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Modification of Optimal Auxiliary Fractional Method to Harry Dym equation

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ABSTRACT: In this paper, we modify and utilised the Optimal Axillary Function Method (OAFM) for non-linear Partial Differential Equations (PDEs). The general formulation is modified for the general PDEs and tested numerically on non-linear time-fractional Harry Dym equations. The OAFM yields a rapidly convergent series solution, which is then validated by comparison with other results. From the comparison of solution, it is concluded that OAFM is operative, simple and unambiguous. The study shows that Optimal Auxiliary Fractional method is applicable in easy way, holds very short computational work and quickly converges to the exact solution of the problem.

Key Words: OAFM, PDEs, Harry Dym Equation, Exact solution.

Contents

1. Introduction

The integer order or fractional order calculus is deal with Fractional calculus. In early Fractional calculus assumptions have no such type of application in physical-world, However, later due to the relevance and applications of fractional calculus (e.g., acoustic wave propagation in inelastic porous materials [1], ultrasonic wave propagation in humanoid bones], this A proof-of-concept [2], visco-elasticity in living tissues [3], and tracing in vehicles [4]), fractional calculus assumptions are now applicable to real-world problems. Due to its extensive use in areas including electromagnetism, physics, visco-elasticity, and materials science, fractional calculus has drawn the attention of scholars recently [5]-[9]. It plays an important role in exploring the exact solution of non-linear problems. For this case, most fractional partial differential equations do not have exact solutions, but we need something more robust. Originally, the switching basis method [10]-[13] was used to deal with such complicated and extensive problems. By using these methods, we can transformed a complex problem into a simple one problem. Scientists use perturbation and other analytical methods [14]-[17] to solve non-linear problems. This method requires a small assumed parameter or initial guess. Incorrect selection of these options can affect accuracy. The theory of Homotopy is actually presented into perturbation method, and the Homotopy Perturbation Method (HPM) [18]-[20] and the Homotopy analysis method (HAM) [21] are developed to deal with small parameter problems. These approaches require an initial guess and have greater flexibility to control the area of convergence. The ptimal homotopy asymptotic method (OHAM) was developed by Marinca and Herisanu to solve the initial guess problem [22]-[26]. H. Ullah et al. [27]-[31] expanded the method to supplementary composite models by incorporating an optimal helper function that does not depend for an initial guess. In order to solve non-linear issues, Herisanu created the Optimal Auxiliary Function Method (OAFM) [33] in 2018. This technique was presented to reduce the amount of computation and obtain an accurate solution at the first iteration. This study aims to adapt the OAFM for PDEs with fractional orders. Complex fractional partial differential equations can

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be solved using FOAFM, which has been found to be a reliable and efficient method. This article is organized in some section for simplicity purpose. In the first section we introduced the history and some introduction of the subject.The formulation is discussed in second section. In third section we presented some numerical example for the testing of accuracy of the method. Graphs and tables are also discussed in the fourth section and similarly in the fifth section result and conclusion are obtained from the article.

1.1 Some Basic Definitions.

Definition 1:
A real valued function $f(\eta)$, $\eta>0$ is in space if $B_\eta, \eta \in \mathbb{R}$ for a real number $\eta<p$, $f(\eta) = \eta^p f_1(\eta)$, where $f_1(\eta) \in B(0,\infty)$, and is in space iff $f^n(\eta) \in B_\eta, n \in \mathbb{N}$.

Definition 2: The Reimann-Liouville fractional integral operator [32]

$$I^\alpha f(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^u (-\tau)^{n-1} f(\tau) d\tau, \tag{1}$$

If,

$$(\eta) = f(\eta)$$

Then,

$$I^\alpha u^\xi = \frac{\Gamma(\xi+\alpha) u^{\alpha+n}}{\Gamma(\xi+\alpha+1)}, \tag{2}$$

Definition 3:
In the Caputo notion of the functions, fractional derivative, $f(u)$ is [33]

$$D_u^\alpha f(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_0^\eta (\eta-\tau)^{n-\alpha-1} f^n(\tau) d\tau, \tag{3}$$

