath+1}-b_{2\jmath+1}^2\right|\leqslant d_1+(\zeta_1+\zeta_2)d_2+\left(\zeta_1^2+\zeta_2^2\right)d_3+(\zeta_1+\zeta_2)^2d_4=\Gamma(\zeta_1,\zeta_2),$$

where

$$\begin{aligned}d_1:=d_1(h)={}&\frac{\beta^2}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})}\left(\frac{\beta^2(\jmath+1)^2}{2(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}})^3}h^4\right.\\
&\left.+\frac{(3+2(1-\beta)(2-\beta))}{6\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h^4+\frac{1}{(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}h(4-\beta h^2)\right),\end{aligned} \tag{4.39}$$

$$\begin{aligned}d_2:=d_2(h)={}&\frac{\beta^2}{4\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h\left(4-h^2\right)\\
&\times\left(\frac{\beta(5\jmath+4)}{2\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h+\frac{1}{(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}})}((1-\beta)+h)\right),\end{aligned} \tag{4.40}$$

$$d_3:=d_3(h)=\frac{\beta^2}{8\left(1-\lambda+\lambda[\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)\left(1-\lambda+\lambda[3\jmath+1]_{\mathfrak{p},\mathfrak{q}}\right)}h\left(4-h^2\right)(h-2), \tag{4.41}$$



(a) d_1, d_2, d_3, d_4 for $\beta = 1$

(b) d_1, d_2, d_3, d_4 for $\beta = 0.1$

(c) d_1, d_2, d_3, d_4 for $\beta = 0.2$

(d) d_1, d_2, d_3, d_4 for $\beta = 0.9$

Figure 5: Graph of d_1, d_2, d_3, and d_4 for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

and

$$d_4 := d_4(h) = \frac{\beta^2}{16\left(1 - \lambda + \lambda[2\jmath + 1]_{\mathfrak{p},\mathfrak{q}}\right)^2}\left(4 - h^2\right)^2. \tag{4.42}$$

Figure 5 demonstrates that d_1, d_2, and d_4 are non-negative, while d_3 is non-positive.

Currently, our objective is to maximize $\Gamma(\zeta_1, \zeta_2)$ within the square region $[0,1] \times [0,1]$. To achieve this, we need to determine the maximum value of $\Gamma(\zeta_1, \zeta_2)$ for the following cases:

Case 1. For $h \in (0,2)$, we find that $d_3 < 0$ and $d_3 + 2d_4 > 0$ (see Fig. 6). Consequently, we deduce that $\gamma_{\zeta_1\zeta_1}\gamma_{\zeta_2\zeta_2} - (\gamma_{\zeta_1\zeta_2})^2$ is negative. This implies that Γ does not attain a local maximum within the interior of the square. Therefore, we examine the boundary of the square. For $\zeta_1 = 0$ with $0 \leqslant \zeta_2 \leqslant 1$ (and similarly, for $\zeta_2 = 0$ with $0 \leqslant \zeta_1 \leqslant 1$), we obtain

$$\Gamma(0, \zeta_2) = B(\zeta_2) = (d_3 + d_4)\zeta_2^2 + d_2\zeta_2 + d_1.$$

Subcase 1. If $d_3 + d_4 \geqslant 0$, we obtain for $0 < \zeta_2 < 1$ we have that

$$B'(\zeta_2) = 2\left(d_3 + d_4\right)\zeta_2 + d_2 > 0,$$

which indicates that $B(\zeta_2)$ is an increasing function. As a result, the maximum value of $B(\zeta_2)$ is attained at $\zeta_2 = 1$, leading to

$$\max B(\zeta_2) = B(1) = d_1 + d_2 + d_3 + d_4.$$

Subcase 2. If $d_3 + d_4 < 0$, we note that $d_2 + 2\left(d_3 + d_4\right) \geqslant 0$ (see Fig. 7). This leads to the following inequality

