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# Average Approximate Controllability of a Nonlinear Fractional Integrodifferential System

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ABSTRACT: In this paper, we study the average approximate controllability of a nonlinear fractional integrodifferential system. We derive adequate conditions for establishing the average approximate controllability of our system. These results are obtained through the use of the resolvent operator techniques, employing the Schauder fixed point theorem. An example is provided to illustrate the application of the obtained theory.

Keywords: Average approximate controllability, Caputo derivative, integrodifferential systems, Schauder fixed point theorem.

## Contents



## 1. Introduction

Fractional differential equations have been the object of interest researchers because of their applicability in many scientific disciplines such as chemistry, physics, biology, aerodynamics, fitting of experimental data, signal and image processing, economics, and control theory. Particularly, boundary value problems defined by fractional differential equations have been utilized to study the qualitative properties of solutions of many models describing various biological and physical phenomena; we refer to the monographs [13,15,19,28] and the papers [3,12,7,8,9].

One of these systems is the fractional control system with all its branches such as stability, controllability, and observability. In recent years, many investigations on the controllability problems of fractional behaviour have extensively appeared with various applications on linear and nonlinear systems. Particularly, the researchers have focused on exact (complete) and approximate controllability (see the papers [22,23,24,29,30]).

Average controllability has a wide range of applications in various fields: Robotics, Aerospace, Pharmacology, Physics and Chemistry..., particularly in systems where precise control is challenging or unnecessary, and the focus is on achieving control over average behavior, for example, controlling the average position or trajectory of robotic arms or autonomous vehicles, managing the average power flow in electrical grids or stabilizing average voltage levels, controlling the average altitude or trajectory of aircraft or spacecraft, regulating the average concentration of drugs in the bloodstream for therapeutic purposes.....

The average controllability property has already been tackled by E. Zuazua et al [32,18,20,31] for some relevant PDE models, it analysis the problem of controlling the expected or averaged value of the systems states starting from a given and known initial datum and using a single control, independent of the parameters involved, see [1,2,14,32,31].

Average controllability is equivalent, by duality, to a property of averaged observability in which the goal is to estimate the norm of the data of the parameter-dependent adjoint system, out of partial

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measurements done on the averages concerning the unknown parameters. This property is of interest on its own when dealing with the observability of parameter-dependent systems. The actual realization of the system depends on the parameter being unknown, it is natural to address the problem based on the measurements done on averages.
The notion of average control is weaker than the classical one of simultaneous control introduced in Lions [16].

E. Zuazua in [32] analysed the problem of controlling parameter-dependent systems. He introduced the notion of average control according to which the quantity of interest is the average of the states with respect to the parameter. He considers the system

$$\begin{cases} x^{\cdot}(t) = A(\nu)x(t) + B(\nu)u(t), \quad t \in I = [0,T] \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $A(\nu)$ is an $(n \times n)$ matrix governing its free dynamics and $u = u(t)$ is an m-component control vector in $\mathbb{R}^m$, $m \leq n$, entering and acting on the system through the control operator $B(\nu)$, an $n \times m$ parameter-dependent matrix. The matrices A and B are assumed to depend on a parameter $\nu$ in a measurable manner.

On the other hand, M.Matar in [22] study the approximate controllability of a fractional nonlinear hybrid differential system

$$\begin{cases} ({}^cD^{\alpha} + A)z(t) = g(t, z(t))I^{1-\alpha}(Bu(t) + f(t, z(t))), \quad t \in I = [0, b] \\ z(0) = z_0 \end{cases} \tag{1.2}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha$ such that $0 < \alpha < 1$, $u \in L^2([0,T], U)$, $I^{1-\alpha}$ denotes the $(1-\alpha)$ order fractional integral and $f : [0, b] \times E \to E$ is given function.
The results are obtained by using resolvent and a sectorial operators technique via Dhage fixed point theorem.
Motivated by the above papers, we analyze the following average approximate controllability for a fractional integro-differential system depending on unknown parameter