Definition 4:

$$If,\ n-1<\alpha \le n,\ \ n \in \mathbb{N},\ and\, f \in B_\eta^n,\ \eta \ge -1,$$

Then, [32]

$$D_\alpha^\alpha I_\alpha^\alpha f(\eta) = f(\eta) = f(\eta) - \sum_{l=0}^{n-1} f^l \frac{(\eta-\alpha)}{l!}, \eta>0 \tag{4}$$

2. Modification of OAFM for PDEs with fractional orders

Now let understand the OAFM to nonlinear ODE;

$$\frac{\partial^\alpha \Psi(\eta,t)}{\partial t^\alpha} = A(\Psi(\eta,t)) + s(\eta), \qquad \alpha > 0, \tag{5}$$

Where mathematicial expression, $\frac{\partial^\alpha}{\partial t^\alpha}$ is called the Caputo/ Riemann-Liouville fractional derivative operator, where $A = \ell + \aleph$ is said to be differential operator, Similarly the linear part is ℓ nonlinear part is $\aleph$,where s where is called source function,At this stage,$\Psi(\eta)$ is an unidentified function, the temporial independent variable is t while α is the parameter donating the fractional derivatives.
Here, the initial conditions are;

$$\begin{aligned}
&D_0^{\alpha-r}(\eta,0) = g_r(\eta),\\
&D_0^{\alpha-s}(\eta,0) = 0,\\
&D_0^{r}(\eta,0) = h_r(\eta),\\
&D_0^{s}(\eta,0) = 0,
\end{aligned}$$

while;

$$r = 0, 1, 2, ..., s-1$$
$$s = [\alpha]$$
$$r = 0, 1, 2, ..., s-1$$
$$s = [\alpha]\,.$$

Now selecting,

$$\Psi\left(\eta, t, G_k\right) = \Psi_0\left(\eta, t\right) + \Psi_1\left(\eta, t, G_k\right), \tag{6}$$
$$K = 1, 2, ..., s$$

Using Eq. (7) in Eq. (5), we find the zeroth approximation, which is defined as;

$$\frac{\partial^{\alpha}\left(\Psi_0\left(\eta, t\right)\right)}{\partial t^{\alpha}} - s(\eta) = 0, \tag{7}$$

$$\Psi_0\left(\eta, 0\right) = g_r\left(\eta\right),$$

where,

$$r = 0, 1, 2, ..., s-1$$

The first approximation is found;

$$\frac{\partial^{\alpha}\left(\Psi_1\left(\eta, t, G_k\right)\right)}{\partial t^{\alpha}} + N\left(\Psi_0\left(\eta, t\right) + \Psi_1\left(\eta, t, G_k\right)\right) = 0, \tag{8}$$

$$\Psi_1\left(\eta, 0\right) = h_r\left(\eta\right),$$

$$r = 0, 1, 2, ..., s-1$$

Since Eqs. (8)- (7), comprise the time fractional derivatives,
Hence by applying I^{α} operator, we get

$$\Psi_0\left(\eta, t\right) = I^{\alpha}\left[s(\eta)\right] = 0, \tag{9}$$

and

$$\Psi_1\left(\eta, t, G_k\right) = I^{\alpha}\left[\aleph\left(\Psi_0\left(\eta, t\right) + \Psi_1\left(\eta, t, G_k\right)\right)\right] = 0, \tag{10}$$

Where the nonlinear term is expressed as;

$$\aleph\left(\Psi_0\left(\eta, t\right) + \Psi_1\left(\eta, t, G_k\right)\right) = \aleph\left(\Psi_0\left(\eta, t\right)\right) + \sum_{l=1}^{\infty} {\Psi^l}_1\left(t, G_k\right) \aleph^l\left(\Psi_0\left(\eta, t\right)\right). \tag{11}$$

Eq. (11) can be written as

$$\ell\left(\Psi_1\left(\eta, t, G_k\right)\right) + D_1\left(\left(\Psi_0\left(\eta, t\right), G_m\right) \Psi\left(\aleph\left(\Psi_0\left(\eta, t\right)\right)\right)\right) + D_2\left(\Psi_0\left(\eta, t\right), G_n\right) = 0, \tag{12}$$

$$B\left(\Psi_1(\eta, t, G_k), \frac{d\Psi_1(\eta, t, G_k)}{d\xi}\right) = 0,$$

Where $n = 1, 2, ..., q$ and $m = q + 1, q + 2, ...s$

Convergence of the Method

As we know the, Method of Least Squares is used to determine the ideal constants; so,

$$K\left(G_s\right) = \int_I R^2\left(\eta, G_s\right)\, d\eta, \tag{13}$$

Where I is called the equation domain.