$$d_2 + 2\left(d_3 + d_4\right) < 2\left(d_3 + d_4\right)\zeta_2 + d_2 < d_2.$$

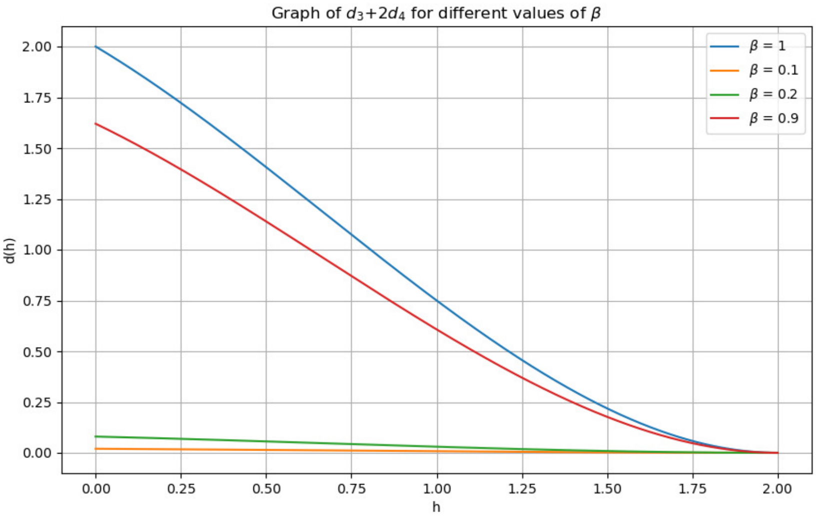


Figure 6: Graph of $d_3 + 2d_4$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

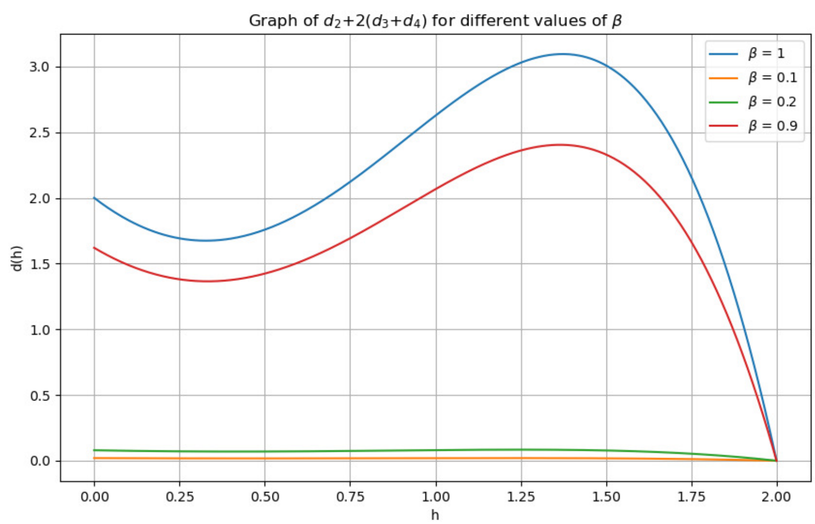


Figure 7: Graph of $d_4 + 2(d_3 + d_4)$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$.

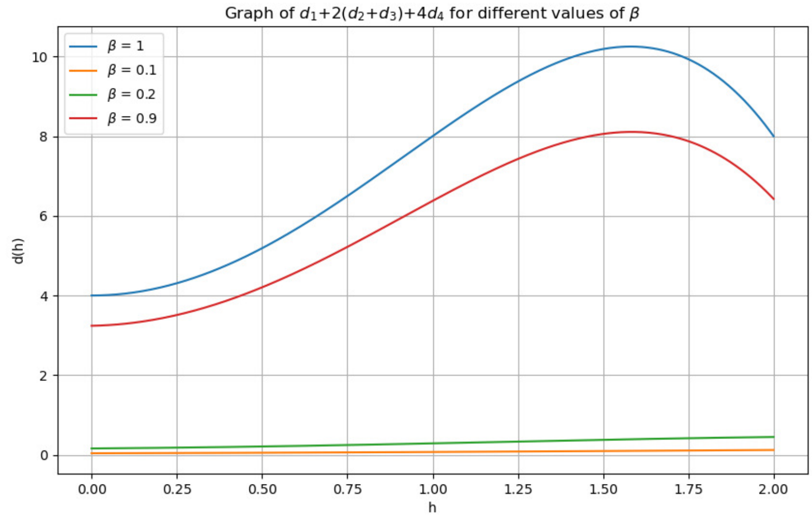


Figure 8: Graph of $d_1 + 2(d_2 + d_3) + 4d_4$ for $\lambda = \jmath = 0$ and $0 < \mathfrak{q} < \mathfrak{p} \leqslant 1$