$$\begin{cases} {}^c\mathfrak{D}^{\alpha}y(t) + \Lambda_{\sigma}y(t) = \varphi(t, y(t)) + \mathcal{I}^{1-\alpha}B_{\sigma}u(t), \quad t \in [0,T], \\ y(0) = y_0 \end{cases} \tag{1.3}$$

where ${}^c\mathfrak{D}^{\alpha}$ is the Caputo fractional derivative of order $\alpha$ such that $0 < \alpha < 1$, $\Xi$ and $U$ are two real Hilbert spaces, $\Lambda_{\sigma} : D(\Lambda_{\sigma}) \subseteq \Xi \to \Xi$ is the infinitesimal generator of an operator noted by $S_{\alpha,\sigma}(t)$ and $\varphi : [0,T] \times \Xi \to \Xi$ is a nonlinear continuous given function.
$B_{\sigma} : U \to \Xi$ is a bounded linear operator with $0 < \sigma < 1$, the control function $u \in L^2([0,T], U)$ and the initial data $y_0$ are assumed to be independent of the parameter $\sigma$, $\mathcal{I}^{1-\alpha}$ denotes the $(1-\alpha)$ order fractional integral.

The main contributions of our manuscript are as follows. In Section 2, we present some basic definitions, notations, and preliminary results. In Section 3, we study the existence of the solution by using Shauder fixed point theorem. The average approximate controllability results of the fractional differential equation (1.3) are presented in section 4. In particular, the controllability problem is transformed into a fixed point problem for an appropriate nonlinear operator in a function space. In Section 5, an example is provided to illustrate the application of the obtained theory.

## 2. Preliminaries

In this section, we introduce some essential preliminaries that will be used throughout the paper. Specifically, we define the fractional integral and Caputo derivative and provide some related lemmas and definitions.

**Definition 2.1** *[15] The fractional integral of order $\alpha$ for an integrable function $z \in L^1(0,T)$ is defined as*

$$\mathcal{I}^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \ \forall t \in [0,T], \tag{2.1}$$

*such that $n-1 < \alpha < n, n \in \mathbb{N}^*$.*

**Definition 2.2** *[15] The Caputo derivative of order $\alpha$ for a function $z \in C^n[0,T]$ is given by*

$${}^c\mathfrak{D}^\alpha z(t) = \mathcal{I}^{n-\alpha} z^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} z^{(n)}(s) ds, \ \forall t \in [0,T], \tag{2.2}$$

*where $n \in \mathbb{N}^*$ such that $n-1 < \alpha < n$.*

**Lemma 2.1** *[15] For $\alpha > 0$, the following relation holds*

$$\mathcal{I}^\alpha({}^c\mathfrak{D}^\alpha z(t)) = z(t) - d_0 - d_1 t - .... - d_{n-1} t^{n-1}, \tag{2.3}$$

*where $d_i \in \mathbb{R}, \ i = 0, ..., n-1$.*

**Definition 2.3** *[15] The Laplace transform of the Caputo fractional derivative is given by*

$$\begin{cases} \mathcal{L}\left\{{}^c\mathfrak{D}^\alpha_{t_0} x(t)\right\}(s) = s^\alpha \mathcal{L}(x(t)) - s^{\alpha-1} y(0), \quad (n-1 < \alpha \leq n), \\ \mathcal{L}\left\{\mathcal{I}^{1-\alpha} x(t)\right\}(s) = s^{\alpha-1} \mathcal{L}(x(t)). \end{cases} \tag{2.4}$$

**Definition 2.4** *: [22]*
*Let $\Lambda_\sigma$ be a closed and densely operator on $\Xi$. A resolvant generated by $\Lambda_\sigma$, is the linear bounded operator $S_{\alpha,\sigma}$ given by*

$$S_{\alpha,\sigma} = \mathcal{L}^{-1}(\delta^{\alpha-1} R(\delta, \Lambda_\sigma)), \tag{2.5}$$