The unknown constants are establish as;

$$\partial_{G_1} K = 0,\ \partial_{G_2} K = 0,\ ....\partial_{G_s} K = 0. \tag{14}$$

For finding the approximated solution, we used the values of equations as;

$$\Psi\left(\eta, t\right) = \Psi_0\left(\eta, t\right) + \Psi_1\left(\eta, t\right) \tag{15}$$

3. Numerical example

To show the effectiveness and accuracy of the FOAFM technique, we find that approximate solution of non-linear time-fractional Harry Dym equation and then compare with the exact solution of the problem. [33]

$$D_t^{\alpha} u(x,t) - u^3(x,t) D_x u(x,t) = 0, \tag{16}$$

$$0 < \alpha \le 1,$$

with initial conditions; $u(x,0) = (a - \frac{3\sqrt{b}}{2}x)^{\frac{2}{3}}$.

First take initial condition; $u_0(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}}$

which gives,

$$u_o(x,0) = \left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3} \tag{17}$$

Let consider,

$$NL = -\{u_0(x,t)\}^3 \partial_x u_0(x,t)$$

$$NL = \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \tag{18}$$

From the method,

$$A_1 = c_1\left(\left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3}\right)^2 + c_2\left(\left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3}\right)^4, \tag{19}$$

$$A_2 = c_3\left(\left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3}\right)^6 + c_4\left(\left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3}\right)^8, \tag{20}$$

$$\tag{21}$$

$$(u_1)^{(0,\alpha)}(x,t) = (A_1 NL + A_2) \tag{22}$$

$$(u_1)^{(0,\alpha)}(x,t) = c_3\left(a - \frac{3\sqrt{b}x}{2}\right)^4 + c_4\left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3}\left(c_1\left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} + c_2\left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3}\right) \tag{23}$$

$$u_1 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \left((A_1 NL + A_2) \right) \mathrm{d}r \tag{24}$$

$$u_1 = \frac{t^\alpha \left(c_3 \left(a - \frac{3\sqrt{b}x}{2}\right)^4 + c_4 \left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \left(c_1 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} + c_2 \left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3} \right) \right)}{\alpha\Gamma(\alpha)}; \tag{25}$$

Now take,

$$u(x,t) = u_0(x,t) + u_1 \tag{26}$$

Put the values of, u_0 and u_1 in (24), we can get;

$$= \left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3} + \frac{t^\alpha \left(c_3 \left(a - \frac{3\sqrt{b}x}{2}\right)^4 + c_4 \left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \left(c_1 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} + c_2 \left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3} \right) \right)}{\alpha\Gamma(\alpha)} \tag{27}$$

Now we solve, $\partial_t u(x,t)$,

$$= \frac{t^{-1+\alpha} \left(c_3 \left(a - \frac{3\sqrt{b}x}{2}\right)^4 + c_4 \left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \left(c_1 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} + c_2 \left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3} \right) \right)}{\Gamma(\alpha)} \tag{28}$$

$$R = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \left(\partial_t u(x,t) \right) \mathrm{d}r - \{u_0(x,t)\}^3 \partial_x u_0(x,t) \tag{29}$$

$$R = \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} + \tag{30}$$

$$\frac{c_3 \left(a - \frac{3\sqrt{b}x}{2}\right)^4 + c_4 \left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \left(c_1 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} + c_2 \left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3} \right)}{(1-\alpha)\,\Gamma(1-\alpha)\Gamma(\alpha)}; \tag{31}$$