Since $B'(\zeta_2) > 0$, this confirms that the maximum of $B(\zeta_2)$ also occurs at $\zeta_2 = 1$.

For $\zeta_1 = 1$ and $\zeta_2 \in [0, 1]$ (similarly, for $\zeta_2 = 1$ and $\zeta_1 \in [0, 1]$, where we omit the specific details of this case), we deduce

$$\Gamma(1, \zeta_2) = \mho(\zeta_2) = (d_3 + d_4)\,\zeta_2^2 + (d_2 + 2d_4)\,\zeta_2 + d_1 + d_2 + d_3 + d_4.$$

Using a similar argument to that in **Subcases 1** and **2**, we obtain

$$\max \mho(\zeta_2) = \mho(1) = d_1 + 2d_2 + 2d_3 + 4d_4.$$

Case 2. For $h = 2$, we obtain

$$\Gamma(\zeta_1, \zeta_2) = \frac{\beta^2}{2\,(1 - \lambda + \lambda[\jmath + 1]_{\mathfrak{p},\mathfrak{q}})} \left(\frac{8\beta^2(\jmath + 1)^2}{(1 - \lambda + \lambda[\jmath + 1]_{\mathfrak{p},\mathfrak{q}})^3} + \frac{8(\beta(2\beta - 9) + 10)}{3\,(1 - \lambda + \lambda[3\jmath + 1]_{\mathfrak{p},\mathfrak{q}})} \right). \tag{4.43}$$

Considering the consistent value in (4.43), we obtain

$$\max\{\Gamma(\zeta_1, \zeta_2) : \zeta_1 \in [0, 1], \zeta_2 \in [0, 1]\} = \Gamma(1, 1) = d_1 + 2(d_2 + d_3) + 4d_4.$$

Case 3. For $h = 0$, we derive

$$\Gamma(\zeta_1, \zeta_2) = \frac{\beta^2(\zeta_1 + \zeta_2)^2}{(1 - \lambda + \lambda[2\jmath + 1]_{\mathfrak{p},\mathfrak{q}})^2}.$$

It follows that the maximum of $\Gamma(\zeta_1, \zeta_2)$ occurs at $\zeta_1 = \zeta_2 = 1$, leading to

$$\max\{\Gamma(\zeta_1, \zeta_2) : \zeta_1 \in [0, 1], \zeta_2 \in [0, 1]\} = \Gamma(1, 1) = d_1 + 2(d_2 + d_3) + 4d_4.$$

Since $d_1 + 2(d_2 + d_3) + 4d_4 \geqslant 0$ (see Fig. 8), combining all cases, we conclude

$$\max\{\Gamma(\zeta_1, \zeta_2) : \zeta_1 \in [0, 1], \zeta_2 \in [0, 1]\} = \Gamma(1, 1).$$

Consider the function $\delta : [0, 2] \to \mathbb{R}$ defined as

$$\delta(h) = \max \Gamma(\zeta_1, \zeta_2) = \Gamma(1, 1) = d_1 + 2d_2 + 2d_3 + 4d_4. \tag{4.44}$$

Substituting the value of d_1, d_2, d_3, and d_4 from (4.39), (4.40), (4.41), and (4.42), respectively, into (4.44), we obtain

$$\delta(h) = \frac{\beta^2}{2}\left[\chi_1(h)h^4 + \chi_2(h)h^2\right] + \frac{8}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2},$$

where $\chi_1(h)$ and $\chi_2(h)$ are defined in (4.28) and (4.29), respectively.