*where $S_{\alpha,\sigma}(0) = I$ ( $I$ is the identity operator) and*

$$R(\delta, \Lambda_\sigma) = (\delta I + \Lambda_\sigma)^{-1}, \ \delta \in \mathbb{C}. \tag{2.6}$$

**Theorem 2.1** *(Schauder fixed point Theorem) Let $F = (F, \|.\|_F)$ be a Banach space, let $B$ be a closed convex bounded subset of $F$. Let $\Psi : B \to B$ be a continuous and compact mapping. Then,$\Psi$ admits a fixed point belonging to $B$.*

Next, we will present various definitions of an average controllable system.

**Definition 2.5** *System (1.3) is said to be approximately controllable on $[0,T]$ if for every desired final state $y_d \in \Xi$ and $\varepsilon > 0$, there exists a control $u \in L^2([0,T], U)$ independent of $\sigma$ such that $y$ satisfies*

$$\|y(T,u) - y_d\|_\Xi < \varepsilon.$$

**Definition 2.6** *System (1.3) is said to be average controllable on $[0,T]$ if for every desired final state $y_d \in \Xi$, there exists a control $u \in L^2([0,T], U)$ such that $y$ satisfes*

$$\int_0^1 y(T,\sigma,u) d\sigma = y_d.$$

*it means that the quantity of interest is the average of the states with respect to the parameter $\sigma$.*

**Definition 2.7** *System (1.3) is said to be average approximately controllable on $[0,T]$ if for every desired final state $y_d \in \Xi$ and $\varepsilon > 0$, there exists a control $u \in L^2([0,T], U)$ independent of $\sigma$ such that*

$$\| \int_0^1 y(T,\sigma,u) d\sigma - y_d\|_\Xi < \varepsilon,$$

## 3. Existence results

In this section, we establish and prove conditions for the existence of the solution of integro-differential fractional control nonlinear systems by applying Schauder's fixed point theorem.

**Lemma 3.1** *Let $\Im(t)$ be a continuous function on $[0,T]$. The linear fractional system*

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha}y(t)+\Lambda_{\sigma}y(t)=I^{1-\alpha}B_{\sigma}u(t)+\Im(t), \quad t\in[0,T], \\ y(0)=y_0 \end{cases} \tag{3.1}$$

*has an integral solution given by*

$$y(t)=S_{\alpha,\sigma}(t)y_0+\int_0^t S_{\alpha,\sigma}(t-s)B_{\sigma}u(s)ds \tag{3.2}$$

$$+\int_0^t (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\Im(s)ds, \forall t\in[0,T].$$

**Proof:**
Applying the Laplace transform to system (3.1), we get

$$\mathcal{L}({}^{c}\mathfrak{D}^{\alpha}y(t))(\delta)+\mathcal{L}(\Lambda_{\sigma}y(t))(\delta)=\mathcal{L}(I^{1-\alpha}B_{\sigma}u(t))(\delta)+\mathcal{L}(\Im(t))(\delta).$$

By using the equation (2.4), we get

$$\delta^{\alpha}\mathcal{L}(y(t))-\delta^{\alpha-1}y(0)+\Lambda_{\sigma}\mathcal{L}(y(t))=\delta^{\alpha-1}\mathcal{L}B_{\sigma}u(t)(\delta)+\mathcal{L}(\Im(t))(\delta), \tag{3.3}$$

that implies

$$\mathcal{L}(y(t))=(\delta^{\alpha}I+\Lambda_{\sigma})^{-1}[\delta^{\alpha-1}y_0+\delta^{\alpha-1}\mathcal{L}(B_{\sigma}u(t))+\mathcal{L}(\Im(t)(\delta)].$$