$$0 < \alpha \leq 1,$$

Used Least Square Method for finding the values of C_i;

$$J = \int_0^1 \int_0^1 R^2 \mathrm{d}x\mathrm{d}t \tag{32}$$

$$\begin{aligned} J = {} & 0.0077064283711064 + 0.006206985974065327 5c_1{}^2 + \\ & 0.005242656518004394 c_2{}^2 + 0.12967742176991875 c_3 + \\ & 0.5516634836956286 c_3{}^2 + c_2 (0.012620549608607161 + \\ & 0.10754637020039384 c_3 + 0.09927167104236266 c_4) + \\ & c_1 (0.01380672568649323 + 0.011388768080345855 c_2 + \\ & 0.11691541135302251 c_3 + 0.10754637020039384 c_4) + \\ & 0.11886097013564825 c_4 + 1.017564020175033 c_3 c_4 + \\ & 0.47082479747052497 c_4{}^2 \end{aligned} \tag{33}$$

Put values of C_i in eq.(30)

$$\begin{aligned} c_1 &= -\,8.277873219002478 \\ c_2 &= -\,32.115729245950966 \\ c_3 &= 3.559686697277607 \\ c_4 &= 0.35827260172344844 \end{aligned}$$

$$u(x,t) = \left(a - \frac{3\sqrt{b}x}{2}\right)^{2/3} + \frac{1}{\alpha\Gamma[\alpha]} t^{\alpha} \left(3.559686697277607 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4} + 0.35827260172344844 \left(a - \frac{3\sqrt{b}x}{2}\right)^{16/3} + \sqrt{b}\left(a - \frac{3\sqrt{b}x}{2}\right)^{5/3} \left(-8.277873219002478 \left(a - \frac{3\sqrt{b}x}{2}\right)^{4/3} - 32.115729245950966 \left(a - \frac{3\sqrt{b}x}{2}\right)^{8/3}\right)\right)$$

As we know that $0 < \alpha \leq 1$, so *for* $\alpha = 1$, we obtained the exact solution of the given problem [33]. $u(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x + bt)\right)^{\frac{2}{3}}$.

4. Figures and Tables

Table 1: Comparison of Solution

x	Exact Solution	OAFM Solution	Absolute Error
0.0	1.0	1.0	3.34261×10^{-7}
0.1	0.999	0.998999	3.33425×10^{-7}
0.2	0.997999	0.997999	3.32591×10^{-7}
0.3	0.996998	0.996997	3.31757×10^{-7}
0.4	0.995996	0.995996	3.30925×10^{-7}
0.5	0.994994	0.994993	3.30093×10^{-7}
0.6	0.993991	0.993991	3.29262×10^{-7}
0.7	0.992988	0.992987	3.28431×10^{-7}
0.8	0.991984	0.991984	3.27602×10^{-7}
0.9	0.990980	0.990979	3.26773×10^{-7}
1.0	0.989975	0.989975	3.25945×10^{-7}

Now we draw the 3D, graph of u, with the help Mathematica.

5. Results Analysis and Conclusion

The mathematical theory of FOAFM offers a very precise solution to the initial value problem of the system presented in Section 3. For the computational work, we used Mathematica software. The consequences found by FOAFM are matched with exact solution of the problems given in Table in 1. The FOAFM is effective and very correct than other analytical methods in literature. The absolute errors of the technique for dissimilar values are presented in Table 1.The solution is once again tested by comparing it with the closed form solutions presented in 3D form in Figs. 1-4. Similarly the absolute errors found by FOAFM are matched with other approaches in the literature and it is determined that the FOAFM consequences are more precise than the other technique. The semi-numerical method FOAFM is used