Additionally, we note that

$$\delta'(h) = \beta^2\left(2\chi_1(h)h^3 + \chi_2(h)h\right).$$

Next, we analyze the function $\delta'(h)$ based on different cases of $\chi_1(h)$ and $\chi_2(h)$.

Result 1. If $\chi_1(h) \geqslant 0$ and $\chi_2(h) \geqslant 0$, then $\delta'(h) \geqslant 0$, meaning that $\delta(h)$ is an increasing function. Therefore,

$$\max\{\delta(h) : h \in (0,2)\} = \delta(2^-),$$

where $\delta(2^-)$ is given by (4.31). Alternatively,

$$\max\{\max\{\Gamma(\zeta_1,\zeta_2) = \zeta_1,\zeta_2 \in [0,1]\} : h \in (0,2)\} = \delta(2^-).$$

Result 2. If $\chi_1(h) \leqslant 0$ and $\chi_2(h) \leqslant 0$, then $\delta'(h) \leqslant 0$ indicating that $\delta(h)$ is a decreasing function. Therefore,

$$\max\{\delta(h) : h \in (0,2)\} = \delta(0^+) = \frac{8}{(1-\lambda+\lambda[2\jmath+1]_{\mathfrak{p},\mathfrak{q}})^2}.$$

Result 3. If $\chi_1(h) > 0$ and $\chi_2(h) < 0$, then $h_0 = \sqrt{\frac{-\chi_2(h)}{2\chi_1(h)}} \in (0,2)$ is critical point. Since $\delta''(h) > 0$, h_0 is a local minimum of $\delta(h)$. Consequently, $\delta(h)$ can not attain have a local maximum.

Result 4. If $\chi_1(h) < 0$ and $\chi_2(h) > 0$, then h_0 is critical point of $\delta(h)$. Given that $\delta''(h) < 0$, $h_0 \in (0,2)$ represents a local maximum. thus, the maximum value occurs at $h = h_0$, leading to

$$\max\{\delta(h) = h \in (0,2)\} = \delta(h_0),$$

where $\delta(h_0)$ is given by (4.30). This concludes the proof. □

5. Corollaries and consequences

By setting $\mathfrak{p} = 1$ and letting $\mathfrak{q} \to 1^-$ for 1-fold symmetric holomorphic functions, we obtain the following results:

Corollary 5.1 *Let* $f \in \mathbb{H}^{\bar{\Sigma}_1}_{\mathfrak{p}=1,\mathfrak{q}\to 1^-}(\alpha,\lambda) \equiv \mathbb{H}^{\bar{\Sigma}}(\alpha,\lambda)$ *be expressed in the form given by* (2.1). *Then,*