So

$$\mathcal{L}(y(t))=\mathcal{L}(S_{\alpha,\sigma}y_0)(\delta)+(\mathcal{L}S_{\alpha,\sigma})(\mathcal{L}B_{\sigma}u)(\delta)+R(\delta,\Lambda_{\sigma})\mathcal{L}(\Im(t)(\delta).$$

Taking the inverse Laplace transform, we get

$$y(t)=S_{\alpha,\sigma}y_0+\mathcal{L}^{-1}\left(\mathcal{L}(S_{\alpha,\sigma}(t)*(B_{\sigma}u(t)))+\mathcal{L}^{-1}\left(\mathcal{L}(I^{1-\alpha}S_{\alpha,\sigma}(t)*(\Im(t))\right)\right. . \tag{3.4}$$

On the other hand, by taking the derivative ${}^{c}\mathfrak{D}^{\alpha}$ of (3.4) the problem (3.1) is obtaind. This finishes the proof.

**Remark 3.1** On the basis of Lemma (3.1), the non-linear problem (1.3) is equivalent to the integral equation

$$y(t)=S_{\alpha,\sigma}(t)y_0+\int_0^t S_{\alpha,\sigma}(t-s)B_{\sigma}u(s)ds+\int_0^t (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds. \tag{3.5}$$

To present and demonstrate the main results, we set the following conditions :
(A1) The bounded linear operators $S_{\alpha,\sigma}$ is strongly continuous on $\mathbb{R}^+$ .
(A2) $S_{\alpha,\sigma}$ is a compact analytic operator such that

$$\theta_{\alpha,\sigma}=sup\{\|S_{\alpha,\sigma}(t)\|,\ t\in[0,T]\}.$$

(A3) $S_{\alpha,\sigma}(t)D(\Lambda_{\sigma})\subset D(\Lambda_{\sigma})$ and $\Lambda_{\sigma}S_{\alpha,\sigma}(t)y=S_{\alpha,\sigma}(t)\Lambda_{\sigma}y, \forall y\in\Xi$.
(A4) $B_{\sigma}:U\to\Xi$ is a linear bounded operator and there exists $\eta>0$ such that $\|B_{\sigma}\|\leq\eta$.
(A5) The function $\varphi:[0,T]\times\Xi\to\Xi$ is continuous, uniformly bounded and there exists $\aleph>0$ such that

$$\|\varphi(t,y)\|\leq\aleph,\ \forall(t,y)\in[0,T]\times\Xi.$$

**Theorem 3.1** *Assume that conditions (A1)-(A5) are satisfied. Then, the system (1.3) has at least one solution on [0,T].*

**Proof:** :
We define an operator $\Psi : \Xi \to \Xi$ by:

$$\Psi y(t) = S_{\alpha,\sigma}(t)y_0 + \int_0^t S_{\alpha,\sigma}(t-s)B_\sigma u(s)ds + \int_0^t (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds, \tag{3.6}$$

and for all $y \in C([0,T],\Xi)$, we have

$$\|\Psi y\| = sup\{\|\Psi y(t)\|, t \in [0,T]\}.$$

Step 1 : The continuity of the operator $\Psi$ is obtained by the continuity of the operators $\varphi$ and $S_{\alpha,\sigma}$.
Step 2 : We show that $\Psi(E)$ is uniformly bounded on $C([0,T],\Xi)$ where $E$ is a subset in $\Xi$ such that:

$$E = \{y \in \Xi,\ \|y\|_\Xi \le \varrho\},$$

for some constant $\varrho > 0$, we have

$$\begin{aligned}
\|\Psi y(t)\| &\le \|S_{\alpha,\sigma}(t)y_0\| + \int_0^t \|S_{\alpha,\sigma}(t-s)B_\sigma u(s)\|ds + \int_0^t \|(I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))\|ds \\
&\le \|S_{\alpha,\sigma}(t)\|\|y_0\| + \int_0^t \|S_{\alpha,\sigma}(t-s)\|\|B_\sigma u(s)\|ds + \int_0^t \|(I^{1-\alpha}S_{\alpha,\sigma}(t-s)\|\|\varphi(s,y(s))\|ds \\
&\le \varrho\theta_{\alpha,\sigma} + \theta_{\alpha,\sigma}\int_0^t \|B_\sigma u(s)\|ds + \frac{\aleph}{\Gamma(2-\alpha)}\|I^{1-\alpha}S_{\alpha,\sigma}(t-s)\|ds \\
&\le \varrho\theta_{\alpha,\sigma} + \theta_{\alpha,\sigma}\eta^2\sqrt{T}\|u\|_{L^2([0,T],U)} + \frac{\aleph}{\Gamma(3-\alpha)}\theta_{\alpha,\sigma}T^{2-\alpha} < +\infty.
\end{aligned}$$

which implies that $\Psi(E)$ is uniformly bounded. Step 3: Now, we show that $\Psi(E)$ is equicontinuous. For any $t_1, t_2 \in [0,T],\ 0 \le t_1 < t_2 \le T$ and $y \in \Xi$, we obtain

$$\begin{aligned}
\|(\Psi y)(t_2) - (\Psi y)(t_1)\| &\le \|(S_{\alpha,\sigma}(t_2)y_0 + \int_0^{t_2} S_{\alpha,\sigma}(t_2-s)B_\sigma u(s)ds) \\
&+ \int_0^{t_2}(I^{1-\alpha}S_{\alpha,\sigma}(t_2-s)\varphi(s,y(s))ds \\
&- \big(S_{\alpha,\sigma}(t_1)y_0 + \int_0^{t_1} S_{\alpha,\sigma}(t_1-s)B_\sigma u(s)ds) \\
&+ \int_0^{t_1}(I^{1-\alpha}S_{\alpha,\sigma}(t_1-s)\varphi(s,y(s))ds\big)\| \\
&\le \|S_{\alpha,\sigma}(t_2) - S_{\alpha,\sigma}(t_1)\|\|y_0\| \\
&+ \|\int_0^{t_1}\big(S_{\alpha,\sigma}(t_2-s) - S_{\alpha,\sigma}(t_1-s)\big)B_\sigma u(s)ds\| \\
&+ \|\int_0^{t_1} I^{1-\alpha}\big((S_{\alpha,\sigma}(t_2-s) - S_{\alpha,\sigma}(t_1-s)\big)\varphi(s,y(s))ds\| \\
&+ \|\int_{t_1}^{t_2} S_{\alpha,\sigma}(t_1-s))B_\sigma u(s)ds\| + \|\int_{t_1}^{t_2}(I^{1-\alpha}S_{\alpha,\sigma}(t_1-s))\varphi(s,y(s)ds\| \\
&\le \|S_{\alpha,\sigma}(t_2) - S_{\alpha,\sigma}(t_1)\|\|y_0\| \\
&+ \eta\left(\int_0^{t_1}\|S_{\alpha,\sigma}(t_2-s) - S_{\alpha,\sigma}(t_1-s)\|^2ds\right)^{1/2}\|u\|_{L^2([0,T],U)} \\
&+ \aleph\int_0^{t_1}\|I^{1-\alpha}(S_{\alpha,\sigma}(t_2-s) - S_{\alpha,\sigma}(t_1-s))\|ds \\
&+ \eta\theta_{\alpha,\sigma}\int_{t_1}^{t_2}\|u(s)\|ds + \frac{\aleph\theta_{\alpha,\sigma}}{\Gamma(2-\alpha)}(t_2-t_1)^{2-\alpha}.
\end{aligned}$$

We conclude that

$$\|(\Psi y)(t_2) - (\Psi y)(t_1)\| \to 0 \ as \ t_2 \to t_1. \tag{3.7}$$

which implies that $\Psi(E)$ is equicontinuous. Thus by the Arzela-Ascoli theorem $\Psi : \Xi \to \Xi$ is completely continuous. Therefore, by Schauder fixed-point theorem, $\Psi$ admits at least one fixed point as a solution of the fractional nonlinear problem (1.3), and this finishes the proof. □

## 4. Average Approximate Controllability

In this section, we formulate and prove conditions for the average approximate controllability of a nonlinear fractional control differential system.