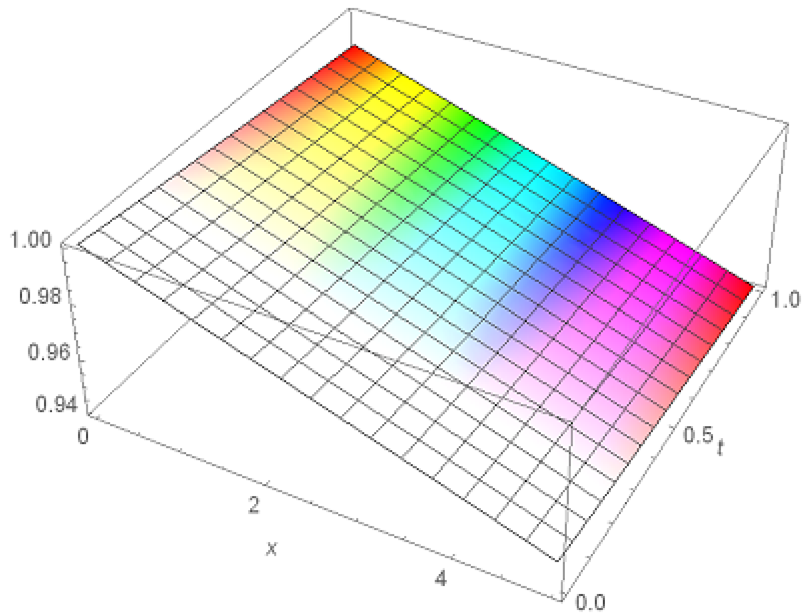

Figure 1: 3D plot of proposed solution

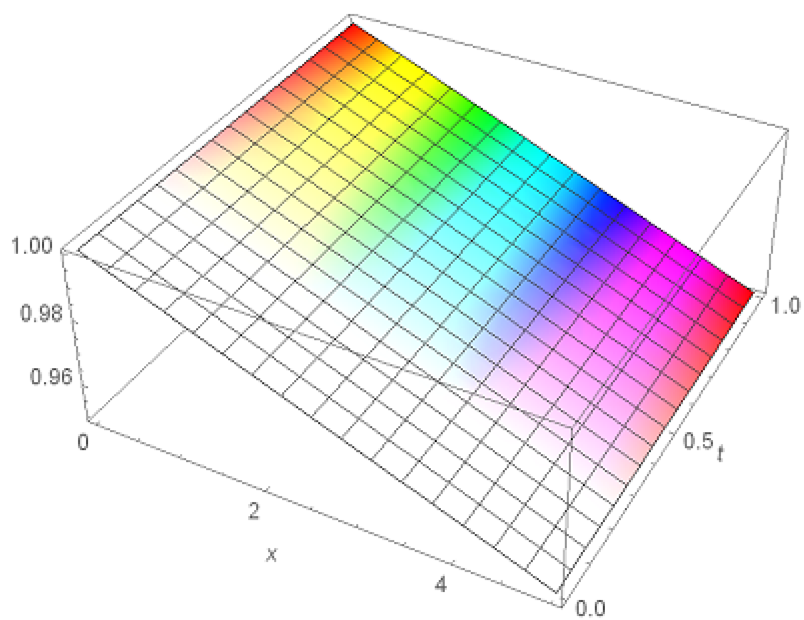

Figure 2: 3D plot of exact solution

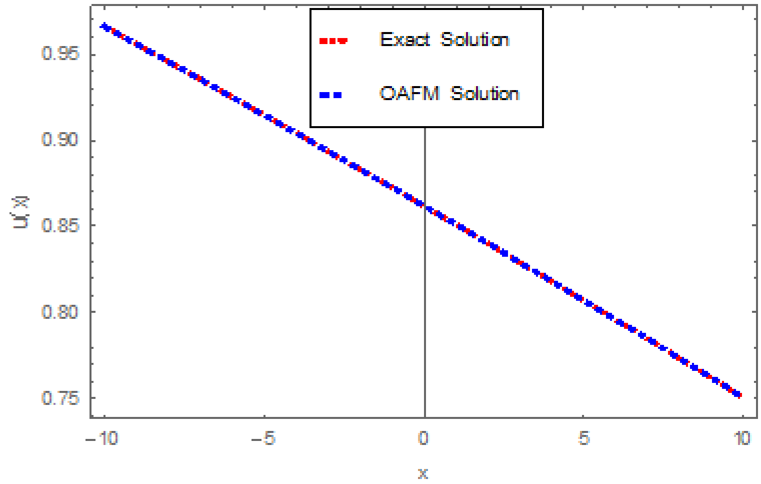

Figure 3: 2D Plot of comparison of Solution

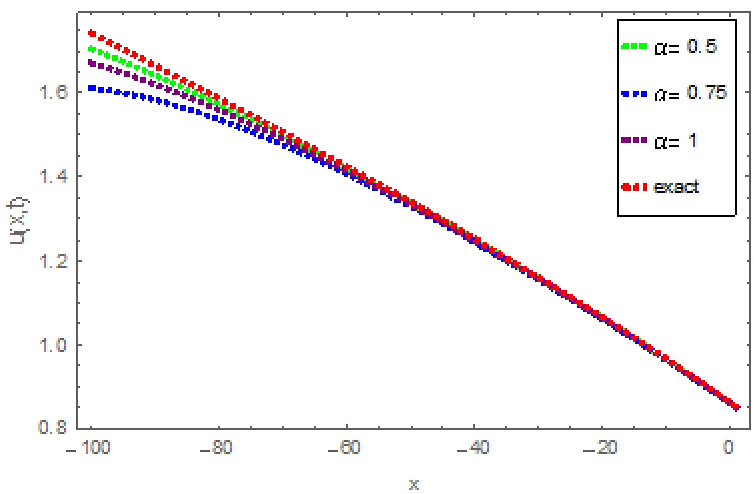

Figure 4: α variation,while $0 < \alpha \leq 1$

to solve the PDE. The series solution for the first iteration is obtained. The accuracy of FOAFM is verified by comparing its results with those available in the literature, showing that FOAFM is simple to apply and can provide accurate results with less computational effort not only for linear PDEs but also for nonlinear PDEs. The convergence of the method is controlled by means of auxiliary functions $D_i, i = 1, 2, 3$. FOAFM does not have any constraints, which allows us to implement it in complex physical problems. All the computational work has been done by Mathematica.