$$
\begin{aligned}
|b_2| &\leqslant \min\left\{\frac{2(1-\alpha)}{1+\lambda}, \sqrt{\frac{1-\alpha}{2(1+2\lambda)}}\right\}, \\
|b_3| &\leqslant \frac{2(1-\alpha)}{1+2\lambda}, \\
|b_4| &\leqslant 2(1-\alpha)\left[\frac{1}{1+3\lambda}+\frac{5(1-\alpha)}{(1+\lambda)(1+2\lambda)}\right], \\
\left|b_3-\mu b_2^2\right| &\leqslant \frac{1-\alpha}{1+2\lambda}\left(\max\left\{1,\left|\frac{1}{(1+\lambda)^2}\left((1+\lambda)^2-(1-\alpha)(1+2\lambda)(\mu(1+\lambda)-1)\right)\right|\right\}\right. \\
&\qquad \left. +\max\left\{1,\left|\frac{1}{(1+\lambda)^2}((1+\lambda)^2-(1-\alpha)(1+2\lambda)(1-\mu(1+\lambda)))\right|\right\}\right),\ \mu\in\mathbb{C} \\
\left|b_2b_4-b_3^2\right| &\leqslant \begin{cases} k(2^-), & \varpi(h)\geqslant 0 \ \textit{and}\ v(h)\geqslant 0, \\ \frac{4(1-\alpha)^2}{(1+2\lambda)^2}, & \varpi(h)\leqslant 0 \ \textit{and}\ v(h)\leqslant 0, \\ \max\left\{\frac{4(1-\alpha)^2}{(1+2\lambda)^2}, k(2^-)\right\}, & \varpi(h)>0 \ \textit{and}\ v(h)<0, \\ \max\left\{k(h), k(2^-)\right\}, & \varpi(h)<0 \ \textit{and}\ v(h)>0, \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\varpi(h) &:= \frac{(1-\alpha)}{2(1+\lambda)^2}\left(\frac{4(1-\alpha)}{(1+\lambda)^2}-\frac{1}{1+2\lambda}\right)+\frac{1}{2(1+2\lambda)^2}-\frac{1}{(1+3\lambda)(1+\lambda)}, \\
v(h) &:= \frac{1}{(1+\lambda)}\left(\frac{2(1-\alpha)}{(1+2\lambda)(1+\lambda)}+\frac{6}{1+3\lambda}\right)-\frac{4}{(1+2\lambda)^2}, \\
k(h_0) &:= \frac{4(1-\alpha)^2}{(1+2\lambda)^2}-\frac{(1-\alpha)^2v^2(h)}{8\varpi(h)};\ h_0 := \sqrt{\frac{-v(h)}{2\varpi(h)}}, \\
k(2^-) &:= \frac{4(1-\alpha)^2}{(1+2\lambda)^2}+(1-\alpha)^2\left(8\varpi(h)+2v(h)\right).
\end{aligned}
$$

Corollary 5.2 *Let* $f\in\mathbb{H}^{\bar{\Sigma}_1}_{\mathfrak{p}=1,\mathfrak{q}\to 1^-}(\beta,\lambda)\equiv\mathbb{H}^{\bar{\Sigma}}(\beta,\lambda)$ *be expressed in the form given by* (2.1). *Then,*

$$
\begin{aligned}
|b_2| &\leqslant \frac{2\beta}{\sqrt{2\beta\,(1+2\lambda)-(\beta-1)\,(1+\lambda)^2}},\\
|b_3| &\leqslant \frac{2\beta}{1+2\lambda}+\frac{2\beta^2}{(1+\lambda)^2},\\
|b_4| &\leqslant \frac{4\beta(\beta-1)(3+(\beta-2))+6\beta}{3\,(1+3\lambda)}+\frac{10\beta^2}{(1+\lambda)\,(1+2\lambda)},\\
\left|b_3-\mu b_2^2\right| &\leqslant \frac{\beta\left(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2\right)-2\mu\beta^2(1+2\lambda)}{(1+2\lambda)\,(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2)}\max\{1,|1-\varepsilon_1|\}\\
&\quad+\frac{\beta\left(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2\right)+2\mu\beta^2(1+2\lambda)}{(1+2\lambda)\,(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2)}\max\{1,|1-\varepsilon_2|\}, \quad \mu\in\mathbb{C},\\
\left|b_2b_4-b_3^2\right| &\leqslant \begin{cases}\delta(2^-), & \chi_1(h)\geqslant 0 \ \textit{and}\ \chi_2(h)\geqslant 0,\\ \frac{4\beta^2}{(1+2\lambda)^2}, & \chi_1(h)\leqslant 0 \ \textit{and}\ \chi_2(h)\leqslant 0,\\ \max\left\{\frac{4\beta^2}{(1+2\lambda)^2},\delta(2^-)\right\}, & \chi_1(h)>0 \ \textit{and}\ \chi_2(h)<0,\\ \max\{\delta(h),\delta(2^-)\}, & \chi_1(h)<0 \ \textit{and}\ \chi_2(h)>0,\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1 &:= \frac{2\beta^2\Big(2(1+2\lambda)\big((\beta-1)(1+\lambda)^2-2\beta(1+2\lambda)\big)\Big)}{4(1+\lambda)^2\Big(\beta\big(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2\big)-2\mu\beta^2(1+2\lambda)\Big)}\\
\varepsilon_2 &:= \frac{2\beta^2\Big(2(1+2\lambda)\big(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2\big)\Big)}{4(1+\lambda)^2\Big(\beta\big(2\beta(1+2\lambda)-(\beta-1)(1+\lambda)^2\big)+2\mu\beta^2(1+2\lambda)\Big)},\\
\chi_1(h) &:= \frac{\beta}{(1+\lambda)^2}\left(\frac{2\beta}{(1+\lambda)^2}-\frac{9}{2(1+2\lambda)}\right)+\frac{(1-\beta)(2-\beta)-3}{3(1+\lambda)(1+3\lambda)}+\frac{1}{2\,(1+2\lambda)^2},\\
\chi_2(h) &:= \frac{2}{(1+2\lambda)}\left(\frac{9\beta}{(1+\lambda)^2}-\frac{2}{1+2\lambda}\right)+\frac{6}{(1+\lambda)(1+3\lambda)},\\
\delta\,(h_0) &:= \frac{4\beta^2}{(1+2\lambda)^2}-\frac{\beta^2\chi_2^2(\beta,h)}{8\chi_1(\beta,h)};\ h_0:=\sqrt{\frac{-\chi_2(\beta,h)}{2\chi_1(\beta,h)}},\\
\delta(2^-) &:= \frac{4\beta^2}{(1+2\lambda)^2}+\beta^2\,(8v_1(h)+2v_2(h)).
\end{aligned}
$$