**Definition 4.1** *We define the controllability operator $H_t : L^2([0,T], U) \longrightarrow \Xi$ by*

$$H_t u = \int_0^1 \int_0^t S_{\alpha,\sigma}(t-s) B_\sigma u(s) ds d\sigma, \ t \in [0,T]. \tag{4.1}$$

*Which is a bounded and linear operator defined on $L^2([0,T], U)$. The adjoint operator*

$$H_T^* : \Xi \longrightarrow L^2([0,T], U).$$

$$H_T^* = \int_0^1 B_\sigma^* S_{\alpha,\sigma}^*(T - .) d\sigma.$$

The controllability Gramnian $W : \Xi \longrightarrow \Xi$ is defined by

$$W = H_T H_T^* = \int_0^T \left( \int_0^1 S_{\alpha,\sigma}(T-s) B_\sigma d\sigma \int_0^1 B_\sigma^* S_{\alpha,\sigma}^*(T-s) d\sigma \right) ds.$$

The control function $u$ is given by [22]

$$u(t) = B_\sigma^* S_{\alpha,\sigma}^*(T-t)(\delta I + W)^{-1} \Phi y, \ t \in [0,T],$$

where

$$\Phi y = y_d - S_{\alpha,\sigma}(T) y_0 + \int_0^T (I^{1-\alpha} S_{\alpha,\sigma}(t-s)) \varphi(s, y(s)) ds. \tag{4.2}$$

**Theorem 4.1** *Assume that all the assumptions of Theorem (3.1) hold. If*

$$\delta(\delta I + W)^{-1} \to 0 \ as \ \delta \to 0^+.$$

*Then the fractional system (1.3) is approximately average controllable on $[0,T]$.*

**Proof:** In virtue of the Theorem (3.1), there exists a solution $y_\sigma \in C([0,T], \Xi)$ such that

$$y_\sigma(t) = S_{\alpha,\sigma}(t) y_0 + \int_0^t S_{\alpha,\sigma}(t-s) B_\sigma u(s) ds + \int_0^t (I^{1-\alpha} S_{\alpha,\sigma}(t-s)) \varphi(s, y(s)) ds,$$

Therefore

$$\begin{aligned}
\int_0^1 y_\sigma(T)d\sigma &= \int_0^1 \Bigg[S_{\alpha,\sigma}(T)y_0 + \int_0^T S_{\alpha,\sigma}(T-t)B_\sigma B_\sigma^* S_{\alpha,\sigma}^*(T-t)(\delta I + W)^{-1} \\
&\times \left(y_d - S_{\alpha,\sigma}(T)y_0 + \int_0^T (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds\right) \\
&+ \int_0^t (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds\Bigg]d\sigma \\
&= \int_0^1 \Bigg[S_{\alpha,\sigma}(T)y_0 + W(\delta I + W)^{-1} \\
&\times \left(y_d - S_{\alpha,\sigma}(T)y_0 + \int_0^T (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds\right)]d\sigma \\
&+ \int_0^t (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds\Bigg]d\sigma \\
&= \int_0^1 \left(y_d - \delta(\delta I + W)^{-1}(y_d - S_{\alpha,\sigma}(T)y_0 + \int_0^T (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds\right)d\sigma \\
\int_0^1 y_\sigma(T)d\sigma - y_d &= -\delta\int_0^1 (\delta I + W)^{-1}(y_d - S_{\alpha,\sigma}(T)y_0 + \int_0^T (I^{1-\alpha}S_{\alpha,\sigma}(t-s))\varphi(s,y(s))ds)d\sigma.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\int_0^1 y_\sigma(T)d\sigma - y_d\| &\le \int_0^1 \|\delta(\delta I + W)^{-1} \\
&\times \left(y_d - S_{\alpha,\sigma}(T)y_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^T\int_0^t (t-s)^{-\alpha}S_{\alpha,\sigma}(s))\varphi(s,y(s))ds)\right)\|d\sigma \\
&\le |\delta|\Bigg(\int_0^1 \|(\delta I + W)^{-1}\|\|y_d\|d\sigma + \int_0^1 \|(\delta I + W)^{-1}\|\|S_{\alpha,\sigma}(T)y_0\|d\sigma \\
&+ \frac{\aleph}{\Gamma(1-\alpha)}\int_0^1 \|(\delta I + W)^{-1}\|\int_0^T \|(t-s)^{-\alpha}\|\|S_{\alpha,\sigma}(t-s))\|ds\Bigg)d\sigma.
\end{aligned}$$