References

1. Z.E.A. Fellah, C. Depollier, M. Fellah, Application of fractional calculus to sound wave propagation in rigid porous materials, validation via ultrasonic measurements, Acta acustic united with acustica, Vol. 88, 34-39, (2002).
2. N. Seba, Z.E.A. Fellah, W. Lauriks and C. Depollier, Applications of fractional calculus to ultrasonic wave propagation in human cancellous bone, Signal proc. Frac.cal. 86, 2668-2677, (2006).
3. F.C. Meral, T.J. Royston, and R. Magin, Fractional calculus in viscoelastic an experimental study, Comm. Non. Sci. Num. Simi. 15, 939-945, (2010).
4. J. I. Sua´rez, B. M. Vinagre, A. J. Calder´on, C. A. Monje, and Y. Q. Chen, Using fractional calculus for lateral and longitudinal control of autonomous vehicles, in Lecture Notes in Computer Science, Springer, Berlin, Germany, 2004.
5. K. B. Oldham and J. Spanier, -e Fractional Calculus, Academic Press, New York, NY, USA, 1974.
6. M. M. Meerschaert, H.-P. Scheffler, and C. Tadjeran, "Finite difference methods for two-dimensional fractional dispersion equation," Journal of Computational Physics, vol. 211, no. 1, pp. 249–261, 2006.
7. R. Metzler and J. Klafter, "*e restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics," Journal of Physic, vol. 37, pp. 161–208, 2004.
8. I. Podlubny, Fractional Differential Equations, Academic Press, New York, NY, USA, 1999.
9. W. R. Schneider and W. Wyess, "Fractional diffusion and wave equations," Journal of Mathematic and Physic, vol. 30, pp. 134–144, 1989.
10. M. Torrisi, R. Tracina, A. Valenti, A group analysis approach for a nonlinear differential system arising in diffusion phenomena. J. Math. Phys. 37 (9) (1996), 4758-4767.
11. P.G. Drzain, R.S. Johnson, An introduction discussion the theory of solution and its diverse applications. Camb. Uni. Pres. 1989.
12. B. Abdel-Hamid, Exact solutions of some nonlinear evolution equations using symbolic computations, Comp. Math. Appl. 40 (2000) 291-302.
13. G. Bluman, S. Kumei, on the remarkable nonlinear diffusion equations. J. Math. Phys. 21 (50 1019-1023.
14. LC. Chun, Fourier series based variational iteration method for a reliable treatment of heat equations with variable coefficients. Int. J. Non-liner. Sci. Num. Simu. 10 (2007) 1383-1388.
15. S.H. Chowdhury, A comparison between the Extended homotopy perturbation method and adomian decomposition method for solving nonlinear heat transfer equations, J, Appl, Sci. 11 (8) (2011) 1416-1420.
16. H. Yaghoobi, M. Tirabi, The application of differential transformation method to nonlinear equation arising in heat transfer. Int. Comm. Heat Mass Transfer 38 (6) (2011) 815-820.
17. D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equation arising in heat transfer, Phy. Lett. A 355 (2006) 337-341.
18. R. Bellman, Perturbation techniques in Mathematics, Phys. and Engg. Holt, Rinehart and Winston, New York, 1964.
19. J.D. Cole, Perturbation Methods in Applied Mathematics, Blaisedell, Waltham, MA. 1968.
20. R.E. O'Malley, Introduction to Singular Perturbation. Acad. Pres. New York. 1974.
21. S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear Problems. PhD thesis, Shanghai Jiao Tong University; 1992. 10
22. V. Marinca, N.Herisanu, I.Nemes, Optimal homotopy asymptotic method with application to thin film flow. Central European Journal of Physics 2008; 6:648–53.
23. N. Herisanu, V, Marinca, Explicit analytical approximation to large amplitude nonlinear oscillation of a uniform contilaver beam carrying an intermediate lumped mass and rotary inertia. Mecc, 2010; 45 847-855.
24. V. Marinca, N.Herisanu, An optimal homotopic asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate. Applied Mathematics Letters 2009; 22:245–51.
25. V. Marinca, N.Herisanu, I.Nemes, A New Analytic Approach to Nonlinear Vibration of An Electrical Machine, Proceeding of the Romanian Academy 9(2008) 229-236.

26. V. Marinca, N.Herisanu, Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the Optimal Homotopy Asymptotic Method , Journal of Sound and Vibration , doi: 10. 1016 / j. jsv. 2009. 11. 005.

27. H. Ullah, S. Islam, M. Fiza, An Extension of the Optimal Homotopy Asymptotic Method to Coupled Schrödinger-KdV E equation. International journal of differential equations. Volume 2014 Article ID 106934, 12 pages.

28. H. Ullah, S. Islam, M. Idrees, M. Arif, Solution of Boundary Layer Problems with Heat Transfer by Optimal Homotopy Asymptotic Method. Abstract and Applied Analysis Volume 2013 Article ID 324869, 10 pages.

29. H. Ullah, S. Islam, M. Idrees, M. Fiza, Application of Optimal Homotopy Asymptotic Method to Doubly Wave Solutions of the Coupled Drinfel d-Sokolov-Wilson Equations. Mathematical Problem in Engineering Volume 2013, Article ID 362816, 8 pages.

30. H. Ullah, S. Islam, S. Sharidan, I. Khan, M. Fiza, Formulation and application of Optimal Homotopy Asymptotic Method for coupled Differential Difference Equations.PLOS ONE 10.137/journal.pone.0120127.

31. H. Ullah, S. Islam, I. Khan, L.C.C. Dennis, M. Fiza, Approximate Solution of Two-Dimensional Nonlinear Wave Equation by Optimal Homotopy Asymptotic Method. Mathematical Problems in Engineering Volume 2015 (2015), Article ID 380104.

32. B. Marinca, V. Marinca, Approximate analytical solutions for thin film flow of fourth grade fluid down a vertical cylinder, Proc. Roman. Acad. Ser. A. 19. (2018).

33. Maitama, S., & Abdullahi, I. (2016). A new analytical method for solving linear and nonlinear fractional partial differential equations. Progr. Fract. Differ. Appl, 2(4), 247-256.

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