Acknowledgments

The authors would like to express their sincere appreciation to the referees and the editor for their positive evaluations and constructive comments.

References

1. P. L. Duren, *Univalent Functions*, Grundlehren der Math. Wiss., vol. 259, Springer, New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
2. M. Lewin, "On a coefficient problem for bi-univalent functions," *Proc. Am. Math. Soc.*, vol. 18, no. 1, pp. 63–68, 1967.
3. D. A. Brannan and J. J. Clunie, eds., *Aspects of Contemporary Complex Analysis*, Academic Press, London, UK; New York, NY, USA, 1980.

4. E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$," *Arch. Ration. Mech. Anal.*, vol. 32, no. 1, pp. 100–112, 1969.

5. D. Tan, "Coefficient estimates for bi-univalent functions," *Chin. Ann. Math., Ser. A*, vol. 5, no. 5, pp. 559–568, 1984.

6. H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Appl. Math. Lett.*, vol. 23, no. 10, pp. 1188–1192, 2010.

7. P. Sharma, S. Sivasubramanian, and N. E. Cho, "Initial coefficient bounds for certain new subclasses of bi-Bazilevič functions and exponentially bi-convex functions with bounded boundary rotation," *Axioms*, vol. 13, no. 1, article 25, 2024.

8. S. Kazimoğlu, E. Deniz, and C. Cotirlă, "Certain subclasses of analytic and bi-univalent functions governed by the Gegenbauer polynomials linked with q-derivative," *Mathematics*, vol. 11, no. 6, article 1192, 2023.

9. P. O. Sabir, "Some remarks on subclasses of bi-univalent functions defined by the Ruscheweyh derivative operator," *Filomat*, vol. 38, no. 4, pp. 1255–1264, 2024.

10. H. M. Srivastava, P. O. Sabir, K. I. Absullah, N. H. Mohammed, N. Chorfi, and P. O. Mohammed, "A comprehensive subclass of bi-univalent functions defined by a linear combination and satisfying subordination conditions," *AIMS Math.*, vol. 8, no. 12, pp. 29975–29994, 2023.

11. F. H. Jackson, "On q-functions and a certain difference operator," *Proc. Edinb. Math. Soc.*, vol. 46, pp. 253–281, 1909.

12. F. H. Jackson, T. Fukuda, and O. J. Dunn, "On q-definite integrals," *Proc. Lond. Math. Soc.*, vol. 41, no. 1, pp. 193–203, 1910.

13. P. Njionou Sadjang, "On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas," *Results Math.*, vol. 73, article 39, 2018.

14. R. George, S. Etemad, I. Stamova, and R. Alubady, "Existence of solutions for $[p, q]$-difference initial value problems: application to the $[p, q]$-based model of vibrating eardrums," *AIMS Math.*, vol. 10, no. 2, pp. 2321–2346, 2025.