Using the compactness of $S_{\alpha,\sigma}$, we can deduce that

$$\|\int_0^1 y_\sigma(T)d\sigma - y_d\| \to 0 \ as \ \delta \to 0^+.$$

which implies that the fractional system (1.3) is approximately controllable on $[0,T]$.
This finishes the proof. □

## 5. Example

We consider the following fractional differential control system

$$\begin{cases} \mathfrak{D}^{0.6}y(x,t) + \sigma\frac{\partial^2}{\partial x^2}y(t,x) = \varphi(t,y(t)) + \mathcal{I}^{0.4}u(t), & x \in (0,\pi), \quad t \in (0,1], \\ y(t,0) = y(t,\pi) = 0, & t \in [0,1], \\ y(0,x) = y_0(x), & x \in (0,\pi). \end{cases} \tag{5.1}$$

Let $\Xi = U = L^2(0,\pi)$, $\varphi(t,y(t)) = \frac{t^2}{5}\sin y(t)$ and $\Lambda_\sigma = \sigma\frac{\partial^2}{\partial y^2}$, where

$$D(\Lambda_\sigma) = \{\psi \in \Xi : \psi \text{ et } \psi' \text{ sont absolument continues}, \psi'' \in \Xi, \psi(0) = \psi(\pi) = 0\}. \tag{5.2}$$

Then $\Lambda_\sigma$ is the infinitesimal generator of an compact analytic operator noted by $S_{\alpha,\sigma}(t)$, $t > 0$, which given by

$$S_{\alpha,\sigma}(t)\varphi = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle \sigma\varphi, e_n\rangle e_n, \quad t > 0, \tag{5.3}$$

where

$$e_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny), \quad n = 1, 2, ...$$

The operator $S_{\alpha,\sigma}(t)$ satisfies the hypothesis (A1)-(A3) such that

$$\theta_{\alpha,\sigma} = \sup_{t>0}|S_{\alpha,\sigma}(t)| = \|\sigma\|. \tag{5.4}$$

On the other hand, simple calculation lead to $\eta = 1$, $\aleph = \frac{1}{5}$ so (A4)-(A5) are both satisfied. Then, all the conditions of the Theorem (3.1) are satisfied. Thus by Theorem (4.1), fractional control system (5.1) is approximately average controllable on $[0, 1]$.

## 6. Conclusion

In this study, we explored the average approximate controllability of integrodifferential nonlinear fractional systems using the Caputo fractional derivative framework. We derived a necessary and sufficient condition for average approximate controllability, expressed in terms of a controllability Grammian matrix. By employing the Laplace transform, fractional calculus, and Schauder's fixed point theorem, we established sufficient conditions for achieving average approximate controllability in nonlinear fractional systems. To highlight the significance of our findings, a relevant example is included in this paper.

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