15. H. M. Srivastava, S. Sivasubramanian, and R. Sivakumar, "Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions," *Int. J. Anal. Math.*, vol. 7, no. 2, pp. 1–10, 2014.

16. E. J. Muthaiyan, "$(\mathcal{P}, \mathcal{Q})$-Lucas polynomial coefficient estimates for new subclasses of m-fold symmetric bi-univalent functions," *Int. J. Anal. Math.*, vol. 36, no. 1, pp. 45–56, 2023.

17. M. Fekete and G. Szegö, "Eine Bemerkung ǔber ungerade schlichte Funktionen," *J. Lond. Math. Soc.*, vol. s1–8, no. 2, pp. 85–89, 1933.

18. S. Biagi, S. Dipierro, E. Valdinoci, and E. Vecchi, "A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators," *Math. Eng.*, vol. 5, no. 1, pp. 1–25, 2023.

19. E. Amini, S. Al-Omari, and J. Al-Omari, "Fekete-Szegö results for certain bi-univalent functions involving q-analogues of logarithmic functions," *Results Nonlinear Anal.*, vol. 7, no. 3, pp. 65–79, 2024.

20. A. Alsoboh, M. Çağlar, and M. Buyankara, "Fekete-Szegö inequality for a subclass of bi-univalent functions linked to q-ultraspherical polynomials," *Contemp. Math.*, vol. 5, no. 2, pp. 22531–2545, 2024.

21. S. D. Jadhav, A. B. Patil, and I. A. Wani, "Initial Taylor-Maclaurin coefficient bounds and the Fekete-Szegö problem for subclasses of m-fold symmetric analytic bi-univalent functions," *TWMS J. Appl. Eng. Math.*, vol. 14, no. 1, pp. 185–196, 2024.

22. P. O. Sabir, A. A. Lupas, S. S. Khalil, P. O. Mohammed, and M. Abdelwahed, "Some classes of Bazilevič-type close-to-convex functions involving a new derivative operator," *Symmetry*, vol. 16, no. 7, article 836, 2024.

23. H. M. Srivastava, P. O. Sabir, S. S. Eker, A. K. Wanas, P. O. Mohammed, N. Chorfi, and D. Baleanu, "Some m-fold symmetric bi-univalent function classes and their associated Taylor-Maclaurin coefficient bounds," *J. Inequal. Appl.*, vol. 2024, article 47, 2024.

24. C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *J. Lond. Math. Soc.*, vol. 41, no. 1, pp. 111–122, 1966.

25. J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean p-valent functions," *Trans. Am. Math. Soc.*, vol. 223, pp. 337–346, 1976.

26. C. Min and Y. J. Chen, "Painlevé IV, Chazy II, and asymptotics for recurrence coefficients of semi-classical Laguerre polynomials and their Hankel determinants," *Math. Methods Appl. Sci.*, vol. 46, no. 15, pp. 15270–15284, 2023.

27. P. O. Sabir, "Sharp bounds on Toeplitz determinants for starlike and convex functions associated with bilinear transformations," *Symmetry*, vol. 16, no. 5, article 595, 2024.

28. S. A. Al-Ameedee, W. G. Atshan, and F. A. Al-Maamori, "Second Hankel determinant for certain subclasses of bi-univalent functions," *J. Phys.: Conf. Ser.*, vol. 1664, no. 1, article 012044, 2020.

29. D. A. Brannan and T. Taha, "On some classes of bi-univalent functions," *J. Math. Anal. Appl.*, vol. 130, no. 2, pp. 568–587, 1988.

30. B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Appl. Math. Lett.*, vol. 24, no. 9, pp. 1569–1573, 2011.

31. E. Deniz, "Certain subclasses of bi-univalent functions satisfying subordination conditions," *J. Class. Anal.*, vol. 2, no. 1, pp. 49–60, 2013.

32. S. Sivasubramanian, R. Sivakumar, S. Kanas, and S. A. Kim, "Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions," *Ann. Polon. Math.*, vol. 113, no. 3, pp. 295–304, 2015.

33. E. Deniz, M. Çağlar, and H. Orhan, "Second Hankel determinant for bi-starlike and bi-convex functions of order β," *Appl. Math. Comput.*, vol. 271, pp. 301–307, 2015.

34. H. Orhan, N. Magesh, and J. Yamini, "Bounds for the second Hankel determinant of certain bi-univalent functions," *J. Inequal. Appl.*, vol. 2015, article 201, 2015.

35. W. J. Koepf, "Coefficients of symmetric functions of bounded boundary rotation," *Math. Proc. Camb. Philos. Soc.*, vol. 105, no. 2, pp. 323–331, 1989.

36. H. M. Srivastava, A. Zireh, and S. Hajiparvaneh, "Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions," *Filomat*, vol. 32, no. 9, pp. 3143–3153, 2018.

37. B. A. Abd and A. K. Wanas, "Coefficient bounds for new families of m-fold symmetric bi-univalent functions defined by Bazilevič convex functions," *Earthline J. Math. Sci.*, vol. 14, no. 1, pp. 105–117, 2024.

38. P. O. Sabir, H. M. Srivastava, W. G. Atshan, P. O. Mohammed, N. Chorfi, and M. Vivas-Cortez, "A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems," *Mathematics*, vol. 11, no. 18, article 3970, 2023.

39. W. Al-Rawashdeh, "Fekete-Szegö functional of a subclass of bi-univalent functions associated with Gegenbauer polynomials," *Eur. J. Pure Appl. Math.*, vol. 17, no. 1, pp. 105–115, 2024.

40. P. O. Sabir, R. P. Agarwal, S. J. Mohammedfaeq, P. O. Mohammed, N. Chorfi, and T. Abdeljawad, "Hankel determinant for a general subclass of m-fold symmetric bi-univalent functions defined by Ruscheweyh operators," *J. Inequal. Appl.*, vol. 2024, article 14, 2024.

41. H. U. Rehman, K. A. Mashrafi, and J. Salah, "Estimating the second-order Hankel determinant for the subclass of bi-close-to-convex functions of complex order," *Malays. J. Math. Sci.*, vol. 18, no. 1, pp. 1–12, 2024.

42. Q. A. Shakir and W. G. Atshan, "On third Hankel determinant for a certain subclass of bi-univalent functions," *Symmetry*, vol. 16, no. 2, article 239, 2024.

43. F. H. Jackson, "On q-difference equations," *Am. J. Math.*, vol. 32, no. 4, pp. 305–314, 1910.

44. G. V. Milovanović, V. Gupta, and N. Malik, "(p,q)-β functions and applications in approximation," *Bol. Soc. Mat. Mex.*, vol. 24, no. 1, pp. 219–237, 2018.

45. B. A. Frasin, S. R. Swamy, A. Amourah, J. Salah, and R. H. Maheshwarappa, "A family of bi-univalent functions defined by (p,q)-derivative operator subordinate to a generalized bivariate Fibonacci polynomials," *Eur. J. Pure Appl. Math.*, vol. 17, no. 4, pp. 3801–3814, 2024.

46. H. M. Srivastava, N. Raza, E. S. Abujarad, G. Srivastava, and M. Abujarad, "Fekete-Szegö inequality for classes of (p,q)-starlike and (p,q)-convex functions," *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math.*, vol. 113, no. 4, pp. 3563–3584, 2019.

47. F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proc. Am. Math. Soc.*, vol. 20, no. 1, pp. 8–12, 1969.

48. U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, 2nd ed., Univ. Calif. Press, 1958.

Shadea Ahmad Abdullah,
Department of Mathematics,
University of Sulaimani,
Sulaymaniyah 46001,
Iraq.
E-mail address: shadea.abdulla@univsul.edu.iq

and

Pishtiwan Othman Sabir,
Department of Mathematics,
University of Sulaimani,
Sulaymaniyah 46001,
Iraq.
E-mail address: pishtiwan.sabir@univsul.edu.iq